Fully Actuated System Approach
A New Framework for Control
— Volume I. Global Fully Actuated Systems

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Preface

Birth of the fully actuated system approach

It is well-known that physical fully actuated systems (FASs) are a perfect type of systems in the sense that their controllers can be easily designed and the resulted closed-loop systems can often be made globally constant linear. A non-FAS is a system that is either a physical underactuated system (UAS) or a system that cannot be classified into a FAS or UAS. If, by any chance, a non-FAS can be converted into a FAS, then the control design of all non-FASs can be systematically solved. However, under physical restrictions, this goal is not practical and in general not achievable.

However, a recent significant achievement on the discovery of the mathematically generalized FAS model of dynamical systems made an important milestone toward this goal. Namely, although a non-FAS cannot be converted into a physical FAS, it can be converted into a mathematically generalized FAS. Moreover, like a physical FAS, the control of a mathematically generalized FAS can also be easily realized. Such facts and logic naturally motivate the author to propose the so-called FAS approach that solves control systems design problems based on generalized FAS models. Specifically, the author proposed the FAS approach in 2020-2021 through publishing two series of papers, the first series of 3 papers was published in *ACTA Automatica Sinica* in 2020, and the second series of 10 papers was published in *International Journal of Systems Sciences*.

By now, many researchers have joined the research on the FAS approach, and eventually there have been hundreds of papers already on this topic. It is nice to see that

- the International *Conference on Fully Actuated System Theory and Applications* (FASTA) has been held three times, with the most recent one, FASTA 2024, receiving over 600 participants;
- the technical committees on Fully Actuated System Theory and Applications have been established with the Chinese Association of Automation (CAA) and the Asian Control Association (ACA);
- the project entitled “Fully Actuated System Control Theory” has been elected as one of the 2023 Ten Major Progresses of Science and Technologies in Chinese Universities.

The proposed FAS approach may be easily misunderstood on the first sight. It is not an approach treating the physical second-order FASs, but one focusing on the math-

ematically generalized FASs. With the generalization of the definition, the mathematically generalized FASs are shown to serve as general models for dynamical systems, and eventually the so-called FAS approach naturally becomes a general methodology for control system analysis and designs.

FAS approach book series

Due to its great advantages, the FAS approach has attracted much attention ever since its emergence. By now, there have been hundreds of papers published on the FAS approach about either FAS theory development or corresponding practical control applications, including but not limited to unmanned aerial vehicles, aerospace vehicles, industrial robots and manipulators, and high-speed trains. Such a situation stimulates us to propose with Springer the following FAS Approach Book Series:

Fully Actuated System Approach for Control and AI

which is affiliated with the International *Conference on Fully Actuated System Theory and Applications* (FASTA).

FAS approach introduction volumes

As part of the above book series, the following FAS Approach Introduction Volumes aim to give an elementary introduction to the FAS approach:

Fully Actuated System Approach: A New Framework for Control

- Volume I. Global Fully Actuated Systems
- Volume II. Sub-fully Actuated Systems
- Volume III. Unidirectionally Connected FASs

These Introduction Volumes are desired to provide the best and the easiest gateway for beginners, to help control scientists enhance both the depth and width of their research work, and to assist control engineers to implement practical control systems in a more advanced and efficient way.

The present Volume I focuses on global FASs, which provides the basic idea and a more fundamental scheme for the FAS approach.

Audience

This book volume mainly has two types of audience.

Firstly, since the book gives a general and systematic introduction to the FAS approach, it can be used as the primary or alternate textbook for some control-related courses for senior undergraduate and post graduate students majoring in the fields of control systems theory and applications, and possibly for those majoring in applied mathematics, mechanical engineering, electrical engineering and aerospace engineering as well.

Secondly, the book can also be used as a reference book by post graduates, researchers, scientists, engineers, and university lecturers in the fields of control systems and applications, applied mathematics, mechanical engineering, electrical engineering and aerospace engineering, etc.

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Taking this opportunity, the author would also like to express his great gratitude to all those researchers who delved into the research on FAS approach immediately after its emergence. It is the active endeavour of this huge group of scholars and researchers that makes this very young subject so prosperous and promising today.

At the last, let me thank in advance all the readers. I would be indeed very grateful if readers could possibly provide feedback about any problems found via E-mail, g.r.duan@hit.edu.cn, or duangr@sustech.edu.cn. Your help will certainly make any future editions of the book much better.

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List of Notations

Notations related to subspaces

$\varnothing$	the null set
$\mathbb{R}$	set of all real numbers
$\mathbb{R}^+$	set of all positive real numbers
$\mathbb{R}^-$	set of all negative real numbers
$\mathbb{C}$	set of all complex numbers
$\mathbb{C}^+$	right-half complex plane
$\mathbb{C}^-$	left-half complex plane
$\mathbb{R}^n$	set of all real vectors of dimension n
$\mathbb{C}^n$	set of all complex vectors of dimension n
$\mathbb{R}^{m\times n}$	set of all real matrices of dimension $m \times n$
$\mathbb{R}^{m\times n}[s]$	set of all polynomial matrices of dimension $m \times n$ with real coefficients
$\mathbb{C}^{m\times n}$	set of all complex matrices of dimension $m \times n$
$\mathbb{C}^{m\times n}[s]$	set of all polynomial matrices of dimension $m \times n$ with complex coefficients
$\mathbb{R}_r^{m\times n}$	set of $m \times n$ real matrices with rank r
$\mathbb{C}_r^{m\times n}$	set of $m \times n$ complex matrices with rank r
$\bar{\mathbb{C}}^+$	the closed right-half complex plane, $\bar{\mathbb{C}}^+ = \{s \mid s \in \mathbb{C},\ \mathrm{Re}(s) \geq 0\}$
$\ker(T)$	kernel of transformation or matrix T
$\mathrm{Image}(T)$	image of transformation or matrix T
$\Omega_1 \backslash \Omega_2$	the complement of the set Ω_2 in set Ω_1
$\Omega_1 \times \Omega_2$	the direct product of sets Ω_1 and Ω_2

Notations related to vectors and matrices

0_n	zero vector in $\mathbb{R}^n$
$0_{m\times n}$	zero matrix in $\mathbb{R}^{m\times n}$
0	zero matrix of appropriate dimension
I_n	identity matrix of order n
I	identity matrix of appropriate dimension
diag	$\mathrm{diag}(d_i, i = 1, 2, ..., n)$, or $\mathrm{diag}(d_1, d_2, \ldots, d_n)$, represents the diagonal matrix with diagonal elements $d_i,\ i = 1, 2, \ldots, n$

A^{-1}	inverse matrix of matrix A
A^{T}	transpose of matrix A
$\bar{A}$	complex conjugate of matrix A
A^{H}	transposed complex conjugate of matrix A
$\mathrm{Re}(A)$	real part of matrix A
$\mathrm{Im}(A)$	imaginary part of matrix A
$\det(A)$	determinant of matrix A
$\mathrm{adj}(A)$	adjoint of matrix A
$\mathrm{trace}(A)$	trace of matrix A
$\mathrm{rank}(A)$	rank of matrix A
$\mathrm{cond}(A)$	condition number of matrix A
$\mathrm{eig}(A)$	set of eigenvalues of matrix A
$\lambda(A)$	set of all eigenvalues of matrix A
$\lambda_i(A)$	i-th eigenvalue of matrix A
$\lambda_{\max}(A)$	maximum eigenvalue of matrix A
$\lambda_{\min}(A)$	minimum eigenvalue of matrix A
$\sigma_i(A)$	i-th singular value of matrix A
$\sigma_{\max}(A)$	maximum singular value of matrix A
$\sigma_{\min}(A)$	minimum singular value of matrix A
$\|A\|_2$	spectral norm of matrix A
$\|A\|_F$	Frobenius norm of matrix A
$\|A\|_1$	row-sum norm of matrix A
$\|A\|_\infty$	column-sum norm of matrix A
$\dot{x}(t)$	first-order derivative of vector x with respect to t
$\ddot{x}(t)$	second-order derivative of vector x with respect to t
$\dddot{x}(t)$	third-order derivative of vector x with respect to t

Notations of relations and manipulations

$\Longrightarrow$	imply
$\Longleftrightarrow$	if and only if
$\in$	belong to
$\subset$	subset
$\cap$	intersection
$\cup$	union
$\forall$	arbitrarily chosen
$\exists$	exist
s.t.	subject to
$A \otimes B$	Kronecker product of matrices A and B

List of Abbreviations

FAS	fully actuated system
sub-FAS	sub-fully actuated system
UC-FAS	unidirectionally connected FAS
HOFAS	high-order FAS
UAS	underactuated system
DOF	degree of freedom
SFS	strict feedback system
sub-SFS	sub-strict feedback system
ESO	extended state observer
MAS	multi-agent system
UAV	unmanned aerial vehicle
LTI	linear time-invariant
LTV	linear time-varying
NHS	nonholonomic system
RoEA	region of exponential attraction
ESA	eigenstructure assignment
GSE	generalized Sylvester equation
RCF	right coprime factorization
MMO	maneuvered moving objects
GUA	globally uniformly asymptotically
LMI	linear matrix inequality
PMD	polynomial matrix description
MIMO	multi-input and multi-output
PVTOL	planar vertical take-off and landing

Chapter 1

Introduction

Physical fully actuated systems (FASs) refer to a type of mechanical systems in which the number of actuators is equal to the degrees of freedom (DOFs). Models of such systems are in general established by applying the Lagrangian Equation, and described by second-order differential equations. It is well-known that physical FASs are a perfect type of systems in the sense that their controllers can be easily designed and the resulted closed-loop systems can often be made globally constant linear.

If, by any chance, every system could be converted into a FAS, then the control designs of all dynamical systems would be systematically solved. However, under the traditional definitions and physical restrictions, this goal is not practical and in general not achievable. Fortunately, a recent significant achievement on the discovery of the mathematically generalized FAS model of dynamical systems made an important milestone toward this goal. Namely, although a non-FAS cannot be converted into a physical FAS, it can be converted into a mathematically generalized FAS. Moreover, like a physical FAS, the control of a mathematically generalized FAS can also be easily realized. Such facts and logic naturally motivate the so-called FAS approach that solves control systems design based on generalized FAS models.

In this introduction chapter, the birth backgrounds of the FAS approach and its difference from the well-known state-space approach are brie y stated. Recent developments of the FAS approach are overviewed. The contents of the book are previewed, and some commonly used symbols and preliminary technical results are presented.

1.1 State-space Approach

1.1.1 The Rising

First-order system approaches for control systems can be traced back to the year of 1750, when Euler proposed the order-reduction method for solving a high-order non-homogeneous ordinary differential equation (Kline (1990); Cui (2010)). Such an early contribution is in fact somehow equivalent to today's problem of state response analysis of a control system represented by a state-space model. This preconceived fact has then

led the way of the development of first-order system approaches for control systems.

In 1892, Aleksandr M. Lyapunov completed his Ph.D. thesis (Lyapunov (1992)), in which he proposed a complete stability theory for systems described by first-order ordinary differential equations, and laid a firm theoretical base for stability analysis and design of control systems. Lyapunov stability theory was proposed for first-order systems, and eventually and basically provides applications within the first-order system framework.

The middle of the last century was a very prosperous period for systems and control. Several very notable and significant contributions emerged.

Firstly, the well-known Bellman's Dynamic Programming was proposed during the period of 1952-1957, which converts the problem of optimal control of discrete-time and continuous-time systems described by first-order difference/differential equations into a problem of solving the well-known HJB (Hamilton-Jacobi-Bellman) equations (Bellman (1952, 1954, 1957)).

Secondly, the well-known Pontryagin's Maximum Principle was proposed in the period of 1956-1958 (Boltyanskii et al. (1956); Pontiyagin (1959, 1960); Boltyanskii et al. (1960)). Parallel to Bellman's Dynamic Programming, Pontryagin's Maximum Principle solves the problem of optimal control of a continuous-time nonlinear system, and the solution is given through a continuous HJB equation (Rozonoer (1959); Pontiyagin et al. (1962)).

Thirdly, several great contributions were conducted by Rudolf E. Kalman. In 1959, he published the epoch-making paper Kalman (1959) (see also Kalman (1960c)), and proposed the celebrated state-space approaches for analysis and design of control systems described by the so-called state-space model represented by a first-order ordinary differential equation. One year later, he published the famous result, known as the Kalman filter, which solves the problem of state estimation in a linear discrete-time stochastic system described also by a first-order state-space model (Kalman (1960b)). This work was later generalized to linear continuous-time stochastic systems by himself and his coauthor, establishing the so-called Kalman-Bucy filtering theory (Kalman and Bucy (1961)). These results have been also extended to nonlinear systems (Sunahara (1970); Julier and Uhlmann (1997)). Rudolf E. Kalman was a pioneer and advocator in bringing mathematics into control systems theory (Kalman et al. (1969); Kalman (2008)).

Guided by the above mentioned significant developments in control systems theory, most later endeavors were also laid on the first-order state-space approaches, and consequently a great deal of contributions were made within the first-order system framework (Isidori (1995); Khalil (2002a); Krstic et al. (1995); Duan and Yu (2013)). State-space approaches have remained dominant for over a half century. To an extent, today's world of control systems is the world of first-order system approaches. Although there are some results which adopt high-order system models, e.g., the polynomial approach for linear systems, they only occupy a rather small portion in the literature as compared with those within the state-space framework.

1.1.2 Deficiencies

A state-space model of a dynamical system emphasizes the state vector by integrating the state variables together, and is probably the best choice for solving the problem of response analysis. However, different from the problem of seeking the state solution, controller design emphasizes the control vector. Toward this goal, a state-space representation does not provide very much convenience. For over a half century of research, results with the first-order state-space approaches have been numerous, but today we are still facing too many unsolved nonlinear control problems (Blongdel et al. (1995)).

Control design

The basic step involved in Lyapunov stability analysis and design is to find a proper Lyapunov function. Since the condition is only sufficient for general nonlinear systems, in certain cases it might not be able to give a definite conclusion. Furthermore, even if a Lyapunov function can be found, often a local stability result is obtained. It is generally difficult to realize global stabilization of a general nonlinear system. The cases with Bellman dynamic programming and Pontryagin's maximum principle are similar, noting that the HJB equations which determine the solutions may not be solvable for complicated nonlinear systems.

Controllability and observability

Like Lyapunov stability, the concepts of controllability and observability are restricted to systems in state-space representations only (Duan (2020a,b,c)). Unlike Lyapunov stability, controllability and observability for systems in non-state-space representations can not be defined by those of the converted state-space models, since it is well-known that a system in an input-output representation may possess, at the same time, a controllable (observable) realization, and also an uncontrollable (unobservable) realization (Duan (2020b)). Therefore, sticking to the state-space approaches, the concepts of controllability and observability for non-state-space systems are vacant. However, a dynamical system in a non-state-space model representation obviously also deserves the concepts of controllability and observability. Such a phenomenon clearly reveals a serious defect of the first-order state-space framework.

Physical background

All practical systems have physical backgrounds. When modeling a system using some physical laws, such as Lagrangian Equation or Theorem of Momentum, a second- or high-order model is obtained, and this original model also re ects in some ways the physical backgrounds of the system. In certain situations, the background information may help in the analysis and design. However, with the first-order system approaches, the system model is reduced to a first-order state-space model, and all the background information is then removed from the original model.

Full-actuation

Physically, there are many FASs in the world, and their original system models established by certain physical laws are of second- or high-order. For such systems, the control can be realized in an extremely simple way, since the full-actuation property allows us to eliminate all the nonlinear dynamics. As a consequence, a desired constant linear closed-loop system can be obtained, and the theories and techniques for analysis and design of linear systems can then be applicable. Furthermore, with the help of the full-actuation feature, many design problems, such as system decoupling, can also be solved very easily. Nevertheless, when converted into first-order state-space models, such systems no longer possess the full-actuation property, and the nonlinear dynamics needs then to be tackled with Lyapunov stability design approaches. In certain complicated cases even local stabilization of the system can not be realized, to say nothing of giving a global stabilization result or obtaining a constant linear closed-loop system.

1.1.3 The Decline

State-space models are, in general, not the original models for dynamical control systems.

The original models of physical dynamical systems are the set of basic equations, either differential or difference, or mixed, established by the system mechanism, or by certain physical laws. They often appear in the form of a set of high-order differential equations. While the well-known first-order state-space models are converted from this set of basic equations based on, essentially, the method of variable extension. State-space models are a type of extreme models which possess the lowest order of differentials but the maximum number of (scalar) equations.

In terms of system control, how well do state-space models work? They worked well for typical linear systems during 1960-1980's, but after that, they encountered insurmountable difficulties in handling nonlinearities, time-varying properties, time delays, system constraints and issues of controllability, and consequently entered into a truly difficult situation, as asserted by many facts including the following.

- It was described, as early as 1982, in a survey paper by Casti (1982), that "seeking a completely general theory of nonlinear systems is somewhat akin to the search for the Holy Grail: a relatively harmless activity full of many pleasant surprises and mild disappointments, but ultimately unrewarding". Because of this, he then made the suggestion: "A far more profitable path to follow is to concentrate upon special classes of nonlinear problems, usually motivated by applications, and to use the structure inherent in these classes as a guide to useful (i.e., applicable) results".

- A book entitled *Rise of The Machines: A Cybernetic History*, written by Rid (2016), and produced by Felicity Bryan Associates Ltd., even made the following appalling and exaggerating claims: "The history of cybernetics holds several warnings. The new field's scientific popularity peaked around 1969. By the 1980s, cybernetics had lost a great deal of academic respectability; it had 'died of dry rot'". "Today, the most serious scholars working on cybernetics

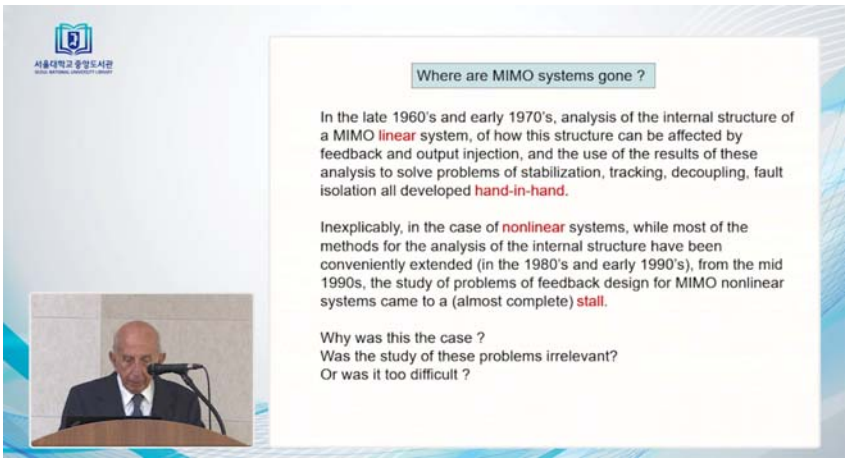


Figure 1.1: Professor Alberto Isidori is giving a talk

are historians of science and cultural scholars, engineers and computer scientists abandoned it in droves, renaming their projects and their degrees. 'Cybernetics', nearly seventy years after the word was coined, sounds oddly old-fashioned and out-of-touch".

- It was also claimed by Alberto Isidori in an open talk given at Seoul National University, South Korea, on 1st September 2017, that "Inexplicably, in the case of nonlinear systems, while most of the methods for the analysis of the internal structure have been conveniently extended (in the 1980's and early 1990's), from the mid 1990s, the study of problems of feedback design for MIMO nonlinear systems came to a (almost complete) stall". He then asked "Where are MIMO systems gone?" "Why was this the case? Was the study of these problems irrelevant? Or was it too difficult?" (see Fig. 1.1).

- As these observations became widely acknowledged, in 2010, the assertion "control is dead" sparked debate within the global control community. Distinguished figures, including Graham Goodwin (University of Newcastle, Australia) and Yu-Chi Ho (Harvard University, USA), were among those who mentioned or explained the assertion. Yu-Chi Ho gave the points: "Any applied scientific field typically go through three stages—breakthrough into a new problem area, building the theoretical foundation, and maturing of the field". "Currently everyone agrees that the field of control is in this mature third stage". He also commented that "Control is dead until it is reborn like Lazarus by a Jesus miracle. In this sense, one can argue CONTROL is certainly not dead and now may be waiting for the 'second coming' of another golden age of control. But chances are it won't be from extending the current mature theory".

"Why was this the case?" Alberto Isidori's question is certainly worth our serious

consideration.

The reason actually lies in the fact that state-space models, which have dominated the modern control theory for centuries, are not the best ones to handle control problems. Such a reason is generally hard to be recognized and also difficult to be accepted because most of us are working with state-space models. As a matter of fact, state-space models were originally proposed to deal with problems directly tied to the states: linear state-space models were firstly used by Euler in early 1750 for solving high-order linear differential equations (giving the state responses), and nonlinear ones were used by Lyapunov in 1892 to analyze the limit behavior of the states. Exceptionally, state-space models have been successfully used in the control of typical linear systems because constant linear systems without delays are easy to handle.

While state-space models are not the best, would there be a better one to perform control tasks?

1.2 FAS Approach

There is no doubt that a model from which the control vector can be explicitly solved would best perform the control. But do such models exist? As a matter of fact, they are all around us but are overlooked because they are overshadowed by the existence of state-space models and our substantial reliance on the state-space approach.

1.2.1 A Better Solution

What makes the problem of nonlinear control so difficult? The answer surely is the nonlinearities. Complex nonlinearities make the analysis of the Lyapunov function very complicated, and also may create insurmountable difficulties in deriving the solutions to the concerned HJB equations. As long as nonlinearities stand, the general problem of nonlinear control can hardly be completely solved with the present state-space approaches, say, in the sense that closed-loop global stability is achieved.

Fortunately, besides the state-space models, there exists another type of system models, namely, the FAS model, proposed by Duan (2020a,b, 2021a), with which the nonlinear term, no matter how complicated, can be easily eliminated using the full-actuation structure as long as the nonlinear term is measurable. Eventually, a constant linear closed-loop system is obtained.

FASs physically exist, but only as a minor part of the set of control systems. However, when this concept is mathematically generalized, a high-order FAS (HOFAS) becomes a general model for control systems. As a matter of fact, most nonlinear systems can be either physically modelled as, or converted into HOFASs. This fact has been demonstrated in Duan (2020a,b).

1.2.2 FASs—From Physical to Mathematical

In certain practical applications, there exists a type of physical systems which are referred to as FASs. They are often modeled as Lagrangian systems (a type of differential equations of second order), and are widely encountered in the mechanical, electrical

and aerospace fields. They are of the type from which the control vectors can be explicitly solved out, and are therefore inherently more straightforward to be controlled. What is more, constant linear closed-loop systems with some initial value constraints can often be obtained. As a matter of fact, many systems originally modeled by physical laws, such as the Newton's Law and the Lagrangian Equations, fall naturally into this category of FASs.

After all, the set of physical FASs is small. How about the control of the so many UASs and the so many systems which are neither FASs nor UASs? Very recently, the physical concept of FAS is generalized mathematically, and a new methodology termed as FAS approach has been proposed. Instead of converting the model of a general dynamical system into a state-space one, the FAS approach converts it into a so-called mathematical FAS model. Then, similar to a physical FAS, the converted mathematical FAS can be easily controlled and a linear closed-loop system can often be obtained with an arbitrarily assignable eigenstructure.

In general, a state-space model is obtained from the original system's fundamental equations by the method of variable extension. As a result, a state-space model has the lowest differential order but the maximum number of (scalar) equations which is in general greater than the number of control variables. What happens if we apply the method of variable elimination to the original set of fundamental equations? —The system order will increase, and the number of (scalar) equations will decrease. Then, how small could the number of equations be? Generally speaking, it is limited by the number of control variables! When the number of equations is less than the number of control variables, an over-actuated system is derived, and when they are equal, a mathematical FAS, or a sub-FAS, is there ready for utilization! Hence, FAS models are another type of extreme system models which possess the highest system orders but the fewest scalar equations.

Technically and mathematically speaking, such a model conversion is a process of identifying an appropriate homeomorphism, under which the original system is equivalent to a FAS or a sub-FAS model. To mention a few, we have proven that the following typical systems in state-space forms all can be equivalently converted into a FAS or a sub-FAS:

- linear controllable systems, both the continuous-time ones and discrete-time ones;
- feedback linearizable systems, both the continuous-time ones and discrete-time ones; and
- strict feedback systems (SFSs) and sub-SFSs, including the continuous-time ones and discrete-time ones, and both with time-varying delays.

Physically, the original concepts of FASs and UASs are determined by the DOFs and the numbers of practical actuators. Thus a UAS can by no means be converted into a FAS without adding practically some necessary actuators in the system. However, with our extended concept of FAS, any system, as long as it is controllable, can be converted into a FAS.

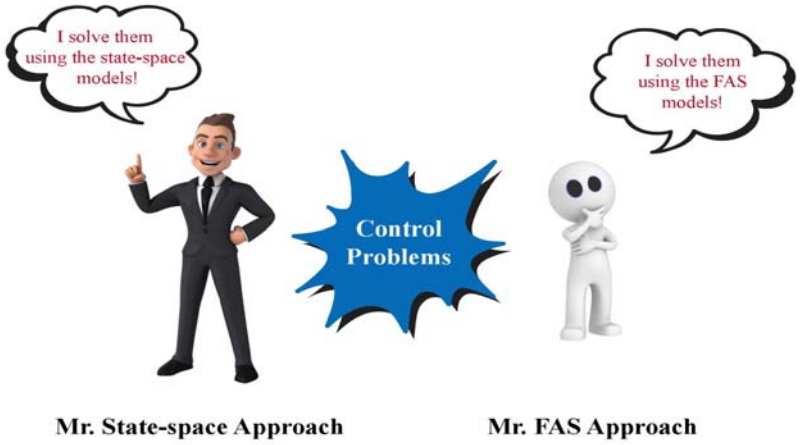


Figure 1.2: State-space approach and FAS approach

1.2.3 Mark of Birth

Besides a series of 3 papers appeared in 2020 in the Chinese Journal *ACTA Automatica Sinica* (Duan (2020a,b,c)), the series of 10 papers on high-order FAS approach, appeared in *International Journal of Systems Science*, namely, Duan (2021a,b,c,d,e,f,g, 2022e,d,f), really marked the birth of the FAS approach. A webpage

https://www.tandfonline.com/journals/tsys20/collections/HOFA_

has been especially set up on the website of *International Journal of Systems Science* to introduce this set of papers. The comments are quoted below:

"*High-order fully actuated (HOFA) systems represent a nuanced frontier in control theory, bridging intricate dynamics with profound control capabilities. Rooted in the realm of systems with advanced differential degrees, these systems, which are often derived from complex physical laws such as Lagrangian mechanics, boast a balanced ratio of control inputs to system outputs. This characteristic, synonymous with being 'fully actuated', ensures a more direct control strategy, rendering them particularly favorable in fields like mechanical engineering, aerospace, and electrical systems. While their high-order nature encapsulates multifaceted behaviors, the full actuation empowers engineers and scientists with a robust platform to achieve precise and desired system responses. Included in this collection is a set of ten seminal papers published in the International Journal of Systems Science, which forms the foundation of the HOFA systems theory. In fact, due to their theoretical significance and practical importance, HOFA systems have already attracted widespread attention with applications to various control problems such as robust control, adaptive control, disturbance rejection, optimal control, and tracking control. It is envisaged that this collection of the pioneering results would help disseminate the HOFA systems theory, which would have a lasting impact on control theory and control engineering.*"

Please note that in later researches, the abbreviation "FAS" is used instead of "HOFA", which is mentioned in the above comments.

The FAS approach, owing to its distinct and well-recognized advantages, has attracted widespread attention since the emergence of the seminal works in two sets of papers Duan (2020a,b,c) and Duan (2021a,b,c,d,e,f,g, 2022e,d,f). By now, over a hundred research groups from at least 70 institutions across the globe have begun to delve into the field. Hundreds of papers on the FAS approach have been published, and a large portion of these has appeared in *Automatica* and various IEEE Transactions (see the overview paper Duan (2024e)).

The FAS approach has been instrumental in resolving many nonlinear control problems, encompassing areas such as robust control, adaptive control, disturbance rejection, optimal control, and tracking control. Several issues, insurmountable or inadequately addressed with the state-space method, have been effectively and satisfactorily resolved using the FAS approach. Frequently, the closed-loop systems derived through the FAS approach have been rendered linear and stable, or globally exponentially stable or sub-stable. The methodology has been extended to diverse domains including discrete-time systems, time-delay systems, and also stochastic systems.

In response to the growing interests and needs of the control community, the following significant events have brought about:

- *Annual International Conference on Fully Actuated System Theory and Applications (FASTA)* This conference has already been organized three times. Notably, FASTA 2022 hosted over 150 participants in Harbin, FASTA 2023 received over 430 participants in Qingdao, and FASTA 2024 received over 610 participants in Shenzhen. Future conferences, FASTA 2025 and FASTA 2026, are planned to be held in Nanjing and Qinhuangdao, respectively. Beginning with FASTA 2023, the conference proceedings are included in the IEEE Xplore database and indexed by EI.

- *Technical Committees on Fully Actuated System Theory and Applications (FASTA)* Firstly, the FASTA technical committee with the Chinese Association of Automation (CAA), China, was established in 2022. Secondly, the FASTA technical committee with the Asian Control Association (ACA) was approved in 2024.

- *Major Progress of Science and Technologies in Chinese Universities* The project entitled "Fully Actuated System Control Theory" has been elected as one of the 2023 Ten Major Progresses of Science and Technologies in Chinese Universities.

While many questions remain to be explored, the FAS approach has already demonstrated its tremendous potential and promises a thriving future. The exploration and application of this approach may stimulate a new era of solutions for control systems, unlocking possibilities previously constrained by conventional methods.

Ever since its emergence, it has gained wide and rapidly increasing attention. The approach has been generalized to time-varying and time-delay systems (Duan (2022b,c, 2023b,c,d,e)), and also discrete-time systems (Duan (2022f,b,c)), and has had fruitful theoretical and practical results (Duan (2024e)) (see, also, the conference proceedings of FASTA2023 and FASTA2024).

The state-space approach solves control problems based on the state-space models. Parallelly, the FAS approach solves control problems based on the FAS models, which are generalizations of the physical FAS model. State-space models and FAS models are actually two extreme models for dynamical systems. The former possesses the lowest differential order but the maximum number of scalar equations, while the latter possesses the highest differential order but the minimum number of scalar equations. The former takes the state vector as the main role and is more suitable to deal with problems tied to the state vector, while the latter takes the control vector as the main role and is more suitable to deal with control problems. As a matter of fact, the state-space model was firstly used by Euler in 1750 (Kline (1990)) to solve a high-order ordinary differential equation (giving the state response of a system), and then used by Lyapunov in 1892 to study the limit behavior of the state vector (Lyapunov (1992)). State-space models do have played a dominant role in control of linear systems but because linear control problems are easier to handle.

There is no doubt that a model which permits the control vector to be explicitly solved will provide great convenience for its controller design. Fortunately, generalized FAS models happen to be such a kind, and also widely exist. Several big classes of systems which have been well-treated with the state-space approach have all been shown to be able to be converted into FASs or sub-FASs:

- **Controllable linear systems** including both the continuous-time ones (Duan (2020b); Duan and Zhou (2022)) and the discrete-time ones (Duan (2022f));
- **Time-delay linear systems** including discrete-time ones with both state and input constant delays (Duan (2022b,c)), and continuous-time ones with a constant input delay (Duan (2023b));
- **Feedback linearizable systems** including both the continuous-time ones (Duan (2020a)) and the discrete-time ones (Duan (2022f));
- **SFSs** including the continuous-time ones (Duan (2020a, 2021b)), the discrete-time ones (Duan (2022f)), both with time-varying delays (Duan (2023b,c)), SFSs with inconsistent dimensions (Xu and Duan (2024)), high-order and mixed-order SFSs (Duan (2021b)), and sub-SFSs (Duan (2024f));
- **Flat systems** including fully at systems and systems with full sets of relative degrees (Duan (2024e));
- **Others** including a particular controllability canonical form for nonlinear systems (Li and Gao (1990)), several types of UASs (Duan (2024g)), some more general types of nonlinear systems (Duan (2020b, 2021a, 2022f)), and many practical systems (Duan (2024e)).

Due to its distinct advantage, like a physical FAS model, a generalized FAS model renders a controller which realizes complete cancellation of all the nonlinearities in the system, and produces a linear time-invariant (LTI) closed-loop system. In the case of a sub-FAS, a sub-stabilizing controller can be designed, which produces an LTI system on a so-called feasibility set (Duan (2021g, 2023e)). Meanwhile, a region of

Table 1.1: Comparison of two extreme dynamical models

	State-space model	**FAS model**
Differential order	lowest	highest
Number of scalar equations	maximum	minimum
Derivation from original physical model	variable extension	variable elimination
Dominant variables	state vector	control vector
Advantageous in solving	state-oriented problems	control-oriented problems

exponential attraction (RoEA) can be found, from which all the trajectories starting are driven by the designed controller to go to the origin exponentially within the feasibility set. Such a kind of stabilization is called sub-stabilization in the FAS approach (Duan (2023a,e,b)), which offers several advantages.

1.2.4 FAS Models vs State-space Models

In general, state-space models are derived from the system's fundamental equations through variable extension, and are eventually a type of extreme models with the lowest order and the maximum number of scalar equations. What happens if the method of variable elimination is employed instead? The system order will increase, and the number of equations will diminish. Generally, this number of equations will not exceed the number of control variables (except in some over-actuated systems). When equal, a mathematical FAS or a sub-FAS is available for utilization. Hence, FAS models constitute another class of extreme system models characterized by the highest system orders but the fewest scalar equations. Technically and mathematically, this model conversion is in fact a process of identifying an appropriate homeomorphism, under which the original system is equivalent to a FAS or a sub-FAS. Specifically, it is shown in the next section that all controllable LTI systems can be equivalently converted into linear FASs (Duan (2020a); Duan and Zhou (2022)), and several very typical nonlinear systems can also be equivalently converted via certain homeomorphisms into nonlinear FASs.

In Table 1.1, a comparison of the FAS models and the state-space models is given. Indeed, as we have seen that the FAS models, especially the multi-order ones, are more complicated than the state-space models. However, the FAS models, as we will soon find out in subsequent sections, do have many advantages over the state-space ones, especially in the process of controller designs.

The state-space approach is based on the state-space models, while the FAS approach is based on the newly discovered general FAS models. If we say, by analogy, the state-space approach is a big tree grown out from the seed of state-space models, then the FAS approach is another tree grown out from the seed of FAS models. However, the former is getting mature, but the latter is growing fast and ourishing.

1.2.5 Present Advances of the FAS Approach

Present contributions of the FAS approach are summarized as follows.

1. A general FAS model for describing dynamic systems has been developed, along with the FAS approach based on this model, which unifies and generalizes a range of typical nonlinear system control theories, including feedback linearization, differential atness, and relative order theory.

2. It is proven that the necessary and sufficient condition for controllability of conventional linear systems is their transformability into high-order FASs. This result has been further extended to discrete-time systems, time-varying systems, and time-delay systems. Additionally, simpler, more direct, and effective methods for eigenstructure assignment (ESA) problems in various types of linear systems have been proposed.

3. A series of FAS control methods has been developed for achieving stabilization and sub-stabilization of complex nonlinear systems. These methods apply to both continuous-time and discrete-time systems, whether deterministic or uncertain, and extend to systems with time-varying parameters and time delays.

4. It has been proven that SFSs can be equivalently transformed into FASs, and iterative recursive formulas for achieving this transformation have been provided. Additionally, the concepts of high-order SFSs and sub-SFSs have been proposed, along with the development of corresponding high-order backstepping methods and sub-stabilization methods.

5. Using the FAS approach, a series of stabilization problems for conventional nonholonomic systems (NHSs) has been addressed. Furthermore, conventional NHSs have been extended to time-varying and time-delay cases, revealing their essential characteristics. Sub-stabilization feedback controllers have also been proposed.

6. Based on the FAS model, a range of complex nonlinear system control problems has been addressed, including robust control, adaptive control, predictive control, optimal control, and tracking control.

1.3 Overview: FAS Approach Theories

Besides the results introduced in this book volume, there have been many theoretical advances on the FAS approach (Duan (2024e)). A brief overview of such advances is given in this section.

1.3.1 Adaptive Control

This section provides a review of a selected list of adaptive control articles which address some key technical issues of adaptive control via the FAS approach.

Nonlinear time-delay systems

The global adaptive control methodology is proposed for nonlinear systems with time delays by utilizing the FAS approach in Ning et al. (2022). Moreover, the unmodeled dynamics and unknown asymmetric dead-zone input problem are also studied in Zhang et al. (2023d) for a class of time-delay nonlinear high-order FASs. Adaptive guaranteed cost control and adaptive asymptotic stabilization are considered respectively in Hu et al. (2023) and Zhao et al. (2023b) for high-order nonlinear systems with unknown parameters and time delays.

Uncertain high-order FASs

Meng et al. (2022) investigate the adaptive event-triggered control problem for uncertain high-order FASs with unknown control matrix function, and Dong et al. (2023) study the adaptive tracking control problem for high-order FASs with actuator faults and full-state constraints. Adaptive iterative learning control for high-order nonlinear systems with different types of uncertainties is investigated in Li et al. (2024e), and adaptive estimation and control for uncertain nonlinear systems are studied in Yan et al. (2023b); Yan and Gu (2022) where the proposed scheme stabilizes the closed-loop system asymptotically for the initial state values over any given bounded set of the state space. For a class of second-order FASs in the presence of internal uncertainties and external disturbances, the adaptive preassigned time stabilization is studied in Shi et al. (2022).

Very recently, a desired compensation adaptive robust control framework is presented by Chen and Duan (2024) for two types of FASs subject to both parametric and nonlinear uncertainties where the proposed adaptive compensation in the controller relies on the noise-free desired trajectory and online parameter estimate only. Chen and Duan (2025) propose a FAS approach integrated with adaptive robust integral of the sign of the error feedback control strategy for multi-input and multi-output (MIMO) high-order FASs in the presence of both external disturbances and parametric uncertainties. An adaptive stabilization control is investigated in Wu et al. (2024b) for high-order SFSs with unknown control directions, in which the Nussbaum gain technique, tuning functions along with the FAS approach are proposed to deal with the unknown control directions and system uncertainties.

Robust adaptive control

Gu et al. (2025) propose a robust adaptive tracking controller for high-order nonlinear systems that have unknown upper bound time-varying uncertainties where the proposed robust adaptive tracking controller is capable of overcoming system uncertainties. A robust adaptive controller is proposed in Zhang and Duan (2025) for FASs with unknown parameters, perturbed input matrices and nonlinear uncertainties, and a parallel extension to the generalized multi-order FAS case is also given.

High-order strict feedback systems

The "differential explosion" problem is considered for high-order SFSs in Cai et al. (2024b); Hou et al. (2023); Liu et al. (2023d,e); Xiao et al. (2024); Liu et al. (2024e) by adaptive dynamic surface control and adaptive command filtered backstepping control, and among them the accurate parameter estimation is studied in Hou et al. (2023); Liu et al. (2023d). The problems of almost disturbance decoupling and adaptive control of high-order nonlinear systems with strict feedback forms have been solved in Wang et al. (2023b). The adaptive control with a tuning function for high-order SFSs is studied in Liu et al. (2023a) to design an adaptive update law without parameter overestimation for unknown mixed high-order FASs. For a specific class of second-order nonlinear SFSs, adaptive control with tuning functions is proposed in Wu et al. (2024c).

Event-triggered adaptive control

The event-triggered adaptive control is considered in Wang et al. (2024g,e,f); Yan et al. (2023a); Wang et al. (2023a). An event-triggered adaptive sliding mode control scheme is developed for a class of uncertain strict feedback nonlinear systems by utilizing the FAS approach in Wang et al. (2024g). A high-order event-triggered adaptive control scheme is proposed for uncertain high-order nonlinear FASs with a completely unknown nonlinear function where the fuzzy logic system technology is applied to approximate the unknown nonlinear function in Yan et al. (2023a). Moreover, a fixed threshold event-triggered adaptive controller is developed by adopting the high-order FAS approach in Wang et al. (2023a) for a class of nonlinear cascade systems.

Besides the above, adaptive containment control of heterogeneous multiagent systems (MASs) has been considered in Liu et al. (2024h), and adaptive neural network control of thermoacoustic instability in Rijke tube has been investigated in Zhao et al. (2022). Recent advances in adaptive control also include adaptive tracking control with input quantization by Liu and Duan (2024) and adaptive decentralized control for second-order large-scale nonlinear systems by Wang and Sun (2025).

1.3.2 Tracking Control

Constrained FASs

Tracking control based on FAS theory has emerged as a pivotal focus of scholarly inquiry. Particularly, the tracking control problem of nonlinear high-order FASs subject to state constraints is studied in Yang and Li (2024), and an explicit reference governor is introduced to ensure the system state with its derivatives remaining within a prescribed constraint set. Additionally, the precise attitude control for space manipulators with system uncertainties and various physical constraints is investigated in Tian et al. (2023), and the high-precision tracking performance, the optimal control gains, and the physical constraints can be ensured simultaneously. Besides, an adaptive disturbance rejection control method is proposed for the tracking control of FASs with state constraints in Gao et al. (2025) by strategically designing composite variables and auxiliary signals, where this method eliminates the need for the stringent conditions and state-dependent transformations required by existing constraint handling techniques.

Model reference tracking control

For FASs with uncertain parameters, Tian and Duan (2023) propose a robust model reference tracking control methodology to ensure the improved tracking performance, and Gao and Duan (2024) study the robust model reference tracking for high-order descriptor linear systems. Furthermore, the problem of position and attitude control for a type of combined spacecraft in discrete-time domain is investigated in Cui et al. (2023a), and a model reference tracking controller is constructed to cope with the uncertain parameters of the combined spacecraft.

Guaranteed cost tracking control

There are also some studies on guaranteed cost tracking control of uncertain high-order FASs in view of FAS theory. Specifically, adaptive guaranteed cost tracking control for high-order nonlinear systems with unknown parameters is investigated in Hu and Duan (2024), and the robust adaptive tracking control problem for high-order FASs with uncertainties is studied in Hu et al. (2023). Based on FAS theory and pulse-like functions, Hu et al. (2024b) present a robust switching adaptive tracking controller to ensure that the system tracking error vector converges into a prescribed neighborhood of equilibrium, which is completely unrelated to the system initial values and the uncertainty. In addition, Hu et al. (2024a) derive a new adaptive guaranteed cost stabilization for two classes of high-order nonlinear systems with unknown parameters (vector) and time delays. Two designed controllers ensure that all of the states of two closed-loop systems are global boundedness, and preset arbitrarily the upper bound of cost functions characterizing the output performance.

1.3.3 Predictive Control

As a thriving new method, predictive control based on FAS theory has emerged as a pivotal area of research. Compared to traditional predictive control methods, integrating FAS theory into the predictive control greatly reduces the complexity of the optimization problems, and has catalyzed significant advancements in both theoretical foundations and practical applications.

Predictive control with multi-step predictor

Liu (2022c) proposes the fully-actuated predictive feedback control methods via multi-step ahead output predictions for nonlinear systems with time-varying delays. In Liu (2022a), the nonlinear MASs are described in a fully-actuated form, and a fully-actuated predictive feedback control is applied to ensure the coordination of MASs. Moreover, a fully-actuated networked predictive controller with multi-step predictor is proposed in Liu (2022b) for nonlinear cyber physical systems with time-varying communication constraints.

Predictive optimization control

For FASs without constraints, by constructing a fully-actuated incremental prediction model, the basic form of predictive control in a discrete-time form is studied, and then multi-step ahead output predictions are established to optimize the tracking control (Zhang et al. (2023b)) and cooperative control (Zhang et al. (2022a)). Parallelly, by introducing the predictive states expressed in Taylor series expansion in FASs, the basic form of predictive control in the continuous-time domain is proposed, and then the explicitly analytical predictive controller is provided in Wang and Duan (2023) via an optimized way for UASs.

For FASs with delays and constraints, a proportional-integral predictive control strategy is developed to enhance the cooperative control performance and offer compensation for fixed communication delays in feedback and forward channels (Zhang et al. (2023c)). By introducing the associated inequality constraints into a cost function of predictive control, the dynamical performance is upgraded, and the input saturation is avoided in Zhang et al. (2023a). Moreover, the work Wang and Duan (2024b) improves real-time performance and eliminates full-state constraints by incorporating nonlinear transformed vector functions into constrained FAS models.

Predictive secure control

For secure control against cyber-attacks, a packet-based fully-actuated predictive controller is used for networked FASs subject to random denial-of-service attack in Zhang and Liu (2024g) and MASs with random deception attacks in Zhang et al. (2024a). Moreover, the work Zhang and Liu (2023b) introduces a sliding-mode variable to defend the random deception attacks by enhancing the robustness of closed-loop systems. In Zhang and Liu (2024e), a high-order FAS approach for predictive sliding-mode control is proposed to achieve the defense for random deception attacks during the cooperative control of MASs.

Observer-based predictive control

For unmeasured states and disturbances, an observer-based predictive controller based on the FAS approach is investigated in Zhang and Liu (2023a) to ensure the realization of tracking control and the compensation of communication constraints and disturbances. Further, the corresponding results are extended to the cooperative control for MASs with time-varying communication constraints in Zhang and Liu (2024d), and an output tracking control problem of uncertain cyber-physical systems is considered in Zhang and Liu (2024a) for a high-order FAS, where a disturbance observer-based high-order fully actuated predictive control is adopted to achieve the desired tracking control performance and compensate for the communication delays in the forward and backward channels.

In addition to the above mentioned research areas, some new perspectives in predictive control have been developed:

- A digital-twin predictive control strategy and a digital-twin control predictor are proposed in Liu (2024) to actively compensate for time delays and communica-

tion delays in the predictive control of nonlinear dynamic systems, enabling the achievement of the desired control performance while greatly simplifying the controller design process.

- The machine learning techniques are seamlessly integrated with stochastic model predictive control in Ning et al. (2025) to address the regulation problem of nonlinear discrete-time step backward high-order FASs with additive disturbance.

- A constrained networked predictive control strategy is proposed by Huang et al. (2025b) for the optimal control problem of complex nonlinear FASs with noises, and the proposed method can effectively deal with nonlinearities, constraints, and noises in the system, optimize the performance metric, and present an upper bound on the stable output of the system.

1.3.4 Fault-Tolerant Control

Deterministic FASs

For fault-tolerant control of a deterministic FAS, a pioneering integrated structure of observation and control is proposed, pointing out a new idea for dynamic data modeling of FASs (Cai et al. (2023a)). This new approach provides a framework for fault diagnosis and active fault-tolerant control of nonlinear systems when they embrace system faults, and shows significant advantages in both incipient fault detection and catastrophic fault compensation.

The actuator is an important part of control instruction execution in dynamic systems. In order to avoid the destruction of the fully-actuated structures resulting from additive and multiplicative actuator faults, robust fault-tolerant control (Gong et al. (2024, 2023)) and adaptive fault-tolerant control (Cai et al. (2023b); Liu et al. (2022c); Li et al. (2023b); Cai et al. (2023c)) have filled the relevant gaps in safety control of FASs. When high-frequency noises exist in measurement equations, a low-power fault-tolerant control strategy is proposed in Cai et al. (2023d) to ensure the practical utility of the FAS theory. In the fault-tolerant control of MASs, a FAS approach is proposed to realize prescribed-time formation control and the convergence speed is accelerated (Ma et al. (2023)). In Ma et al. (2024b), a neural-network-based fault-tolerant control structure is constructed for fully-actuated heterogeneous MASs to cope with model uncertainties. In Li et al. (2024b), the prescribed-time fault-tolerant control for the formation of quadrotors is effectively completed by using the FAS approach. In Cui et al. (2023b), a fuzzy fault-tolerant control method via the FAS approach is proposed to realize nonaffine fault compensation. Moreover, the fault-tolerant control design of nonaffine FASs has been deeply investigated (Cai et al. (2024d)). Besides, the fault-tolerant controllers based on the FAS approach also have a wide range of applications in other aspects, e.g., state constraints (Dong et al. (2023)) and finite-time convergence (Cai et al. (2024c)).

The sensor is the key component of state estimation in dynamic systems. The data given by faulty sensors can seriously damage observation and control. Different from the fault-tolerant control against actuator faults, the core principle of fault-tolerant control against sensor faults is redundancy (Cai et al. (2023e)). In Cai et al. (2023f), the

authors give the conditions for the redundant observability of nonlinear FASs, define the sensor accommodation, and originally propose an outstanding self-healing fault-tolerant control framework based on the redundancy principle.

Recent progresses have been made in fault-tolerant control based on the FAS approach in several fields. In Ma et al. (2024c), a prescribed-time stability criterion and an active fault-tolerant control protocol for mixed-order heterogeneous MASs are proposed within the FAS framework. Jiang et al. (2025b) propose a novel integrated sliding mode fault-tolerant control strategy for nonlinear high-order FAS with component faults and disturbances, and state feedback and output feedback controllers were respectively designed to address the control problem in the case of incomplete elimination of known nonlinearities and simultaneous presence of unknown nonlinearities, disturbances, and faults.

Stochastic FASs

Stochastic FASs, as a significant subject alongside the widely studied deterministic FASs, require urgent research attention. Nevertheless, the scarcity of corresponding high-order mathematical tools has posed a formidable challenge. Recently, a groundbreaking contribution to stochastic FAS theory is presented in Liu et al. (2022d, 2024f), where a novel stochastic integral model is proposed with rigor and applicability.

In addition, the probabilistic ultimate bounds of stochastic FASs with persistent disturbances are revealed in Liu et al. (2023g), which extends the results about stability, and successful implementation of fault-tolerant control for the stochastic FASs with local faults by Liu et al. (2023f) has further underscored the potential of this approach. Furthermore, an adaptive actuator fault-tolerant tracking control is proposed by Liu et al. (2025) for stochastic HOFASs where the designed closed-loop strategy advocates the advantage of the FAS methodology and guarantees performance by ensuring that the tracking error complies with the user-defined probabilistic ultimate bound.

1.3.5 Other Control Topics

Prescribed performance control

In Zhang et al. (2022b), a practical prescribed-time control method is developed for strongly interconnected nonlinear systems using the FAS method, which ensures the output performance of the system. For the second-order FASs, a prescribed error trajectory control method is proposed in Li et al. (2022b) to make the tracking error converge to zero along the prescribed error trajectory.

Disturbance decoupling

In Wang et al. (2024a), a weak disturbance decoupling controller is designed for the nonlinear FAS to achieve the desired output tracking performance and attenuate the in uence of the disturbance. For high-order fully-actuated nonlinear SFSs, an almost disturbance decoupling design method is proposed in Wang et al. (2022) to attenuate the effect of the disturbances on the output.

Saturation and constrained control

In Wang and Jiang (2024), a FAS approach control framework based on anti-windup control is established for a class of input-saturated nonlinear systems with Lipschitz condition. A recent study by Xiao and Chen (2024) focused on controller design for high-order FASs that experience input saturation and are affected by bounded disturbances, and a saturated adaptive controller is developed via the FAS approach and the Lyapunov stability theory to achieve output control and reject disturbances. Regarding the research on constrained control, a high-order adaptive dynamic surface output constraint control method is developed in Cai et al. (2024a) for uncertain high-order fully-actuated SFSs with asymmetric time-varying output constraints.

Impulsive systems

In Huang and Sun (2023), the authors not only extend for the first time the application of the FAS approach to nonlinear impulsive systems, but also obtain a stability criterion for the considered systems. As a further result, the robust stabilization problems of nonlinear uncertain control systems with impulsive effects in both the Lagrangian sense and the Lyapunov sense are solved in Huang et al. (2024a) by combining robust control with the FAS approach. Recently, Huang et al. extend the research results for impulsive systems to manifolds and sub-FASs. The tracking control problem of the fully actuated rigid body system defined on the matrix Lie groups with impulsive effects is considered by Huang and Sun (2024), and a controller containing impulsive input is designed for tracking control based on the high-order FAS approach. Huang et al. (2025a) present a global stabilization of sub-FASs by a hybrid impulsive control method and a hybrid controller is devised via the HOFAS approach and state-dependent impulsive control method, addressing the global stabilization problem without constraints on the initial states. Among the latest developments, Huang et al. (2024b) introduce a novel hybrid control strategy for nonlinear high-order FAS, transforming the stabilization problem into that of impulsive switched systems and the study provides criteria for exponential and asymptotic stabilization using switched Lyapunov functions.

Observer design

For control systems, in some cases, certain states are unknown. At the same time, system disturbances can be considered as a type of unknown state. Therefore, observer-based control for high-order FASs has also been studied. The authors in Jiang et al. (2024b) propose an observer that combines the design ideas of disturbance observer and state observer for FASs with external disturbances where the Lipschitz nonlinearity is converted to the varying linearity by using the linear parameter varying method, and the observer gains are then determined via solving a set of linear matrix inequalities. In Jiang et al. (2024a), a feedback controller is proposed based on the basis of the full-actuation property of FAS and a generalized proportional-integral observer is designed to estimate states and disturbances simultaneously.

Discrete-time systems

Parallel to the continuous-time control problem, discrete-time control for FASs has also attracted the attention of researchers. The structural properties of linear discrete-periodic systems are investigated in Lv et al. (2024b) by using FAS approaches and the related criteria are presented where observability and controllability criteria based on FAS approaches are derived. Xu (2024) proposes high-order FAS models for discrete-time SFSs with nonuniform dimensions, where the subsystem dimensions are assumed to be increasing. Both step-forward and step-backward FAS models for discrete-time SFSs with nonuniform dimensions are given, based on which the exponentially stabilizing controllers can be directly obtained.

Differential atness

It is worth noting that the differential atness method can transform systems under state-space approaches into FASs. This type of problem has also attracted the attention of some researchers. Based on factorizations in the pseudo-polynomials ring, a at output for the single-input linear commensurate time-delay systems is constructed explicitly, and then the time-delay system can be converted into a high-order FAS model with the generalized state as the at output (Li et al. (2024g)). The paper in Liu et al. (2024g) studies transforming the commensurate time-delay systems with multi-inputs to FASs via differential atness where the at output of commensurate time-delay systems with multi-inputs can be constructed explicitly on the basis of the solution to the Bezout identity.

Generalized chained form systems

Inspired by the famous Brockett's example, generalized chained form systems have been extensively studied. Recent research findings indicate that the FAS approach offers significant advantages in handling such systems. A more general type of NHSs is proposed by Duan (2024c), which contains both the two well-known Brockett's chained forms as special cases. To stabilize such systems, a stabilizing controller is designed based on the FAS approach. The study in Zhang and Duan (2024c) examines the stabilization issue of extended chained NHSs with external disturbance where the considered system is transformed into a FAS model, simplifying the stabilizing controller design.

Linear FASs

The FAS approach is primarily used to solve nonlinear system problems. However, due to its unique properties, it also demonstrates distinct advantages in handling linear systems. Recent advances include Zhang et al. (2024c), where a frequency-domain method is proposed to transform high-order linear systems with high-order input derivatives into corresponding FAS models. The paper provides two methods to address the non-square problem and presents explicit transformations based on coprime factorization theory. State responses for several classes of linear systems are investigated in Wu et al. (2024a), and a stabilizing control law is designed for a class of high-order

fully actuated continuous-time linear systems with a single input-delay by using the fundamental-matrix-based approach.

Besides the above, other studies based on the FAS approach include:

- stabilization of discrete-time multi-input nonlinear systems with distinct input delays (Wu et al. (2022));
- controllability of linear discrete-periodic systems together with parametric design of periodic state-feedback control laws (Lv et al. (2024a));
- controllability of a leader-follower MAS over finite fields, and development of the stabilization control protocol (Yang et al. (2024a));
- quadratic-programming-based safety control of a mobile robot system (Wu and Liu (2022));
- iterative learning adaptive sliding mode control for a class of uncertain FASs with repetitive motions (Fang et al. (2024));
- time-varying group formation tracking control for heterogeneous nonlinear MASs with unknown dynamics (Tan et al. (2025));
- converting feedback linearizable nonlinear systems into FAS models for both single-input and multi-input scenarios (Gu et al. (2024));
- coordinated control problems over finite fields (Qin et al. (2024)); and
- synthesis problem of a class of generalized pseudo pure-feedback nonlinear control systems (Zhang et al. (2025a)).

Remark 1.1 In addition, there have been numerous researches on robust control based on the FAS approach. These involve robust stabilization of time-varying nonlinear systems with time-varying delays (Duan (2023d)), robust high-order backstepping (Duan (2021c); Liu et al. (2024d)), robust adaptive control (Duan (2021e)), robust command filtered backstepping design for SFSs (Liu et al. (2022b)), robust adaptive dynamic surface control of high-order SFSs (Cai et al. (2024b)), robust fault-tolerant stabilization of uncertain high-order FASs with actuator faults (Gong et al. (2024)), robust control of impulsive systems (Huang et al. (2024a)), robust tracking control (Tian and Duan (2023); Hu et al. (2023); Gao and Duan (2024); Hu et al. (2024b)), robust control of uncertain multi-order sub-FASs (Zhang and Duan (2024a)), robust control with systems under nonlinear uncertainties and perturbed input matrices (Zhang and Duan (2024b)) and robust control of aircraft and quadrotors (Yang et al. (2024b); Zhang et al. (2024e); Ma et al. (2024a)).

1.4 Overview: FAS Approach Applications

It is great to see that, besides the numerous theoretical developments, there have been many developments in the application of the FAS approach. Particularly, there have been many application results with experimental demonstrations.

1.4.1 Spacecraft Control

Because of general physical laws for describing the attitude and orbit dynamics of spacecraft, the dynamic model of spacecraft control is almost always second-order fully-actuated. As is claimed in the perspective paper Zhang and Duan (2023), FAS theory can be naturally introduced into the control design of spacecraft, which can reduce the design complexities and make full use of the design DOFs to obtain a desired closed-loop performance.

Attitude control

Several results have been proposed to realize the attitude stabilization and maneuvering based on FAS theory. For instance, in Xiao and Chen (2022), the attitude dynamic model of spherical liquid-filled spacecraft by using a second-order FAS model is proposed, and the attitude stabilization via the FAS approach is achieved. In Zhao and Duan (2022b), the FAS approach is applied to implement the attitude stabilization and maneuvering of exible spacecraft with time-varying nonlinear inertia. Furthermore, a FAS theory-based optimal tracking control is designed in Liu et al. (2022a) to realize attitude control of rigid spacecraft under actuator saturation, and in Xiao and Chen (2023), the FAS theory is used to construct a quaternion model for attitude dynamics of rigid spacecraft, and then a fully-actuated controller is designed to achieve attitude stabilization and tracking control. In Liu et al. (2024c), a combination of predictive control with prescribed performance control and disturbance observer is considered to ensure attitude tracking control of second-order fully-actuated exible spacecraft. Finally, a design method for large-angle attitude maneuver controller is proposed in Lu et al. (2024a) by using FAS theory and barrier Lyapunov function to investigate the attitude avoidance problem of spacecraft with attitude pointing constraints and disturbance torque.

Attitude joint orbit control

There are also some studies on dealing with the attitude joint orbit control of spacecraft in the view of FAS theory. Concretely, a fully-actuated controller combined with an extended state observer (ESO) is developed in Zhao and Duan (2022a) to maintain tracking control of attitude and position of 6-DOF spacecraft with unknown disturbances. In Wang and Jin (2024), a convex hull method is used to cope with the actuator saturation of spacecraft, and a controller is developed based on FAS theory and the ESO in Zhao and Duan (2022a) to achieve the spacecraft rendezvous. Furthermore, an appointed-time controller is established in Xing et al. (2024) based on the FAS approach to implement the spacecraft ying around the non-cooperative target in the presence of external disturbances. Different from the above, Zhang and Liu (2024b) transform the spacecraft ying-around problem into a tracking control one of nonlinear FAS, and then develop a predictive controller based on FAS theory to deal with this problem. Following Zhang and Liu (2024b), Zhang and Liu (2024c) further use the FAS predictive control to address the communication constraints in the ying-around task. Again, using the FAS approach, Liu et al. (2024a) present an adaptive prescribed

performance controller to realize the integrated attitude and orbit control of spacecraft with uncertain moment of inertia and spatial disturbance torque.

Besides the above, an appointed-time prescribed performance controller is designed by Yin et al. (2024) based on FAS theory to realize the cooperative ying-around of multiple spacecraft when external disturbances exist and undesired orbital maneuvers happen, and the attitude and orbit integrated control problem of spacecraft is investigated by Jin et al. (2025), in which the attitude and orbit dynamics systems of the spacecraft are transformed into high-order FAS models, and a comprehensive control strategy is designed by means of the high-order FAS method.

Position and attitude control of combined spacecraft

FAS theory has also played an important role in the control of a type of combined spacecraft. For example, the position and attitude control for a type of combined spacecraft in continuous-time domain is considered in Duan and Liu (2022, 2024). Specifically, a FAS theory-based optimal control is proposed in Duan and Liu (2022) to realize stabilization control, and an adaptive FAS prescribed performance controller is utilized in Duan and Liu (2024) to ensure tracking control in the consideration of external disturbances. Meanwhile, position and attitude control for a type of combined spacecraft in discrete-time domain has also been investigated (Cui et al. (2024); Cui and Duan (2024)). In Cui et al. (2024), an adaptive stabilization controller is proposed to cope with the uncertain parameters of the combined spacecraft, and in Cui and Duan (2024), anti-disturbance control of the combined spacecraft under unknown disturbances is addressed by using FAS theory and disturbance observer.

In addition, a robust controller is proposed by Cui et al. (2025), based on discrete-time high-order FAS method, to achieve state convergence for a combined spacecraft simulator subjected to nonlinear uncertainties. Linear matrix inequality and multi-objective optimization methods are employed to improve the control system performance. Furthermore, Li et al. (2024d) investigate the control problem of spacecraft rendezvous with obstacle constraint and the external disturbance forces caused by orbit perturbation. The FAS approach is used to linearize the original nonlinear system, which simplifies the processes of control law design and ensures model accuracy.

1.4.2 Aircraft and Quadrotor Control

Recently, numerous controller design methods have been investigated for aircraft and quadrotors by using the FAS approach. In Liu et al. (2023b), a FAS approach-based control scheme is proposed for the trajectory tracking of a quadrotor unmanned aerial vehicle (UAV), in which a gradient-based optimization method is introduced for tuning these parameter matrices. In Li et al. (2024b), a horizontal position controller is synthesized for the formation of quadrotors by combining the FAS approach with the prescribed-time method and the altitude and attitude fault-tolerant controller. In Ma et al. (2024b), the FAS approach is first applied for the heterogeneous MASs composed of unmanned ground vehicles and UAVs, in which the unmodeled dynamics and actuator faults are handled by integrating radial basis function neural network with the Nussbaum function and adaptive method. The work in Yan et al. (2023c) presents

robust control of underactuated 3-DOF helicopter based on a lower-order disturbance estimator.

To address communication link faults within the formation, a novel distributed fault-tolerant formation control protocol is developed by Meng et al. (2025) for fixed-wing UAVs, in which a distributed state observer is designed for each follower UAV and an improved Hungarian algorithm is introduced to ensure that the total flight distance flown by the UAVs is as short as possible during formation reconstruction. Based on the high-order FAS method, a tracking control scheme with arbitrarily separable structure is designed by Chen et al. (2025) to make the linear and angular tracking errors of the quadrotor UAV converge to the specified range in a prescribed time. Lu et al. (2024b) investigate the fault-tolerant formation control problem for unmanned helicopters with actuator faults under fixed and switching communication topologies. The formation fault-tolerant controller and attitude tracking fault-tolerant controller are designed for the inner and outer loops respectively by using FAS approach. A prescribed-time disturbance observer is constructed to rapidly and accurately estimate the composite disturbances composed of external disturbances and actuator faults. To overcome multiple disturbances such as actuator failure, aerodynamic resistance, coupling effect and nonlinear characteristics, Miao et al. (2024) design two prescribed-time controllers based on incremental FAS method to achieve robust attitude tracking and distributed leader-following formation control performance for multiple helicopters. In contrast to the existing results separating the underactuated model into outer-loop and inner-loop subsystems, Liu et al. (2024b) transform the underactuated unmanned helicopter system into three high-order FASs, and incorporate the high-order FAS approach into the prescribed-time fault-tolerant formation control protocol design procedures to eliminate the influence of failures and preserve the target geometric pattern. In Liu and Li (2025), two controllers are designed based on the FAS method for the position and angle tracking of the quadrotor UAV so that the UAV formation is driven along the given trajectory with enhanced internal stability. An ESO is designed to estimate and compensate the uncertainty such that the control accuracy is improved. Xu et al. (2025) propose a fault-tolerant game control strategy based on FAS approach to achieve quadrotor helicopters' formation under the influence of the bidirectional interactions between the inner and outer loops. An adaptive fault estimation law is proposed for fault compensation, facilitating the controller reconfiguration. Under the designed controllers, an interactive Stackelberg equilibrium is achieved.

1.4.3 Guidance and Control

Aiming at the terminal impact-angle constraint in the interception of large maneuvering targets, Chen et al. (2023) propose a three-dimensional interception guidance law combined with the high-order FAS approach and the cascaded linear ESO. The controller parameter matrix is solved using the desired closed-loop poles, and the convergence is validated by the Lyapunov stability theory. The angle constraint issue in the integrated guidance and control design is investigated in Wang et al. (2025b); Zhang et al. (2024b). By using the high-order FAS approach, a nonlinear robust controller is presented under the framework of the second-order backstepping method. The inverse hyperbolic sine function tracking differentiator is used to estimate the virtual control

command, avoiding the "differential explosion" phenomenon. Wang et al. (2024d) investigate the design of an attitude autopilot for a dual-channel controlled spinning glide-guided projectile. Employing the high-order FAS approach and parametric approach, the nonlinear dynamics of the spinning glide-guided projectile are recast into a constant linear closed-loop system, ensuring that the projectile's attitude asymptotically tracks the given goal with the desired eigenstructure. Shi et al. (2025) propose an adaptive attitude autopilot design method for a class of spinning projectiles with time-varying aerodynamic uncertainties. By combining FAS theory and adaptive control theory, an adaptive attitude autopilot is proposed to restore the linear characteristics of the closed-loop system. A parameter-safe adaptive law is presented using control barrier functions to ensure the boundedness of estimation parameters and improve the robustness of the closed-loop system.

1.4.4 Robot Motion Control

In order to illustrate the effectiveness and superiority of the proposed FAS approach, control of a robot system with elastic joint has been considered as an example in Duan (2021b). Based on this pioneering work, various methods have been developed for robot motion control based on the FAS approach. In Sun et al. (2022a), trajectory tracking control of a 6-DOF manipulator is investigated based on FAS models and a direct parametric method. In Li et al. (2022b), prescribed error trajectory control is proposed for second-order FASs, with a 2-DOF robot manipulator applied as an illustrative example. In addition, many theoretical research results on the FAS approach have employed the single-link robot system as the simulation example for effectiveness validation, such as adaptive guaranteed cost control (Hu et al. (2024b)), robust adaptive dynamic surface control (Cai et al. (2024b)), adaptive fault-tolerant control (Dong et al. (2023)), adaptive stabilization control (Wu et al. (2024c)), robust command filtered backstepping control (Liu et al. (2022b)), robust model reference tracking control (Tian and Duan (2023)), and adaptive containment control (Liu et al. (2024h)).

1.4.5 Underactuated Systems

UASs are a type of systems that have less number of controls than the DOFs, and have been a very active scientific research area with broad applications, e.g. spacecraft, helicopters, mobile robots, underactuated manipulators, underwater vehicles, and surface vessels. Duan (2024g) investigates four general types of time-varying UASs for which FAS models can be established. Under very mild assumptions, all these types of UASs have been transformed into FAS models or sub-FAS models under corresponding homeomorphism transformations. The sets of feasibility are explicitly characterized and sub-stabilizing controllers for the proposed systems are designed. Based on the FAS framework, a control strategy is applied in Zhang et al. (2025d) for nonlinear underactuated ships with two piecewise continuous or even smooth stabilization controllers. The sufficient conditions for the existence of these controllers are provided. The controller extension method is adopted to construct some intermediate auxiliary controllers that can link the piecewise controllers given in advance continuously or even smoothly.

1.4.6 Underwater Vehicles and Vessels

Unmanned underwater vehicles have been widely used in military, underwater resource exploration, underwater search and other fields. The trajectory tracking control problem of unmanned underwater vehicles with unknown dead-zone inputs is studied by Zhang et al. (2024d). A novel error constraint FAS approach is proposed by incorporating the tracking error dependent normalized function and barrier function along with time-varying scaling. Based on the proposed FAS approach, an adaptive radial basis function neural networks is designed to deal with the model uncertainties. Moreover, a positive time-varying integral function is introduced to completely eliminate the effect of the residual effect caused by unknown dead-zone inputs. It is proved that the trajectory tracking error converges to zero asymptotically by using the Lyapunov method. Xu et al. (2024) investigate the problem of tracking the trajectory of dynamic positioning vessels under sudden surges of elevated sea states. A predefined-time control term is designed based on FAS approach to maintain the convergence time of the trajectory tracking error within a specified range. A linear control matrix is developed to eliminate the need for estimating the upper bound of perturbation after system switching. This approach enhances control accuracy and robustness by configuring a larger dominant pole following the smooth switching.

1.4.7 Experimentally Demonstrated Applications

In this subsection, all the works on the FAS approach have been verified by experimental applications, and all the experimental results have well demonstrated the effectiveness and superiority of the proposed approaches.

Quadrotor and helicopter control

In Wang and Duan (2024a), a novel scheme based on the FAS approach is proposed for comprehensive reconstruction and predictive control of quadrotor UAV, which significantly relieves computational demands and achieves satisfactory real-time performance. In Lu et al. (2023a), a modified high-order FAS approach is introduced for the nonlinear 3-DOF quadrotor modeling and control, without relying on pseudo strict feedback forms required by the original high-order FAS approach. This modified FAS approach is further extended to the control design of 6-DOF nonlinear quadrotor systems with recursive actions (Lu et al. (2023b)). Furthermore, the work in Zhang et al. (2024e) presents robust cooperative tracking control for close formation of fixed-wing UAVs. The work in Chen et al. (2024a) considers control of a novel coaxial drone with two contra-rotating rotors, and applies the FAS approach to produce the desired control force. By combining FAS and tensor product model, two-loop hybrid control schemes are proposed by Chang et al. (2024) for quadrotor system, in which the control scheme based on FAS method deals with the position subsystem, and the fuzzy controller under tensor product manages the attitude subsystem. A superdrive UAV system is established by combining the biaxial inclined thrust vector quadrotor and serial manipulator in Li et al. (2024f), and a control strategy based on the high-order FAS method is designed to overcome the limitation of underactuated UAVs.

Robotic manipulator control

The work in Yan et al. (2024) proposes an adaptive fuzzy dynamic event-triggered tracking control scheme for a exible-joint robot system, which not only reduces the computational complexity, but also saves resources. In Kang et al. (2023), a sliding mode-based adaptive tube model predictive control is proposed for optimal tracking control for robotic manipulators with state constraints and uncertain dynamics, which can achieve fast and efficient trajectory tracking. To better handle the uncertain dynamics and external disturbances, Tian et al. (2024) present an optimal FAS approach for the trajectory tracking control of robot manipulators, in which a high-order disturbance observer is used in the inner loop to enhance the rejection to disturbances, and a gradient-based optimal parameter tuning method is applied to tune the controller parameters with physical constraints. Moreover, Jiang et al. (2025a) propose a preassigned time composite fuzzy adaptive trajectory tracking control for a class of uncertain exible-joint manipulator systems via FAS approach. A noteworthy contribution of this work is the first development of a composite fuzzy adaptive control scheme which makes all the tracking errors of the uncertain system ultimately converge to zero rather than being only bounded. Zou et al. (2025) investigate the prescribed-time control for a novel tendon-driven space manipulator with uncertainties and deformations in the extensive operations. Using an equivalent transformation, the dynamical model of the space manipulator is transformed into a class of second-order fully actuated nonlinear systems. A practical prescribed-time controller is designed using the high-order FAS approach, which achieves high tracking accuracy and fast dynamic response.

Underactuated mechanical system control

In recent years, control of underactuated mechanical systems has attracted much attention from the robotics and control community. Yao et al. (2024) present a HOFAS approach to control of underactuated overhead cranes with unknown dynamics. The specific structure and advantages of the established HOFAS model are utilized for controller design, with a nonlinear observer designed to estimate the unknown dynamics. Both simulation studies and experimental results verify the better performance of the proposed framework over existing methods. Guaranteeing state constraints is critical for maintaining system safety and reliability. Gao et al. (2024) introduce a new solution for underactuated overhead cranes subject to state constraints based on FAS approach. The main feature of the proposed method is that it ensures full-state constraints by enforcing the constraint on a composite variable. This technique allows for the direct development of the controller, eliminating the need for the recursive procedure inherent in traditional methods. Consequently, the resulting controllers exhibit low-complexity, and the asymptotic stability is strictly proven by theoretical analysis. The experimental result shows that the proposed control strategy has a favorable control performance.

Nonholonomic system control

Many mechanical systems exhibit non-integrable velocity relationships, classifying them as NHSs. Notable examples include wheeled mobile cars, rigid spacecraft, hopping robots, and vertical wheels on rotating tables. Zhang and Duan (2024c) examine

the stabilization issue of extended chained NHSs with external disturbance. Unlike the existing approaches, this work transforms the considered system into a FAS model, simplifying the stabilizing controller design. The FAS approach, the fixed-time disturbance observer, and the terminal sliding mode control method are comprehensively applied to facilitate the control development. Under the proposed stabilizing control strategy, all system trajectories starting from any initial values converge to zero exponentially or within a finite time. Furthermore, the FAS-based anti-disturbance stabilization control of NHSs is investigated by Gao et al. (2025) under disturbances and model uncertainties. The original system is a cascaded form with two subsystems. The first subsystem is designed to be exponentially stable, while the second is reformulated into a FAS with some states remaining unavailable. An effective observer based on Fourier transform technique is presented to simultaneously estimate states and disturbances. Both simulation and experimental results are presented to illustrate the efficacy of the proposed strategy.

Active suspension system control

Active suspension systems have received great attention due to their important contribution to the vehicle industry of improving ride comfort and road holding. By applying the FAS approach, Zhang et al. (2025c) propose a direct robust control method for uncertain quarter-vehicle active suspension systems. The robust stabilizing control method is constructed to guarantee that the system states finally converge globally into an arbitrarily small region. The closed-loop stability is proven strictly based on the Lyapunov stability theory. Incorporating a bioinspired reference model and a FAS approach, a neuroadaptive control method is proposed by Zhang et al. (2025b) for quarter-vehicle active suspension systems, in which the radial basis function neural networks are employed to approximate the systematic uncertainties and input dead zones, and the bioinspired reference model is deliberately employed to take advantage of favorable nonlinear stiffness and damping effects. Different from existing FAS approaches, the designed control method can guarantee the state converges to zero rather than a small region. The experimental results confirm that the designed control method achieves significant enhancements in transient performance.

Servo system control

In Li and Duan (2024), a FAS approach is proposed for servo motor systems, in which the mechanical subsystem and electrical subsystem in cascade are transformed into a high-order FAS model to facilitate the controller design. To handle the mechanical resonance, the FAS approach for exible servo systems has also been investigated in Li and Duan (2023a) and Li and Duan (2023b). Especially in Li and Duan (2023b), singular perturbation method is introduced to transform the exible servo system into two FASs for the controller design based on two time-scale properties, which possesses many advantages such as simplicity, intuition and effectiveness. Recently, Li et al. (2025) further study the output tracking control of exible servo systems with uncertainties and disturbances via the FAS approach and the ESO technique. The controller and observer gains are explicitly solved via the bandwidth parameterization method,

which gains a huge benefit for practical use. In addition, an extra disturbance observer is designed to suppress the state estimation errors, which can further improve the control accuracy. In order to save communication resources in networked control systems, Li et al. (2024c) also study the event-triggered control of servo motor systems based on FAS approach, which adopts an event-triggered control scheme with a fixed threshold strategy and uses a hyperbolic tangent function to ensure the exponential stability of the control system.

Power electronics system control

In Yu et al. (2024a), a decoupled direct current microgrid model is firstly proposed to design a secondary predictive control strategy via the FAS approach, which not only provides a comprehensive representation of the power topology information and the actual electrical characteristics of the direct current microgrid, but also has a very simple and facilitated structure for the design of secondary controllers. In Yu et al. (2024b), a distributed iterative learning algorithm-driven secondary control is proposed based on the FAS approach to achieve the voltage restoration and current sharing of direct current microgrids. The derived FAS microgrid model allows the iterative learning algorithm to optimize the future state trajectories in a distributed fashion over a finite time window, offering the design freedom for the desired dynamic response. Based on this work, a distributed optimal cooperative secondary control strategy is designed in Yu et al. (2024c) with the established FAS microgrid model, which can effectively solve the model-based distributed optimal regulation problem in islanded direct current microgrids. Furthermore, Yu et al. (2025) propose a distributed optimal fully actuated secondary predictive control strategy based on dynamic compensator, in which the DOF of secondary regulation is enhanced by setting the variable weights in the cost function. This strategy effectively achieves precise voltage recovery and current sharing, demonstrating excellent performance.

Remark 1.2 Except for the above several main categories, the FAS approach has also been applied in many other practical systems. Examples include, but are not limited to, roll autopilots (Wang et al. (2025c, 2024c)), intelligent vehicle systems (Gao et al. (2023); Zhang and Chen (2023)), chemical reactor train systems (Li et al. (2024g)), heterogeneous MASs (Ma et al. (2023); Zhang and Liu (2024f)), thermoacoustic unstable systems in Rijke tube (Zhao and Ma (2024); Zhao et al. (2023b, 2022)), civil turbofan engines (Peng et al. (2023)), proton exchange membrane fuel cell air feed systems (Li et al. (2024a)), rolling mill systems (Lin et al. (2023); Wang et al. (2024b)), and so on.

1.5 About the Book

1.5.1 The FAS Approach Book Series

This book is the first one in the following FAS Approach Book Series:

Fully Actuated System Approach for Control and AI

which is to be published with Springer (see http://www.fasta.org.cn for the proposal). The executing period of this book series is from 2025 to 2030. It is affiliated with the following FASTA conference which has been held three times:

Title: *International Conference on Fully Actuated System Theory and Applications*

Websites: http://fasta2024.fasta.org.cn; http://fasta2025.scimeeting.cn

As planned, each year within the executing period there will be 4 to 6 books on this topic published. The following titles are planned to appear in 2025:

1. Fully Actuated System Approach
   —A New Framework for Control,
   by Guang-Ren Duan, Harbin Institute of Technology;

2. Predictive Cooperative Control for Fully Actuated Multiagent Systems,
   by Da-Wei Zhang, Guo-Ping Liu, Southern University of Science and Technology;

3. Fault-Tolerant Control for Fully Actuated Systems,
   by Dong-Hua Zhou, Miao Cai, Tsinghua University;

4. Adaptive Constrained Control for Fully Actuated Nonlinear Systems,
   by Chang-Chun Hua, Liu-Liu Zhang, Peng-Ju Ning, Yanshan University;

5. Cooperative Safety Control of Multiagent Systems
   —A Fully Actuated System Approach,
   by Ke Zhang, Bin Jiang, Yong-Hao Ma, Yu-Hang Xu, Yuan Lu,
   Nanjing University of Aeronautics and Astronautics;

6. Modeling and Control of Renewable Power Systems
   —A Fully Actuated System Approach,
   by Yi Yu, The Hong Kong Polytechnic University,
   Guo-Ping Liu, Southern University of Science and Technology.

1.5.2 The FAS Approach Introduction Volumes

Volume Titles

The above first book is actually divided into three volumes, as shown below:

- *Fully Actuated System Approach—A New Framework for Control*,
  by Guang-Ren Duan, Harbin Institute of Technology

Volume I. Global Fully Actuated Systems (Duan (2025f))

Volume II. Sub-Fully Actuated Systems (Duan (2025g))

Volume III. Unidirectionally Connected FASs (Duan (2025h))

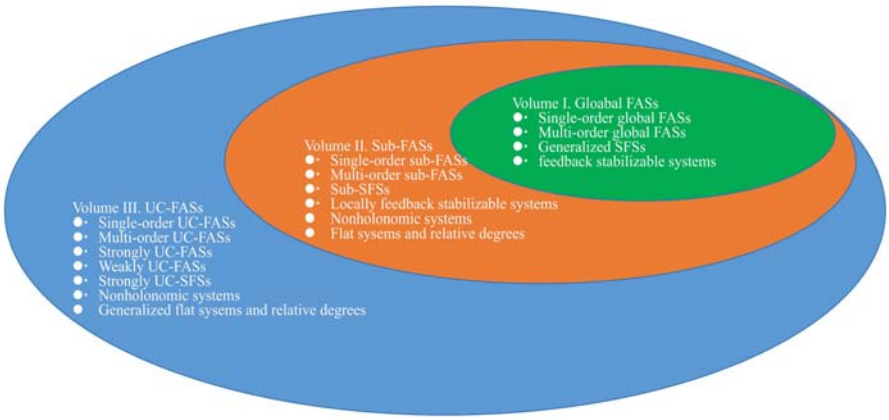


Figure 1.3: Relations between gloabl FASs, sub-FASs and UC-FASs

This FAS Approach Introduction Volumes aim to provide an introduction to the most fundamental results in the FAS approach. These three volumes go deeper in a natural sequence (see Fig. 1.3). More specifically, sub-FASs contain global FASs as special cases, and describe a much more general and complicated class of dynamical systems than global FASs. Meanwhile, unidirectionally connected FASs (UC-FASs), proposed in the set of papers, Duan (2025b,c,d), contain sub-FASs as special cases, and also describe a much more general and complicated class of dynamical systems than sub-FASs. In each volume, the problem of converting dynamical systems into the class of FAS models is considered, and the problem of controller design based on the obtained FAS models is solved. Examples with both theoretical and practical backgrounds, e.g., quadrotors, maneuvered moving objects (MMOs) and planar vertical take-off and landing (PVTOL) planes, are provided to demonstrate the FAS approach proposed. As a consequence, the control techniques proposed in a former volume happen to be special cases of those proposed in a latter volume.

Features

This book volume possesses the following features:

- Provides, as the first book in the world on the FAS approach, written by the proposer of the general approach, a comprehensive elementary introduction to the FAS approach for analysis and control of nonlinear dynamical systems.
- Presents a whole new framework parallel to the state-space approach, which unifies and generalizes several dominant existing techniques for nonlinear control, while proposing essential developments ranging from models, concepts to control techniques.
- Demonstrates by both theoretical developments and practical examples such as, control of MMOs, PVTOL planes, quadrotors, and spacecraft rendezvous and

attitude control, that the introduced FAS approach solves many problems that the state-space approach can not solve or does not solve well.

- Claims that, as a new and systematical approach, the FAS approach is almost parallel to the state-space approach but does better than the state-space approach in terms of performing control since it is born for solving control problems and more suitable in dealing with complex systems.

We remark that the FAS approach is superior to deal with time-delay systems, especially time-varying time-delay systems. However, as an elementary introduction to FAS approach, the particular book volumes do not include contents related with time-delay systems. Readers who are interested in FAS approach for time-delay systems may refer to Duan (2022b,c, 2023b,c,d).

Controllability and stabilizability of dynamical systems are very important issues, and the FAS approach provides a much more reasonable and solid theory for this topic. However, again as a basic introduction, this part is not included in this set of volumes. An illustrative introduction to this topic can be found in Duan (2021g).

Besides the above, the FAS approach for discrete-time systems is neither included in this book volumes. Readers who are interested are suggested to refer to Duan (2022f,b,c).

1.5.3 Contents of Volume I of the Book

This volume I is divided into four parts (see Fig. 1.4).

Part I. Basic theories

This part is composed of two chapters, namely, Chapters 2 and 3. It aims to introduce the basic models for global FASs and simultaneously provide the basic control laws for the introduced global FASs.

Chapter 2 concentrates on the FAS models. Firstly, starting with modelling of a second-order physical mass-spring-damper system and a first-order attitude system of spacecraft, Chapter 2 introduces the single-order global FASs. Differences and relations between physical FASs and the proposed generalized FASs are discussed. Secondly, by investigating a motivating example taken from Treangle et al. (2019), the more general models of multi-order FASs, including affine multi-order FASs and non-affine multi-order FASs are introduced. Thirdly, as a typical example, linear systems are treated with the FAS approach. It is shown that both controllable LTI and lexicography controllable linear time-varying (LTV) systems can be equivalently converted into linear FASs, which permits immediately extremely simple strategies for ESA and stabilization of linear systems. Last but not the least, the first type of nonlinear systems, namely, globally feedback linearizable systems, are investigated. Specifically, it is shown that an affine and a nonaffine nonlinear system is feedback linearizable if and only if it can be converted into a global FAS.

Chapter 3 concentrates on the state feedback control of FASs. Firstly, the huge advantage of the FAS approach of arbitrary assignment of closed-loop dynamics, offered by the full-actuation feature of the proposed single-order and multi-order global FASs,

is revealed, with a particular demonstration of finite-time stabilization for single-order and multi-order global FASs. Secondly, the popular problem of replacement of the open-loop nonlinear dynamics with a desired linear one is discussed, a general and unified control approach for global FASs is proposed, which produces an LTI closed-loop system with an arbitrarily desired eigenstructure. Thirdly, it is reasoned that, with the FAS approach, many nonlinear problems can be converted into linear ones, and the feature is stressed for both deterministic systems and uncertain systems. Fourthly, the elementary problem of FAS approach, to which the control of almost all systems treated with the FAS approach is finally reduced, is examined, and a complete and explicit parametric solution is introduced. Finally, a basic procedure for the FAS approach is provided, which is illustrated by two examples.

Part II. Uncertain FASs

This part is composed of two chapters, namely, Chapters 4 and 5. It handles the control of FASs with external disturbances and dynamical uncertainties.

Chapter 4 addresses the central issue of the disturbance attenuation and decoupling. Two types of FAS models are introduced, one is subject to deterministic disturbances, and the other is subject to dynamical disturbances. The problem of disturbance attenuation via state feedback for the type of FASs with deterministic disturbances is firstly dealt with, and a parametric solution is provided, which allows us to optimize the design parameters F and Z to improve certain performance of the designed system. The problem of asymptotic disturbance decoupling via output feedback for the type of FASs with dynamical disturbances is then handled, and, again, a parametric solution is proposed, with which additional performance requirements of the closed-loop system can be insured by optimizing the design DOFs. Finally, a coarse-fine tracking system is treated as an illustrative example to demonstrate the effect of the proposed approach.

Chapter 5 treats three types of control problems of FASs with uncertainties, namely, robust control, adaptive control and robust adaptive control. Firstly, for a high-order FAS with nonlinear uncertainties, robust stabilizing and the robust tracking controllers are proposed. Secondly, adaptive stabilization and adaptive tracking controllers are proposed for a high-order FAS with an unknown parameter vector. Thirdly, a high-order FAS with both nonlinear uncertainties and an unknown parameter vector is considered, and a robust adaptive stabilizing controller and a robust adaptive tracking controller are designed. In all the above three situations, certain stability of the designed systems is ensured, and some special cases even render the global exponential stability. Finally, the simulation results of an illustrative example are carried out to show that for a FAS with nonlinear uncertainties and/or time-varying unknown parameters, a robust adaptive control law exists under very mild conditions, which guarantees the global stability (boundedness) of the closed-loop system.

Part III. Generalized strict-feedback systems

This part is constituted by fine chapters, namely, Chapters 6∼10. It investigates the control of generalized SFSs via FAS approach.

Both Chapters 6 and 7 are concerned with the problem of converting equivalently a certain type of SFSs into global FASs. The difference is that the former treats SFSs with uniform dimensions, while the latter handles SFSs with increasing dimensions. As a result, it is shown that all SFSs with uniform dimensions can be equivalently converted into single-order FASs, while all SFSs with increasing dimensions can be equivalently converted into multi-order FASs. In both chapters, besides the normal first-order SFSs, the extended second- and mixed-order SFSs are also proposed and considered. For all the considered SFSs, the derived equivalent FAS models can be simply obtained via a set of recursive formulas, which reduce to explicit ones in some special cases. Furthermore, an additional principle is proposed for mixed-order SFSs with uniform dimensions, which reveals a very important property of the equivalent FAS models of generalized SFSs. Several examples are given to demonstrate the proposed approach.

Chapter 8 proposes a high-order FAS approach for a generalized form of nonaffine SFSs. Firstly, a generalized form of first-order nonaffine SFS is introduced, and a recursive solution is proposed to transform equivalently it into a high-order FAS model under certain common assumptions. Particularly, two special cases for first-order nonaffine SFSs are studied, and their recursive results are directly obtained based on the generalised results. Secondly, the FAS approaches for second- and mixed-order nonaffine SFSs are also presented. It is shown that using the feature of FAS models, the control design for the class of generalized nonaffine SFSs can be easily realized and better solved in the sense of providing a constant and linear closed-loop system with a desired eigenstructure. Lastly, illustrative examples are worked out to demonstrate the design procedure and show the efficacy of the proposed approach.

Both Chapters 9 and 10 propose high-order backstepping for high-order SFSs. More specifically, Chapter 9 concentrates on the high-order backstepping for robust control of uncertain SFSs, while Chapter 10 concentrates on high-order backstepping for adaptive control of high-order SFSs with parametric uncertainties. In both chapters, two types of high-order SFSs are considered, namely, the second-order SFSs subject to uncertain dynamics or unknown parameter vectors, and the mixed-order SFSs subject to uncertain dynamics or unknown parameter vectors. Correspondingly, the direct second-order backstepping strategy and the direct mixed-order backstepping strategy are proposed. The proposed high-order backstepping method is a great contribution of the FAS approach which greatly reduces the possibility of "differential explosion" of the normal first-order method of backstepping. Effect of the proposed high-order backstepping is demonstrated via a numerical example.

Part IV. Output feedback and Type I systems

This part is composed of Chapters 11 and 12. The former treats the problem of observer-based control design of FASs, and the latter focuses on the control of systems containing FASs as subsystems, demonstrating the fact that the FAS approach can not only study FASs, but also can treat systems that can not be completely turned into FASs.

Specifically, Chapter 11 proposes observer-based control of FASs with partial state feedback, both full-order observers and reduced-order observers are considered. For full-order observer-based control, a dynamical output feedback control scheme con-

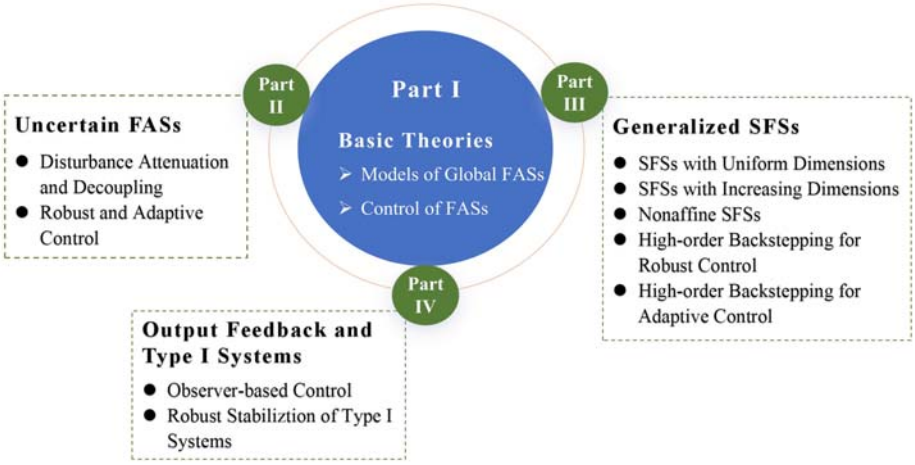


Figure 1.4: Structure of Volume I of the book

sists of a full-order observer and a feedback controller with full-actuation property is proposed for a FAS satisfying Lipschitz assumption. For reduced-order observer-based control, a dynamical output feedback controller is also proposed based on a reduced-order observer for a FAS satisfying partial Lipschitz assumption, both general and special cases are discussed. For both designs, the closed-loop stability is proven, particularly with the observation errors converging to zero. Illustrative examples demonstrate the solution procedure and the effect of the proposed FAS approaches.

Chapter 12 studies the stabilization of time-varying nonlinear uncertain type I systems. Firstly, two motivating examples are presented in order to introduce the so-called type I systems. By definition, a type I system is composed of a sub-FAS and an unforced stable subsystem. Secondly, the robust problem formulation for the uncertain type I system is proposed. That is to find a full-state feedback controller such that the uncertain type I system is robustly globally uniformly asymptotically stabilized. Thirdly, the relation between the robust input-to-state stabilization and the robust stabilization of the uncertain type I system is discussed, and a robust stabilization with full state feedback is proposed for the uncertain type I system, which includes cases of single-order FASs and multi-order FASs. Fourthly, the further issues about robust stabilization with partial state feedback are addressed, including the case of perturbed $B(\cdot)$ matrix and robust control of FASs. Finally, two illustrative examples of full state feedback and partial state feedback are carried out to demonstrate the effect of the proposed approaches.

Last but not least, let us point out that most of the problems treated in this first volume can be generalized to the sub-FAS case. However, due to length requirement, not all the generalizations are covered in Duan (2025g).

Remark 1.3 Although this Volume I is not like the other two volumes which solve

control problems of much more complicated and difficult systems, it is of great importance in the following senses:

1. it provides a suitable beginner's entry, and helps the readers to understand the contents in the other two volumes;

2. it demonstrates that many problems can be solved better by utilizing the high-order FAS models than by the first-order state-space models; and

3. it introduces results like high-order SFSs and high-order backstepping designs, which are not obtainable within the first-order state-space frame.

1.6 Preliminaries

1.6.1 Special Notations for FAS Approach

Notations

For $x, x_i \in \mathbb{R}^m$, and $A_i \in \mathbb{R}^{m\times m}$, $i = 0, 1, \ldots, n-1$, the following typical symbols are used in this book (see also Duan (2021a,b,c,d,e,f,g, 2022e,d)):

$$x^{(0\sim n)} = \begin{bmatrix} x \\ \dot{x} \\ \vdots \\ x^{(n)} \end{bmatrix},\ x_{i\sim j} = \begin{bmatrix} x_i \\ x_{i+1} \\ \vdots \\ x_j \end{bmatrix},\ i \le j,$$

$$x_{i\sim j}^{(0\sim n)} = \begin{bmatrix} x_i^{(0\sim n)} \\ x_{i+1}^{(0\sim n)} \\ \vdots \\ x_j^{(0\sim n)} \end{bmatrix},\ x_k^{(n_k)}|_{k=i\sim j} = \begin{bmatrix} x_i^{(n_i)} \\ x_{i+1}^{(n_{i+1})} \\ \vdots \\ x_j^{(n_j)} \end{bmatrix},\ i \le j,$$

$$x_{i\sim j}^{(0\sim n)} = \begin{bmatrix} x_{i\sim j}^{(0)} \\ x_{i\sim j}^{(1)} \\ \vdots \\ x_{i\sim j}^{(n)} \end{bmatrix},\ x_k^{(n_k)}|_{k=i\sim j} = \begin{bmatrix} x_i^{(n_i)} \\ x_{i+1}^{(n_{i+1})} \\ \vdots \\ x_j^{(n_j)} \end{bmatrix},\ i \le j,$$

$$x_k^{(n_0\sim n_k)}|_{k=i\sim j} = \begin{bmatrix} x_i^{(n_0\sim n_i)} \\ x_{i+1}^{(n_0\sim n_{i+1})} \\ \vdots \\ x_j^{(n_0\sim n_j)} \end{bmatrix},\ i \le j,$$

$$A_{0\sim n-1} = \begin{bmatrix} A_0 & A_1 & \cdots & A_{n-1} \end{bmatrix},$$

$$\Phi(A_{0\sim n-1}) = \begin{bmatrix} 0 & I & & \\ & & \ddots & \\ & & & I \\ -A_0 & -A_1 & \cdots & -A_{n-1} \end{bmatrix}.$$

Usage illustrations

In this subsection, let us illustrate the usages of the above notations, which often appear in the book.

Clearly, using the notations, we can write a linear state-space model as

$$\dot{x}_{1\sim n} = Ax_{1\sim n} + Bu_{1\sim r},$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times r}$, and a nonlinear state-space model as

$$\dot{x}_{1\sim n} = f\left(x_{1\sim n}, u_{1\sim r}\right),$$

where $f\left(x_{1\sim n}, u_{1\sim r}\right)$ is an n-dimensional vector function.

In the following, let us consider two more complicated cases.

Case A. Single-order systems

Consider a linear system in the following form:

$$x^{(m)} + A_{m-1}x^{(m-1)} + \cdots + A_1\dot{x} + A_0x = \upsilon, \tag{1.1}$$

where $x, \upsilon \in \mathbb{R}^r$ and $A_i \in \mathbb{R}^{r\times r}, i = 0, 1, \ldots, m-1$. With our notations, system (1.1) can be compactly written as

$$x^{(m)} + A_{0\sim m-1}x^{(0\sim m-1)} = \upsilon,$$

or equivalently as the state-space form:

$$\dot{x}^{(0\sim m-1)} = \Phi(A_{0\sim m-1})x^{(0\sim m-1)} + B_c\upsilon,$$

where

$$B_c = \begin{bmatrix} 0_{(m-1)r\times r} \\ I_r \end{bmatrix}.$$

Furthermore, the solution to the system is given by

$$\begin{aligned} x^{(0\sim m-1)}(t) &= \exp\left[\Phi(A_{0\sim m-1})t\right]x^{(0\sim m-1)}(0) \\ &\quad + \int_0^t \exp\left[\Phi(A_{0\sim m-1})(t-\tau)\right]B_c\upsilon(\tau)\,\mathrm{d}\tau. \end{aligned}$$

Case B. Multi-order systems

Consider the following system:

$$x_k^{(\mu_k)} + \sum_{j=0}^{\mu_k-1} A_{kkj} x_k^{(j)} + \sum_{\substack{i=1 \\ i\neq k}}^{\eta} \sum_{j=0}^{\mu_j-1} A_{ijk} x_i^{(j)} = v_k,\ k = 1, 2, \ldots, \eta, \tag{1.2}$$

where $1 \leq \eta \leq r$ is an integer, μ_k, $k = 1, 2, \ldots, \eta$, are a set of integers (often distinct), $x_k, v_k \in \mathbb{R}^{r_k}$, $k = 1, 2, \ldots, \eta$, with r_k, $k = 1, 2, \ldots, \eta$, being a set of distinct integers satisfying

$$r_1 + r_2 + \cdots + r_\eta = r. \tag{1.3}$$

Furthermore, $A_{ijk} \in \mathbb{R}^{r_i \times r_j}$, $j = 0, 1, \ldots, \mu_i - 1$, $i, k = 1, 2, \ldots, \eta$.

By using our notations, system (1.2) can be firstly written, more compactly, as

$$x_k^{(\mu_k)} + A_{kk,0\sim\mu_k-1} x_k^{(0\sim\mu_k-1)} + \sum_{\substack{i=1 \\ i\neq k}}^{\eta} A_{ik,0\sim\mu_i-1} x_i^{(0\sim\mu_i-1)} = v_k,\ k = 1, 2, \ldots, \eta. \tag{1.4}$$

Secondly, denote

$$A_k = \begin{bmatrix} A_{1k,0\sim\mu_1-1} & A_{2k,0\sim\mu_2-1} & \cdots & A_{\eta k,0\sim\mu_\eta-1} \end{bmatrix},\ k = 1, 2, \ldots, \eta.$$

Then system (1.2) can be further written as

$$x_k^{(\mu_k)} + A_k x_p^{(0\sim\mu_p-1)}|_{p=1\sim\eta} = v_k,\ k = 1, 2, \ldots, \eta,$$

or

$$\begin{bmatrix} x_1^{(\mu_1)} \\ x_2^{(\mu_2)} \\ \vdots \\ x_\eta^{(\mu_\eta)} \end{bmatrix} = \begin{bmatrix} A_1 x_p^{(0\sim\mu_p-1)}|_{p=1\sim\eta} \\ A_2 x_p^{(0\sim\mu_p-1)}|_{p=1\sim\eta} \\ \vdots \\ A_\eta x_p^{(0\sim\mu_p-1)}|_{p=1\sim\eta} \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_\eta \end{bmatrix}. \tag{1.5}$$

Finally, denote

$$A_E = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_\eta \end{bmatrix},\ v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_\eta \end{bmatrix}.$$

Then system (1.2) can be written as

$$x_k^{(\mu_k)}\Big|_{k=1\sim\eta} = A_E\ x_k^{(0\sim\mu_k-1)}\Big|_{k=1\sim\eta} + v. \tag{1.6}$$

In the special case of

$$A_k = A_{k,0\sim\mu_k-1},\ k = 1, 2, \ldots, \eta,$$

or, equivalently,

$$A_E = \operatorname{diag}\left(A_{k,0\sim\mu_k-1},\ k=1,2,\ldots,\eta\right),$$

system (1.5), or (1.6), becomes the following set of independent ones:

$$x_k^{(\mu_k)} = A_{k,0\sim\mu_k-1}x_k^{(0\sim\mu_k-1)} + \upsilon_k,\ k=1,2,\ldots,\eta, \tag{1.7}$$

whose state-space representations are

$$\dot{x}_k^{(0\sim\mu_k-1)} = \Phi\left(A_{k,0\sim\mu_k-1}\right)x_k^{(0\sim\mu_k-1)} + B_{c,k}\upsilon_k,\ k=1,2,\ldots,\eta. \tag{1.8}$$

In this case, the state response of the system is given by

$$\begin{aligned} x_k^{(0\sim\mu_k-1)}(t) &= \exp\left[\Phi\left(A_{k,0\sim\mu_k-1}\right)t\right]x_k^{(0\sim\mu_k-1)}(0) \\ &\quad + \int_0^t \exp\left[\Phi\left(A_{k,0\sim\mu_k-1}\right)(t-\tau)\right]B_{c,k}\upsilon_k(\tau)\,\mathrm{d}\tau, \\ k &= 1,2,\ldots,\eta. \end{aligned} \tag{1.9}$$

1.6.2 Parametric Generalized Eigenstructure Assignment

Consider the linear system:

$$\dot{x} = Ax + Bu, \tag{1.10}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^r$ are the state vector and the control input vector, respectively; $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times r}$ are real matrices satisfying the following assumptions.

Assumption 1.1 (A, B) *is controllable.*

Assumption 1.2 $\operatorname{rank} B = r$.

When a controller for the system is chosen as

$$u = Kx, \tag{1.11}$$

the closed-loop system is obtained as

$$\dot{x} = (A+BK)x. \tag{1.12}$$

The purpose of this subsection is to present solutions to the following generalized ESA problem.

Problem 1.1 *For an arbitrarily chosen $F \in \mathbb{R}^{n\times n}$, find all the matrices $V \in \mathbb{R}^{n\times n}$ and $K \in \mathbb{R}^{r\times n}$ satisfying* $\det V \neq 0$ *and*

$$A + BK = VFV^{-1}. \tag{1.13}$$

As a consequence, the closed-loop system (1.12) possesses the same eigenstructure of matrix VFV^{-1}. As long as F is stable, the closed-loop system is stable.

For the matrix pair (A, B), according to Duan (2015) (Chapter 3), there exist a pair of right coprime polynomial matrices $N(s) \in \mathbb{R}^{n\times r}[s]$ and $D(s) \in \mathbb{R}^{r\times r}[s]$ satisfying the following right coprime factorization (RCF):

$$(sI - A)^{-1} B = N(s) D^{-1}(s). \tag{1.14}$$

If we denote $D(s) = \left[d_{ij}(s)\right]$ and

$$\omega = \max\left\{\deg\left(d_{ij}(s)\right),\ i, j = 1, 2, \ldots, r\right\},$$

then $N(s)$ and $D(s)$ can be written in the form of

$$\begin{cases} N(s) = \sum_{i=0}^{\omega} N_i s^i,\ N_i \in \mathbb{R}^{n\times r} \\ D(s) = \sum_{i=0}^{\omega} D_i s^i,\ D_i \in \mathbb{R}^{r\times r}. \end{cases} \tag{1.15}$$

The following lemma provides a complete general parametric solution to the generalized ESA problem (Duan (2015)).

Theorem 1.1 *Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$ be two matrices satisfying Assumptions 1.1 and 1.2, and $N(s)$ and $D(s)$ be a pair of right coprime polynomial matrices given by (1.15) and satisfy RCF (1.14). Then, all the solution to Problem 1.1 are given by*

$$\begin{cases} K = WV^{-1} \\ V = N_0 Z + N_1 ZF + \cdots + N_\omega ZF^\omega \\ W = D_0 Z + D_1 ZF + \cdots + D_\omega ZF^\omega, \end{cases} \tag{1.16}$$

where $Z \in \mathbb{R}^{r\times n}$ is an arbitrary parameter matrix satisfying the following constraint:

$$\det\left(N_0 Z + N_1 ZF + \cdots + N_\omega ZF^\omega\right) \neq 0. \tag{1.17}$$

The meaning of equation (1.13) is very clear. If F is chosen to be a stable matrix, then K turns to be a stabilizing gain matrix for the state feedback controller (1.11). Therefore, Theorem 1.1 has given all the stabilizing state feedback controllers for system (1.10). The parameter matrix Z in the above lemma represents the design DOFs in the design, and can be properly utilized to achieve additional system performance (Duan (1992a, 1993a); Duan et al. (2002); Duan (2024a,b, 2025a)).

1.6.3 Nondefective Eigenstructure Assignment

Nondefective ESA - Solution based on RCF

In this subsection, let us consider the important case that the matrix F is in the diagonal form of

$$F = \text{diag}(s_i, i = 1, 2, \ldots, n), \tag{1.18}$$

where $s_i, i = 1, 2, \ldots, n$, are clearly the closed-loop eigenvalues to be assigned.

This subsection treats the case that the closed-loop eigenvalues are allowed to be undetermined and sought later according to some system requirements.

Theorem 1.2 *Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$ be two matrices satisfying Assumptions 1.1 and 1.2, and $N(s)$ and $D(s)$ be a pair of right coprime polynomial matrices satisfying RCF (1.14). Then, for a diagonal matrix $F \in \mathbb{C}^{n\times n}$ in the form of (1.18), all the solutions to Problem 1.1 are given by*

$$\begin{cases} K = WV^{-1} \\ V = \begin{bmatrix} N(s_1)z_1 & N(s_2)z_2 & \cdots & N(s_n)z_n \end{bmatrix} \\ W = \begin{bmatrix} D(s_1)z_1 & D(s_2)z_2 & \cdots & D(s_n)z_n \end{bmatrix}, \end{cases} \tag{1.19}$$

where $z_i \in \mathbb{C}^r$, $i = 1, 2, \ldots, n$, are a group of parameter vectors satisfying the following Constraints:

Constraint 1.1 $\det\begin{bmatrix} N(s_1)z_1 & N(s_2)z_2 & \cdots & N(s_n)z_n \end{bmatrix} \neq 0,$

Constraint 1.2 *$z_i = \bar{z}_l$, if $s_i = \bar{s}_l$ and $\mathrm{Im}(s_i) \neq 0$, $i, l = 1, 2, \ldots, n$.*

The above general complete parametric solution formula given in (1.19) to the generalized ESA problem for the case of F being a diagonal matrix turns out to be just extremely simple. Yet this case is also the mostly encountered one in applications. What is more, with the help of similarity transformations, this diagonal case can also be applied to the case that F is nondefective since a nondefective matrix is similar to a diagonal matrix. For the same reason, the system resulted in by the above controller (1.19) is nondefective.

Remark 1.4 The DOFs existed in this solution (1.19) are composed of two aspects. On one side, the set of vectors z_i, $i = 1, 2, \ldots, n$, which are required to satisfy the two constraints, namely, Constraints 1.1 and 1.2, obviously gives one part of the DOFs. On the other side, please note that this solution allows the eigenvalues of the matrix F to be set undetermined, but restricted to be within designed regions on the complex plane. Particularly, in the case that (A, B) is controllable, the closed-loop eigenvalues $s_i \in \mathbb{C}$, $i = 1, 2, \ldots, n$, can be arbitrarily chosen, and hence form another part of DOFs. These two aspects of DOFs may give great convenience and advantages to many analysis and design problems in control system theory.

Nondefective ESA - Solution based on singular value decompositions

In the previous subsection, solution to the problem of generalized ESA with the matrix F being diagonal is considered, where the eigenvalues of the matrix F are not needed until the very last step to express the solution. Consequently, the matrix F can be set undetermined and used as part of the DOFs.

Different from the previous subsection above, in this subsection, we still treat the case that the matrix F is in the diagonal form of (1.18), but with the eigenvalues s_i, $i = 1, 2, \ldots, n$, being known.

Applying singular value decomposition to $\begin{bmatrix} s_i I - A & -B \end{bmatrix}$, we can obtain two orthogonal matrices P_i, Q_i satisfying the following:

$$P_i \begin{bmatrix} s_i I - A & -B \end{bmatrix} Q_i = \begin{bmatrix} I & 0 \end{bmatrix}, \ i = 1, 2, \ldots, n. \tag{1.20}$$

Further, partition the matrix Q_i as

$$Q_i = \begin{bmatrix} * & \hat{N}_i \\ * & \hat{D}_i \end{bmatrix}, \ \hat{N}_i \in \mathbb{R}^{n \times r}, \ \hat{D}_i \in \mathbb{R}^{r \times r}. \tag{1.21}$$

Then we obtain two series of matrices $\hat{N}_i$ and $\hat{D}_i$, $i = 1, 2, \ldots, n$, satisfying

$$(s_i I - A)\hat{N}_i - B\hat{D}_i = 0, \ i = 1, 2, \ldots, n, \tag{1.22}$$

and

$$\operatorname{rank} \begin{bmatrix} \hat{N}_i \\ \hat{D}_i \end{bmatrix} = r, \ i = 1, 2, \ldots, n. \tag{1.23}$$

With the above preparation, we have the following result about a particular solution to the generalized ESA problem.

Theorem 1.3 *Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times r}$ be two matrices satisfying Assumptions 1.1 and 1.2, $F \in \mathbb{C}^{n \times n}$ be a diagonal matrix in the form of (1.18). Further, let $\hat{N}_i \in \mathbb{R}^{n \times r}$ and $\hat{D}_i \in \mathbb{R}^{r \times r}$, $i = 1, 2, \ldots, n$, be given by (1.22)-(1.23). Then all the solutions to Problem 1.1 are given by*

$$\begin{cases} K = WV^{-1} \\ V = \begin{bmatrix} \hat{N}_1 z_1 & \hat{N}_2 z_2 & \cdots & \hat{N}_n z_n \end{bmatrix} \\ W = \begin{bmatrix} \hat{D}_1 z_1 & \hat{D}_2 z_2 & \cdots & \hat{D}_n z_n \end{bmatrix}, \end{cases} \tag{1.24}$$

where $z_i \in \mathbb{C}^r$, $i = 1, 2, \ldots, n$, are a group of parameter vectors satisfying Constraint 1.2 and

Constraint 1.3 $\det \begin{bmatrix} \hat{N}_1 z_1 & \hat{N}_2 z_2 & \cdots & \hat{N}_n z_n \end{bmatrix} \neq 0.$

Obviously, the above general complete parametric solution (1.24) for the case of F being a known diagonal matrix turns out to be even simpler. This is of course a huge advantage. Besides simplicity, another huge advantage of this solution is its numerical reliability ensured by the well-known numerical stability property of singular value decompositions. Since s_i, $i = 1, 2, \ldots, n$, are used to derive $\hat{N}_i, \hat{D}_i$, $i = 1, 2, \ldots, n$, they are now no longer DOFs. Nevertheless, this solution still provides complete DOFs beyond the eigenvalues of F represented by the group of arbitrary parameter vectors $z_i \in \mathbb{C}^r$, $i = 1, 2, \ldots, n$.

1.6.4 A Particular Eigenstructure Assignment Result

This subsection looks into the problem of seeking a matrix $A_{0\sim m-1}$ such that $\Phi(A_{0\sim m-1})$ is stable. Since

$$\Phi(A_{0\sim m-1}) = \Phi(0_{0\sim m-1}) - B_c A_{0\sim m-1},$$

the problem can be in fact solved via solving a generalized ESA problem associated with the matrix pair $(\Phi(0_{0\sim m-1}), B_c)$. Therefore, solution to this problem can be obtained by applying Theorem 1.1. The result can be stated as follows and has been widely used in the FAS approach (see, Duan (2020a,b, 2021a,b,c,d,f)).

Proposition 1.1 *For an arbitrarily chosen $F \in \mathbb{R}^{mn\times mn}$, all the matrix $A_{0\sim m-1}$ and the nonsingular matrix $V \in \mathbb{R}^{mn\times mn}$ satisfying*

$$\Phi(A_{0\sim m-1}) = VFV^{-1}$$

are given by

$$\begin{aligned} A_{0\sim m-1} &= -ZF^nV^{-1}(Z,F), \\ V &= V(Z,F) = \begin{bmatrix} Z \\ ZF \\ \vdots \\ ZF^{n-1} \end{bmatrix}, \end{aligned}$$

where $Z \in \mathbb{R}^{m\times mn}$ is an arbitrary parameter matrix satisfying

$$\det V(Z,F) \neq 0.$$

Remark 1.5 It is seen from the above result that as long as the matrix F is chosen stable, the closed-loop system is stable. Clearly, there are quite some DOFs in the selection of F, while on the other hand we also have the parameter matrix Z, which provides m^2n DOFs. All these DOFs can be further utilized to achieve additional performance of the system (see, e.g., Duan (1992a, 1993a); Duan et al. (2002); Duan (2024a,b, 2025a)).

Remark 1.6 In Appendix A, several very commonly used special cases of the above Proposition 1.1 are provided. These results present, in a sense, a database for users of the FAS approach to derive a specific parameter $A_{0\sim m-1}$ for the designed controller and the closed-loop systems.

Part I

Basic Theories

Chapter 2

Models of Global FASs

This chapter introduces the FAS models. Firstly, through investigating some motivating examples, the single-order FAS models are introduced, which are direct generalizations of the physical FAS models. Secondly, again by investigating certain motivating examples, the more general multi-order affine and nonaffine FAS models are proposed.

Like the state-space models, the FAS models also generally exist. As elementary demonstrations, in this chapter, it is shown that the following three basic types of systems can all be equivalently converted into FASs:

1. all controllable linear time-invariant (LTI) systems;
2. all lexicography-fixedly controllable linear time-varying (LTV) systems; and
3. all globally feedback linearizable nonlinear systems.

It should be noted that state-space models are not the first-place ones for dynamical systems. As a matter of fact, modelling of dynamical systems using physical laws often produces non-state-space models. Such models can often be directly converted into FAS models instead of the state-space ones.

2.1 Physical FASs

2.1.1 Traditional Definition

"Full-actuation" is originally a physical concept. Physically, a FAS is of the following form:

$$M(x,\dot{x},t)\ddot{x}+D(x,\dot{x},t)\dot{x}+K(x,\dot{x},t)x=u, \tag{2.1}$$

where $x,u\in\mathbb{R}^r$ are the elementary state vector and the control vector, respectively, $M(\cdot)$, $D(\cdot)$ and $K(\cdot)\in\mathbb{R}^{r\times r}$ are, respectively, the generalized mass matrix, the generalized damping matrix, and the generalized stiffness matrix. This type of systems describes a variety of practical systems in the field of robot control, spacecraft control (Duan (2020d,e)), etc, and can be generally modelled by the well-known Lagrangian Equation.

In many cases, the generalized mass matrix is nonsingular, that is,

$$\det M(x,y,t) \neq 0,\ \forall x,y \in \mathbb{R}^r,\ t \geq 0.$$

Under such a condition, system (2.1) can be converted into a general second-order system as follows:

$$\ddot{x} = f(x,\dot{x},t) + B(x,\dot{x},t)u, \tag{2.2}$$

where

$$f(x,\dot{x},t) = -M^{-1}(x,\dot{x},t)[D(x,\dot{x},t)\dot{x} + K(x,\dot{x},t)x], \tag{2.3}$$

$$B(x,\dot{x},t) = M^{-1}(x,\dot{x},t). \tag{2.4}$$

We also call system (2.2)-(2.4) a FAS because $B(\cdot)$ satisfies the following full-actuation condition:

$$\det B(x,y,t) \neq 0,\ \forall x,y \in \mathbb{R}^r,\ t \geq 0.$$

A huge advantage of a FAS lies in its control. In fact, applying the following controller

$$u = -B^{-1}(x,\dot{x},t)\left[f(x,\dot{x},t) + A_1\dot{x} + A_0x - \upsilon\right] \tag{2.5}$$

to system (2.2), gives the LTI closed-loop system

$$\ddot{x} + A_1\dot{x} + A_0x = \upsilon, \tag{2.6}$$

where $\upsilon \in \mathbb{R}^r$ is a reference input, A_0 and A_1 are two $r \times r$ matrices making system (2.6) stable.

We mention that a great deal of practical systems can be modeled by physical laws as in (2.1), and treated as above. But there are also systems which are originally represented by first-order differential equations, while can be converted into second-order FASs, as illustrated in the following subsection.

2.1.2 Attitude Control of Spacecraft

It is well-known that the dynamical equation of the attitude system for a rigid spacecraft is described by the following Euler's equations of motion:

$$\begin{cases} J_1\dot{\omega}_1 = (J_2 - J_3)\omega_2\omega_3 + u_1 \\ J_2\dot{\omega}_2 = (J_3 - J_1)\omega_3\omega_1 + u_2 \\ J_3\dot{\omega}_3 = (J_1 - J_2)\omega_1\omega_2 + u_3, \end{cases} \tag{2.7}$$

where J_i, $i = 1,2,3$, are the principal moments of inertia, ω_i, $i = 1,2,3$, are the angular velocities, and u_i, $i = 1,2,3$, are the torque inputs acting on the spacecraft. Denote

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix},\ u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \tag{2.8}$$

and

$$J = \mathrm{diag}(J_1, J_2, J_3),\ S(\omega) = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}, \tag{2.9}$$

then equation (2.7) can be compactly written as

$$J\dot{\omega} = S(\omega) J\omega + u. \tag{2.10}$$

Next, denote

$$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} \phi \\ \vartheta \\ \psi \end{bmatrix}, \tag{2.11}$$

where

$$\theta_1 = \phi,\ \theta_2 = \vartheta,\ \theta_3 = \psi,$$

are the Euler angles. Then the kinetic equation of the spacecraft can be given by

$$\dot{\theta} = T(\theta_1, \theta_2)\omega, \tag{2.12}$$

where

$$T(\theta_1, \theta_2) = \begin{bmatrix} 1 & \sin\theta_1 \tan\theta_2 & \cos\theta_1 \tan\theta_2 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 \sec\theta_2 & \cos\theta_1 \sec\theta_2 \end{bmatrix}. \tag{2.13}$$

As a normal requirement, let us restrict

$$-\frac{\pi}{2} < \theta_2 = \vartheta < \frac{\pi}{2}.$$

Then we have

$$\det T(\theta_1, \theta_2) = \frac{1}{\cos\theta_2} \neq 0. \tag{2.14}$$

Thus $T^{-1}(\theta_1, \theta_2)$ exists.

Differentiating (2.12), and submitting (2.10) into the results, yield

$$\begin{aligned} \ddot{\theta} &= \dot{T}(\theta_1, \theta_2)\omega + T(\theta_1, \theta_2)\dot{\omega} \\ &= \dot{T}(\theta_1, \theta_2) T^{-1}(\theta_1, \theta_2)\dot{\theta} + T(\theta_1, \theta_2) J^{-1}[S(\omega) J\omega + u] \\ &= \dot{T}(\theta_1, \theta_2) T^{-1}(\theta_1, \theta_2)\dot{\theta} \\ &\quad + T(\theta_1, \theta_2) J^{-1} S\left(T^{-1}\dot{\theta}\right) JT^{-1}\dot{\theta} + T(\theta_1, \theta_2) J^{-1} u, \end{aligned}$$

which gives

$$\ddot{\theta} = f\left(\theta, \dot{\theta}\right) + B(\theta_1, \theta_2) u, \tag{2.15}$$

where

$$\begin{aligned} f\left(\theta, \dot{\theta}\right) &= \dot{T}(\theta_1, \theta_2) T^{-1}(\theta_1, \theta_2)\dot{\theta} \\ &\quad + T(\theta_1, \theta_2) J^{-1} S\left(T^{-1}\dot{\theta}\right) JT^{-1}\dot{\theta}, \end{aligned} \tag{2.16}$$

$$B(\theta_1, \theta_2) = T(\theta_1, \theta_2) J^{-1}. \tag{2.17}$$

This is the FAS model for the attitude system of a rigid spacecraft since $\det B(\theta_1, \theta_2) \neq 0$ due to (2.14).

Applying the following controller

$$u = -B^{-1}(\theta_1, \theta_2)\left[f\left(\theta, \dot{\theta}\right) + A_1\dot{\theta} + A_0\theta - \upsilon\right] \tag{2.18}$$

to system (2.15), gives the LTI closed-loop system

$$\ddot{\theta} + A_1\dot{\theta} + A_0\theta = \upsilon, \tag{2.19}$$

where A_0 and A_1 are two 3×3 matrices making system (2.19) stable, and υ is some reference signal which is chosen zero when only attitude stabilization is considered.

Once θ and $\dot{\theta}$ are solved via (2.19), we can get

$$\omega = T^{-1}(\theta_1, \theta_2)\dot{\theta}. \tag{2.20}$$

2.2 Single-order FASs

Let us give two motivating examples before presenting the general high-order FASs.

2.2.1 Motivating Examples

Example 2.1 Consider the following system:

$$\begin{cases} \ddot{x} = \mathrm{tg}\theta, \quad |\theta| < \frac{\pi}{2} \\ \ddot{\theta} = u, \end{cases} \tag{2.21}$$

where u is a scalar input. Using the first equation in (2.21), we obtain

$$\theta = \mathrm{arctg}\,(\ddot{x}), \tag{2.22}$$

from which we can easily derive

$$\dot{\theta} = \frac{1}{1+\ddot{x}^2}\dddot{x}, \tag{2.23}$$

and further

$$\begin{aligned} \ddot{\theta} &= \left(\frac{1}{1+\ddot{x}^2}\right)'\dddot{x} + \frac{1}{1+\ddot{x}^2}x^{(4)} \\ &= \frac{1}{1+\ddot{x}^2}x^{(4)} - \frac{2}{\left(1+\ddot{x}^2\right)^2}\ddot{x}\,\dddot{x}^2. \end{aligned} \tag{2.24}$$

Substituting the above equation into the second equation in (2.21), yields

$$x^{(4)} - \frac{2}{1+\ddot{x}^2}\ddot{x}\,\dddot{x}^2 = \left(1+\ddot{x}^2\right)u. \tag{2.25}$$

This is the FAS model of system (2.21). Since $1+\ddot{x}^2 \neq 0$, the above system (2.25) is called globally fully actuated.

For control of the above FAS (2.25), we can design the controller

$$u = -\frac{1}{1+\ddot{x}^2}\left(\frac{2}{1+\ddot{x}^2}\ddot{x}\dddot{x}^2 + \sum_{i=0}^{3} a_i x^{(i)} - \upsilon\right), \tag{2.26}$$

where $a_i, i = 0, 1, 2, 3$, are some real scalars, and υ is an external scalar input. Then the closed-loop system is

$$x^{(4)} + \sum_{i=0}^{3} a_i x^{(i)} = \upsilon, \tag{2.27}$$

where $a_i, i = 0, 1, 2, 3$, can be properly chosen such that the above system possesses a set of desired poles. Therefore, all $x^{(i)}, i = 0, 1, 2, 3$, can be determined by the above system, while $\theta, \dot{\theta}$ and $\ddot{\theta}$ can then be determined by (2.22)-(2.24), with the restriction of $|\theta| < \frac{\pi}{2}$.

Example 2.2 Consider the following pseudo strict feedback system (SFS):

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) + x_3 \\ \dot{x}_3 = u, \end{cases} \tag{2.28}$$

where $x_i \in \mathbb{R},\ i = 1, 2, 3$, and $u \in \mathbb{R}$ are the state variables and control input, respectively, and $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ are two differentiable functions, and $f_1(x_1, x_2)$ satisfies the condition

$$\frac{\partial f_1}{\partial x_2} \neq 0,\ \forall x_1, x_2 \in \mathbb{R}. \tag{2.29}$$

Following our treatment, by taking the differential of the second equation in (2.28), and substituting the third one into the result, we obtain the equivalent system

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \ddot{x}_2 = \dot{f}_2(x_1, x_2) + u. \end{cases} \tag{2.30}$$

Taking the differential of the first equation in (2.30), yields

$$\ddot{x}_1 = \frac{\partial f_1}{\partial x_1}\dot{x}_1 + \frac{\partial f_1}{\partial x_2}\dot{x}_2, \tag{2.31}$$

which further gives

$$\begin{aligned} \dddot{x}_1 &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial f_1}{\partial x_1}\dot{x}_1\right] + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial f_1}{\partial x_2}\dot{x}_2\right] \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial f_1}{\partial x_1}\dot{x}_1\right] + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial f_1}{\partial x_2}\right]\dot{x}_2 + \frac{\partial f_1}{\partial x_2}\ddot{x}_2 \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial f_1}{\partial x_1}\dot{x}_1\right] + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial f_1}{\partial x_2}\right]\dot{x}_2 + \frac{\partial f_1}{\partial x_2}\dot{f}_2(x_1, x_2) + \frac{\partial f_1}{\partial x_2}u \\ &= g(x_1^{(0\sim 2)}, x_2, \dot{x}_2) + \frac{\partial f_1}{\partial x_2}u, \end{aligned} \tag{2.32}$$

where

$$g(x_1^{(0\sim 2)}, x_2, \dot{x}_2) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial f_1}{\partial x_1}\dot{x}_1\right] + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial f_1}{\partial x_2}\right]\dot{x}_2 + \frac{\partial f_1}{\partial x_2}\dot{f}_2(x_1, x_2). \tag{2.33}$$

In view of (2.29), by the well-known Theorem of Inverse Functions, the state variable x_2 can be explicitly and uniquely solved from the first equation in (2.28) as follows:

$$x_2 = g_1(x_1, \dot{x}_1). \tag{2.34}$$

Meanwhile, from (2.31) we obtain, in view of condition (2.29),

$$\dot{x}_2 = \left[\frac{\partial f_1}{\partial x_2}\right]^{-1}\left(\ddot{x}_1 - \frac{\partial f_1}{\partial x_1}\dot{x}_1\right). \tag{2.35}$$

With the help of the above relations (2.34) and (2.35), we can convert $g(x_1^{(0\sim 2)}, x_2, \dot{x}_2)$ and $\frac{\partial f_1}{\partial x_2}$ into functions of $x_1^{(0\sim 2)}$ only, that is, we have

$$\begin{cases} f(x_1^{(0\sim 2)}) \triangleq g(x_1^{(0\sim 2)}, x_2, \dot{x}_2) \\ B(x_1^{(0\sim 1)}) \triangleq \frac{\partial f_1(x_1, x_2)}{\partial x_2}. \end{cases}$$

Therefore, it follows from (2.32) that the original system (2.28) is equivalently transformed into the following third-order model:

$$\dddot{x}_1 = f(x_1^{(0\sim 2)}) + B(x_1^{(0\sim 1)})u. \tag{2.36}$$

This turns to be a third-order sub-FAS (to be examined in the second volume Duan (2025g)) if there exists a nonempty set $\mathbb{F}$ such that

$$\det B(X) \neq 0, \ \forall X \in \mathbb{F} \subset \mathbb{R}^2.$$

Particularly, it is a third-order global FAS if $\mathbb{F} = \mathbb{R}^2$, and in this case, a controller can be designed as

$$u = -B^{-1}(x_1^{(0\sim 1)})\left(a_{0\sim 2}x_1^{(0\sim 2)} + f(x_1^{(0\sim 2)}) - v\right), \tag{2.37}$$

which gives the closed-loop system

$$\dddot{x}_1 + a_{0\sim 2}x_1^{(0\sim 2)} = v,$$

where v is a scalar external signal, while the parameter vector

$$a_{0\sim 2} = \begin{bmatrix} a_0 & a_1 & a_2 \end{bmatrix}$$

can be easily chosen to make the closed-loop system stable.

Finally, it is pointed out that as long as the state of the original system is measurable, the above controller is realizable since the needed variables can be obtained through

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \ddot{x}_1 = \frac{\partial f_1}{\partial x_1}\dot{x}_1 + \frac{\partial f_1}{\partial x_2}(f_2(x_1, x_2) + x_3). \end{cases} \tag{2.38}$$

Remark 2.1 This example system does not obey the triangular structure of an SFS, hence the well-known method of backstepping is generally not applicable. Furthermore, with only the assumption (2.29), the conditions for the well-known method of feedback linearization are also not met. Yet with the proposed FAS approach the problem is well solved in the sense that a constant linear closed-loop system is obtained with a desired eigenstructure.

2.2.2 Single-order FASs

Affine FASs

The so-called single-order affine FASs, firstly introduced in Duan (2020a, 2021a), appear in the following form:

$$x^{(n)} = f(x^{(0\sim n-1)}, t) + B(x^{(0\sim n-1)}, t)u, \tag{2.39}$$

where $n \geq 1$ is an integer, $x \in \mathbb{R}^r$ and $u \in \mathbb{R}^r$ are the state vector and system (input vector, respectively, $f(x^{(0\sim n-1)}, t) \in \mathbb{R}^r$ is a piece-wise continuous vector function, and $B(x^{(0\sim n-1)}, t) \in \mathbb{R}^{r\times r}$ is a piece-wise continuous matrix function.

The above system (2.39) is called a global FAS due to the following assumption.

Assumption 2.1 $\det B(X,t) \neq 0$, $\forall X \in \mathbb{R}^{nr}$, $t \geq 0$.

When the above assumption is not met, we can get a sub-FAS which is treated in the second volume of the book Duan (2025g).

Nonaffine FASs

The so-called single-order nonaffine FASs, firstly introduced in Duan (2023b,c), appear in the following form:

$$x^{(n)} = f(x^{(0\sim n-1)}, t) + g(x^{(0\sim n-1)}, u, t), \tag{2.40}$$

where x, u and $f(x^{(0\sim n-1)}, t)$ are as stated above, while $g(x^{(0\sim n-1)}, u, t) \in \mathbb{R}^r$ is a piece-wise continuous vector function.

The above nonaffine system (2.40) is called a global FAS due to the following assumption.

Assumption 2.2 *The following mapping*

$$\tilde{u} = g(X, u, t) \tag{2.41}$$

forms a differential homeomorphism from u to $\tilde{u}$ for all $X \in \mathbb{R}^{nr}$ and $t \geq 0$.

Again, when the above assumption is not met, we can get a nonaffine sub-FAS which is treated in the second volume of the book Duan (2025g).

Traditional treatments

In traditional context of related research, for the affine FAS (2.39), due to Assumption 2.1, the following control transformation is often applied:

$$\hat{u} = f(x^{(0\sim n-1)}, t) + B(x^{(0\sim n-1)}, t)u. \tag{2.42}$$

Parallelly, for the nonaffine FAS (2.40), due to Assumption 2.2, the following control transformation is often applied:

$$\hat{u} = f(x^{(0\sim n-1)}, t) + g(x^{(0\sim n-1)}, u, t). \tag{2.43}$$

Based on the above two control transformations, both the affine FAS (2.39) and the nonaffine FAS (2.40) are reduced to the following linear system:

$$x^{(n)} = \hat{u}.$$

Remark 2.2 In our context of FAS approach, the above treatment is not accepted because of the following reasons:

1. the more general forms (2.39) and (2.40) are easier to treat some more control problems, such as robust and adaptive control by simply adding uncertain terms of function $f(x^{(0\sim n-1)}, t)$ and $B(x^{(0\sim n-1)}, t)$ or $g(x^{(0\sim n-1)}, u, t)$;

2. maintaining the general forms gives a FAS approach which happens to the special case of that for sub-FASs treated in the second volume of the book Duan (2025g), and that for unidirectionally connected FASs (UC-FASs) treated in the third volume of the book Duan (2025h); and

3. the more general forms (2.39) and (2.40), like the state-space models, actually represent general models for dynamical systems, especially, when the above two assumptions are not met. The above treatment turned away too early to discover the present general FAS approach for control.

2.2.3 From Physical to Mathematical

Once we face a physical control system with a clear mechanism, denoted by Σ_{physical} for convenience, to establish the model of the system, we need firstly to derive a set of basic equations.

To model a physical system via mechanism, we need to first derive a set of equations by applying certain physical principles governing the dynamics of the system. These include, but not limited to, the Newton's Law, the Theorem of Linear and Angular Momentum, the Lagrangian Equation, and the Kirchhoff's Law of Voltage or Current. This set of equations obtained by applying these laws are called basic equations. For convenience, let the basic equations be generally denoted by

$$\alpha_i\left(z^{(0\sim n_1)}, u^{(0\sim n_2)}, t\right) = 0, \quad i = 1, 2, \ldots, \kappa, \tag{2.44}$$

where $\alpha_i(\cdot)$, $i = 1, 2, \ldots, \kappa$, are a set of vector functions (often scalar ones), z is a vector of dynamical variables in the system, while u is a vector of control variables. Let us particularly point out that, with the above-mentioned physical laws, most of the established original set of basic equations in (2.44) are of second-order, that is, $n_1 = 2$, and $n_2 \leq 2$ (see Duan (2020a)).

When a set of basic equations for a physical control system are obtained, we can further derive the desired type of models by rearranging the set of basic equations. There are basically two methods to fulfill this task, namely, the method of variable extension and the method of variable elimination. The former produces a first-order state-space model for a control system, while the latter gives a high-order model.

Due to the popularity of the state-space approaches, almost all models for physical control systems in the literature are finally converted into state-space forms, which are essentially a set of first-order differential equations. Since most of the original set of basic equations in (2.44) are of second-order, the state-space approach for modelling is essentially a method of variable extension. Without the extended variables, a set of high-order differential equations can not at all be equivalently turned into a set of first-order differential equations.

For over half a century of dominance, the state-space approaches have been completely accepted by system (and control community. State-space models have been taken universal. Control scientists and engineers have for long been so accustomed to establishing a state-space model for any theoretical or practical system encountered, despite the fact that, in most cases, the set of basic equations in (2.44) originally established by the physical laws are of second-order, and even may be already fully actuated.

It is not that state-space models are of the first priority, especially when dealing with the problem of control. It is not that high-order models have to be obtained from state-space models, many of these models should be directly obtained from the set of basic equations in the first place. Practically, most FAS models should be obtained directly from the modelling process.

too add

Remark 2.3 The original physical FASs and underactuated systems (UASs) are both of the second-order. They are determined by the number of actuators (or number of control inputs) and the degrees of freedom (DOFs) existing in a physical (often mechanical) system. As a consequence, FASs and UASs can not be transformed mathematically from one to another. However, with the newly defined mathematical FASs, many UASs can be converted into FASs by applying proper differential homeomorphisms. Furthermore, FASs, under the generalized meaning, are no longer restricted to second-order ones. They are in most cases of higher orders, but in certain extreme cases, we may also have first-order FASs.

2.3 Multi-order Global FASs

Again, let us start with an example.

2.3.1 Motivating Example

Example 2.3 Consider the following dynamical system:

$$\begin{cases} \dot{x}_1 = x_2 - 0.1x_1 + u_1 \\ \dot{x}_2 = x_3 - 0.01x_2 - 0.5x_2^3 - x_1 + u_2 \\ \dot{x}_3 = -0.3x_3 + 0.1\tanh(x_3) - x_2, \end{cases} \tag{2.45}$$

which is obtained from a system studied in Treangle et al. (2019) by discarding the output equation and the external term $\varepsilon(t)$.

From the third equation in (2.45), we can get

$$x_2 = -0.3x_3 + 0.1\tanh(x_3) - \dot{x}_3. \tag{2.46}$$

Substituting this into the first equation in (2.45), produces

$$\dot{x}_1 = f_1\left(x_1, x_3^{(0\sim1)}\right) + u_1, \tag{2.47}$$

where

$$f_1\left(x_1, x_3^{(0\sim1)}\right) = -0.3x_3 + 0.1\tanh(x_3) - \dot{x}_3 - 0.1x_1. \tag{2.48}$$

Meanwhile, taking the derivative of the third equation in (2.45), and into the result substituting the second equation in (2.45), yield

$$\begin{aligned} \ddot{x}_3 &= -0.3\dot{x}_3 + 0.1\left(1 - \tanh^2 x_3\right)\dot{x}_3 - \dot{x}_2 \\ &= -0.3\dot{x}_3 + 0.1\left(1 - \tanh^2 x_3\right)\dot{x}_3 - x_3 + 0.01x_2 + 0.5x_2^3 + x_1 - u_2. \end{aligned}$$

Again, substituting (2.46) into the above equation, gives

$$\ddot{x}_3 = f_2\left(x_1, x_3^{(0\sim1)}\right) - u_2, \tag{2.49}$$

where

$$f_2\left(x_1, x_3^{(0\sim1)}\right) = x_1 + \frac{1}{100}\left(f_{21}(x_3) + f_{22}\left(x_3^{(0\sim1)}\right)\right), \tag{2.50}$$

with

$$\begin{aligned} f_{21}(x_3) &= 1.35x_3^2(\tanh x_3 - x_3) + 0.05\tanh^3 x_3 \\ &\quad - \left(0.45\tanh^2 x_3 + 100.3\right)x_3 + 0.1\tanh x_3, \end{aligned} \tag{2.51}$$

$$\begin{aligned} f_{22}\left(x_3^{(0\sim1)}\right) &= -50\dot{x}_3^3 - 45x_3\dot{x}_3^2 - \left(13.5x_3^2 + 21\right)\dot{x}_3 \\ &\quad + (9x_3 + 15\dot{x}_3)\dot{x}_3\tanh x_3 - 11.5\dot{x}_3\tanh^2 x_3. \end{aligned} \tag{2.52}$$

Combining (2.47) and (2.49), gives

$$\begin{bmatrix} \dot{x}_1 \\ \ddot{x}_3 \end{bmatrix} = \begin{bmatrix} f_1\left(x_1, x_3^{(0\sim1)}\right) \\ f_2\left(x_1, x_3^{(0\sim1)}\right) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{2.53}$$

Applying the following controller

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \left(\begin{bmatrix} f_1\left(x_1, x_3^{(0\sim 1)}\right) \\ f_2\left(x_1, x_3^{(0\sim 1)}\right) \end{bmatrix} + \begin{bmatrix} a_1 x_1 \\ b_1 \dot{x}_3 + b_0 x_3 \end{bmatrix} \right) \tag{2.54}$$

to system (2.53), gives the linear closed-loop system

$$\begin{cases} \dot{x}_1 + a_1 x_1 = 0 \\ \ddot{x}_3 + b_1 \dot{x}_3 + b_0 x_3 = 0, \end{cases} \tag{2.55}$$

where a_1 and b_0, b_1 are all positive scalars. Once x_1 and $x_3^{(0\sim 1)}$ are obtained from (2.55), we can then get x_2 from (2.46), which clearly converges to zero since both x_1 and $x_3^{(0\sim 1)}$ do.

2.3.2 Multi-order FASs

Affine multi-order FASs

System (2.53) is called the multi-order FAS model of system (2.45). In general, an affine multi-order FAS, firstly introduced in Duan (2021g), is in the following form:

$$\begin{bmatrix} x_1^{(\mu_1)} \\ x_2^{(\mu_2)} \\ \vdots \\ x_\eta^{(\mu_\eta)} \end{bmatrix} = \begin{bmatrix} f_1\left(x_k^{(0\sim \mu_k - 1)}|_{k=1\sim\eta}, t\right) \\ f_2\left(x_k^{(0\sim \mu_k - 1)}|_{k=1\sim\eta}, t\right) \\ \vdots \\ f_\eta\left(x_k^{(0\sim \mu_k - 1)}|_{k=1\sim\eta}, t\right) \end{bmatrix} + B\left(x_k^{(0\sim \mu_k - 1)}|_{k=1\sim\eta}, t\right) u, \tag{2.56}$$

where μ_k, $k = 1, 2, \ldots, \eta$, are a set of integers, $x_k \in \mathbb{R}^{r_k}$, $k = 1, 2, \ldots, \eta$, are a set of vectors of proper dimensions, with r_k, $k = 1, 2, \ldots, \eta$, being a set of distinct integers satisfying

$$r_1 + r_2 + \cdots + r_\eta = r. \tag{2.57}$$

Further, $f_k(\cdot) \in \mathbb{R}^{r_k}$, $k = 1, 2, \ldots, \eta$, are a set of piece-wise continuous nonlinear vector functions, and $B(\cdot) \in \mathbb{R}^{r\times r}$ is a piece-wise continuous matrix function, both with respect to $x_k^{(0\sim \mu_k - 1)}|_{k=1\sim\eta}$, and $t \geq 0$.

Introduce

$$\varkappa_0 = r_1\mu_1 + r_2\mu_2 + \cdots + r_\eta\mu_\eta.$$

Then the above system (2.56) is called a global multi-order FAS if the following assumption is satisfied:

Assumption 2.3 $\det B(X, t) \neq 0$ *for* $X \in \mathbb{R}^{\varkappa_0}$, *and* $t \geq 0$.

If we denote

$$f\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta},t\right)=\left[\begin{array}{c} f_1\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta},t\right) \\ f_2\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta},t\right) \\ \vdots \\ f_\eta\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta},t\right) \end{array}\right],$$

then system (2.56) can be compactly written as

$$x_k^{(\mu_k)}|_{k=1\sim\eta}=f\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta},t\right)+B(\cdot)u. \tag{2.58}$$

Clearly, in the case of $\eta=1$, the above FAS model (2.56), or (2.58), reduces to the form of (2.39).

When nonlinear uncertainties, unknown parameters and disturbances are added to these basic FASs, robust and adaptive control as well as disturbance rejection control have been investigated in Duan (2021c,d,e,f) (see also Chapters 4 and 5).

Nonaffine multi-order FASs

A nonaffine multi-order FAS, firstly introduced in Duan (2023b,c), is in the following form:

$$\left[\begin{array}{c} x_1^{(\mu_1)} \\ x_2^{(\mu_2)} \\ \vdots \\ x_\eta^{(\mu_\eta)} \end{array}\right]=\left[\begin{array}{c} f_1\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta},t\right) \\ f_2\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta},t\right) \\ \vdots \\ f_\eta\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta},t\right) \end{array}\right]+g\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta},u,t\right), \tag{2.59}$$

where $g(\cdot)\in\mathbb{R}^r$ is a piece-wise continuous nonlinear vector function, while all the other variables are stated as above.

This system is called a global nonaffine multi-order FAS if the following assumption is assumed:

Assumption 2.4 *The following mapping*

$$\tilde{u}=g(X,u,t), \tag{2.60}$$

forms a differential homeomorphism from u to $\tilde{u}$ for all $X\in\mathbb{R}^{\varkappa_0}$ and $t\geq 0$.

Similarly, the nonaffine FAS (2.59) can also be compactly written as

$$x_k^{(\mu_k)}|_{k=1\sim\eta}=f\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta},t\right)+g\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta},u,t\right). \tag{2.61}$$

Clearly, in the case of $\eta=1$, the above FAS model (2.59), or (2.61), reduces to the form of (2.40).

Remark 2.4 Regarding modelling of FASs, there are generally two ways. One is through modelling by mechanism using existing physical laws, such as, Newton's Second Law, Lagrangian Equation of Motion, Kirchhoff's Law, etc. The other is through converting state-space models into FAS models. The latter is a key step in the FAS approach, and will be demonstrated frequently in the book.

2.4 Linear Time-invariant FASs

2.4.1 FAS Controllability Canonical Form

Consider a linear system in the following state-space form:

$$\dot{x} = Ax + Bu, \tag{2.62}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^r$ are the state vector and input vector, respectively, and $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times r}$ are the coefficient matrices.

Single-input Case

Consider the LTI single-input system in the form of

$$\dot{x} = Ax + bu, \tag{2.63}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$ are the system state vector and the system input, respectively, and $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ are the system matrices. Denote the characteristic polynomial of system (2.63)

$$\det(sI - A) = \alpha(s) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0. \tag{2.64}$$

Then we have the following well-known result.

Proposition 2.1 *System (2.63) is controllable if and only if the following controllability matrix*

$$Q_c = \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix}$$

is invertible. Further, define

$$P = Q_c \begin{bmatrix} \alpha_1 & \cdots & \alpha_{n-1} & 1 \\ \vdots & \iddots & \iddots & \\ \alpha_{n-1} & \iddots & & \\ 1 & & & \end{bmatrix}, \tag{2.65}$$

then under the coordinate transformation $\bar{x} = P^{-1}x$, system (2.63) can be converted into the Luenberger controllability canonical form

$$\dot{\bar{x}} = A_c\bar{x} + b_c u, \tag{2.66}$$

where

$$A_c = \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n-1} \end{bmatrix}, \; b_c = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{2.67}$$

Based on the above proposition, we can further obtain the following result which reveals the relation between controllability of system (2.63) and its FAS presentation.

Theorem 2.1 *System (2.63) is controllable if and only if it can be equivalently converted into the following linear FAS:*

$$z^{(n)} + \alpha_{n-1} z^{(n-1)} + \cdots + \alpha_1 \dot{z} + \alpha_0 z = u, \tag{2.68}$$

with α_i, $i = 0, 1, \ldots, n-1$, *given by (2.64).*

Proof. Firstly, according to Proposition 2.1, system (2.63) is controllable if and only if the coordinate transformation $x = P\bar{x}$, with P given by (2.65) being invertible, and under which system (2.63) is reduced into the Luenberger controllability canonical form (2.66)-(2.67).

Secondly, rewrite the above the Luenberger controllability canonical form (2.66)-(2.67) into the form of

$$\begin{cases} \dot{\bar{x}}_1 = \bar{x}_2 \\ \dot{\bar{x}}_2 = \bar{x}_3 \\ \quad \vdots \\ \dot{\bar{x}}_{n-1} = \bar{x}_n \\ \dot{\bar{x}}_n = -\alpha_0 \bar{x}_1 - \alpha_1 \bar{x}_2 - \cdots - \alpha_{n-1} \bar{x}_n + u. \end{cases} \tag{2.69}$$

If we denote $z = \bar{x}_1$, then the first $n-1$ equations of (2.69) yield

$$\bar{x}_i = z^{(i-1)}, \; i = 1, 2, \ldots, n. \tag{2.70}$$

Substituting (2.70) into the last equation of (2.69), gives FAS (2.68). Finally, in view of the fact that the above process is invertible, the proof is completed. ■

The above result states that the above single-order FAS (2.68) can be viewed as a controllability canonical form for the single-input LTI (2.63).

Multi-input Case

Let us consider the LTI multi-input system

$$\dot{x} = Ax + Bu, \tag{2.71}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^r$ are the state vector and the input vector, respectively, and $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times r}$ are the system matrices.

Parallel to Proposition 2.1, we have the following result.

Proposition 2.2 *System (2.71) is controllable if and only if there exists an invertible state transformation $\hat{x} = Px$, which converts system (2.71) into the following Luenberger controllability canonical form:*

$$\dot{\hat{x}} = \hat{A}_c\hat{x} + \hat{B}_c u, \tag{2.72}$$

where

$$\hat{A}_c = \begin{bmatrix} \hat{A}_{11} & \cdots & \hat{A}_{1r} \\ \vdots & \ddots & \vdots \\ \hat{A}_{r1} & \cdots & \hat{A}_{rr} \end{bmatrix},\ \hat{B}_c = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \\ \vdots \\ \hat{B}_r \end{bmatrix}, \tag{2.73}$$

with

$$\begin{aligned}
\hat{A}_{ii} &= \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ -\alpha_{ii}^0 & -\alpha_{ii}^1 & \cdots & -\alpha_{ii}^{\mu_i-1} \end{bmatrix} \in \mathbb{R}^{\mu_i\times\mu_i}, \\
\hat{A}_{ij} &= \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ -\alpha_{ij}^0 & \cdots & -\alpha_{ij}^{\mu_j-1} \end{bmatrix} \in \mathbb{R}^{\mu_i\times\mu_j},\ i \neq j, \\
\hat{B}_i &= \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & & 0 & 1 & b_{i,i+1} & \cdots & b_{i,r} \end{bmatrix} \in \mathbb{R}^{\mu_i\times r}.
\end{aligned}$$

In fact, a procedure for finding such a transformation matrix P can be given as follows.

Algorithm 2.1 Let μ_i, $i = 1, 2, \ldots, r$, be the controllability indices of system (2.71).

Step 1. Reorder the columns in the controllability matrix

$$Q_c = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \tag{2.74}$$

to obtain

$$\begin{cases} \bar{Q}_c = \begin{bmatrix} \bar{Q}_{c1} & \bar{Q}_{c2} & \cdots & \bar{Q}_{cr} \end{bmatrix} \\ \bar{Q}_{ci} = \begin{bmatrix} b_i & Ab_i & \cdots & A^{\mu_i-1}b_i \end{bmatrix},\ i = 1, 2, \ldots, r, \end{cases} \tag{2.75}$$

with b_i being the i-th column of the input matrix B.

Step 2. As a consequence, $\bar{Q}_c$ given by (2.75) is nonsingular. Find

$$\bar{Q}_c^{-1} = \begin{bmatrix} \bar{q}_{10}^{\mathrm{T}} \\ \vdots \\ \bar{q}_{1,\mu_1-1}^{\mathrm{T}} \\ \vdots \\ \bar{q}_{r0}^{\mathrm{T}} \\ \vdots \\ \bar{q}_{r,\mu_r-1}^{\mathrm{T}} \end{bmatrix}, \tag{2.76}$$

with $\bar{q}_{ij} \in \mathbb{R}^n$, $i = 1, 2, \ldots, r,\ j = 0, 1, \ldots, \mu_i - 1$.

Step 3. Construct the transformation matrix P as

$$P = \begin{bmatrix} \bar{q}_{1,\mu_1-1}^{\mathrm{T}} \\ \bar{q}_{1,\mu_1-1}^{\mathrm{T}} A \\ \vdots \\ \bar{q}_{1,\mu_1-1}^{\mathrm{T}} A^{\mu_1-1} \\ \vdots \\ \bar{q}_{r,\mu_r-1}^{\mathrm{T}} \\ \bar{q}_{r,\mu_r-1}^{\mathrm{T}} A \\ \vdots \\ \bar{q}_{r,\mu_r-1}^{\mathrm{T}} A^{\mu_r-1} \end{bmatrix}. \tag{2.77}$$

Parallel to Theorem 2.1, we also have the following result for the LTI multi-input system (2.71).

Theorem 2.2 *System (2.71) is controllable if and only if it can be equivalently converted into the following linear multi-order FAS:*

$$z_k^{(\mu_k)}\Big|_{k=1\sim r} = \tilde{L}\, z_k^{(0\sim\mu_k-1)}\Big|_{k=1\sim r} + \tilde{B}u, \tag{2.78}$$

where μ_k, $k = 1, 2, \ldots, r$, are a set of integers satisfying

$$\mu_1 + \mu_2 + \cdots + \mu_r = n,$$

$z = [z_1, z_2, \cdots, z_r]^{\mathrm{T}} \in \mathbb{R}^r$ is some vector satisfying

$$x = T z_k^{(0\sim\mu_k-1)}|_{k=1\sim r}, \tag{2.79}$$

for some nonsingular matrix $T \in \mathbb{R}^{n\times n}$. Furthermore, if system (2.71) is reduced to the

Luenberger controllability canonical form (2.72)-(2.73), then

$$\tilde{L} = \begin{bmatrix} -\alpha_{11}^{0} & \cdots & -\alpha_{11}^{\mu_1-1} & \cdots & -\alpha_{1r}^{0} & \cdots & -\alpha_{1r}^{\mu_r-1} \\ -\alpha_{21}^{0} & \cdots & -\alpha_{21}^{\mu_1-1} & \cdots & -\alpha_{2r}^{0} & \cdots & -\alpha_{2r}^{\mu_r-1} \\ \vdots & & \vdots & & \vdots & \cdots & \vdots \\ -\alpha_{r1}^{0} & \cdots & -\alpha_{r1}^{\mu_1-1} & \cdots & -\alpha_{rr}^{0} & \cdots & -\alpha_{rr}^{\mu_r-1} \end{bmatrix}, \tag{2.80}$$

$$\tilde{B} = \begin{bmatrix} 1 & b_{12} & b_{13} & \cdots & b_{1r} \\ & 1 & b_{23} & \cdots & b_{2r} \\ & & 1 & \cdots & b_{3r} \\ & & & \ddots & \vdots \\ & & & & 1 \end{bmatrix},$$

and the state variables $z_k^{(0\sim\mu_k-1)}\Big|_{k=1\sim r}$ *is given by*

$$z_k^{(0\sim\mu_k-1)} = \hat{x}_{k,1\sim\mu_k},\ k = 1,2,\ldots,r. \tag{2.81}$$

Proof. Firstly, according to Proposition 2.2, system (2.71) is controllable if and only if it can be equivalently the coordinate transformation $x = P\bar{x}$, with P given by (2.77) being invertible, and under which system (2.71) is reduced into the Luenberger controllability canonical form (2.72)-(2.73).

Secondly, in accordance with canonical form (2.72)-(2.73), the state vector $\hat{x}$ and the control vector u can be partitioned into

$$\hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \vdots \\ \hat{x}_r \end{bmatrix},\ u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{bmatrix},$$

with

$$\hat{x}_i = \begin{bmatrix} \hat{x}_{i1} & \hat{x}_{i2} & \cdots & \hat{x}_{i\mu_i} \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{\mu_i}$$

and μ_i, $i = 1,2,\ldots,r$, satisfying

$$\mu_1 + \mu_2 + \cdots + \mu_r = n.$$

Thus we can rewrite (2.72)-(2.73) into the following explicit forms:

$$\begin{cases} \dot{\hat{x}}_{k1} = \hat{x}_{k2} \\ \dot{\hat{x}}_{k2} = \hat{x}_{k3} \\ \quad\vdots \\ \dot{\hat{x}}_{k,\mu_k-1} = \hat{x}_{k\mu_k} \\ \dot{\hat{x}}_{k\mu_k} = -\sum_{i=1}^{\mu_k} \alpha_{kk}^{i-1} \hat{x}_{ki} - \sum_{i=1,i\neq k}^{r} \sum_{j=1}^{\mu_i} \alpha_{ki}^{j-1} \hat{x}_{ij} \\ \qquad\quad +u_k + \sum_{l=k+1}^{r} b_{kl} u_l,\ k = 1,2,\ldots,r. \end{cases} \tag{2.82}$$

It then follows from the first $\mu_k - 1$ equations of (2.82) that the following relations hold:

$$\begin{cases} \hat{x}_{k2} = \dot{\hat{x}}_{k1} \\ \hat{x}_{k3} = \ddot{\hat{x}}_{k1} \\ \quad \vdots \\ \hat{x}_{k\mu_k} = \hat{x}_{k1}^{(\mu_k-1)}, \ k = 1, 2, \ldots, r, \end{cases} \tag{2.83}$$

Substituting the above into the last equation of (2.82) gives

$$\hat{x}_{k1}^{(\mu_k)} = -\sum_{i=1}^{\mu_k} \alpha_{kk}^{i-1} \hat{x}_{k1}^{(i-1)} - \sum_{i=1, i\neq k}^{r} \sum_{j=1}^{\mu_i} \alpha_{ki}^{j-1} \hat{x}_{i1}^{(j-1)} + u_k + \sum_{l=k+1}^{r} b_{kl} u_l, \ k = 1, 2, \ldots, r. \tag{2.84}$$

Finally, define

$$z_k = \hat{x}_{k1}, \ k = 1, 2, \ldots, r.$$

Combining this with the equations in (2.83), gives transformation (2.81), or equivalently,

$$\hat{x}_{kj} = z_k^{(j-1)}, \ j = 1, 2, \ldots, \mu_k, \ k = 1, 2, \ldots, r. \tag{2.85}$$

Substituting these into equation (2.84) produces the multi-order FAS (2.78). Note that the above process is invertible. The proof is then completed. ■

Suppose there are η ($\eta \leq r$) number of distinct integers among μ_k, $k = 1, 2, \ldots, r$, which are assumed, without loss of generality, to be μ_k, $k = 1, 2, \ldots, \eta$. By combining together the equations with the same maximum differential order in (2.78), we can write the linear FAS (2.78) into the following multi-order global FAS:

$$z_k^{(\mu_k)}\Big|_{k=1\sim\eta} = \hat{L}\, z_k^{(0\sim\mu_k-1)}\Big|_{k=1\sim\eta} + \hat{B}u, \tag{2.86}$$

which can also be more explicitly written as

$$\begin{bmatrix} z_1^{(\mu_1)} \\ z_2^{(\mu_2)} \\ \vdots \\ z_\eta^{(\mu_\eta)} \end{bmatrix} = \begin{bmatrix} \hat{L}_1 z_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta} \\ \hat{L}_2 z_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta} \\ \vdots \\ \hat{L}_\eta z_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta} \end{bmatrix} + \hat{B}u, \tag{2.87}$$

where $1 \leq \eta \leq r$ is an integer, μ_k, $k = 1, 2, \ldots, \eta$, are a set of distinct integers, $x_k \in \mathbb{R}^{r_k}$, $k = 1, 2, \ldots, \eta$, are a set of state vectors, with r_k, $k = 1, 2, \ldots, \eta$, being a set of distinct integers satisfying

$$r_1 + r_2 + \cdots + r_\eta = r. \tag{2.88}$$

Further,

$$\hat{L} = \begin{bmatrix} \hat{L}_1 \\ \hat{L}_2 \\ \vdots \\ \hat{L}_\eta \end{bmatrix}, \ \hat{L}_k \in \mathbb{R}^{r_k \times n},$$

and $\hat{B} \in \mathbb{R}^{r\times r}$ is a constant nonsingular matrix.

The above result states that the LTI FAS (2.78), or (2.86), can be viewed as a controllability canonical form for the multi-input LTI (2.71).

The above result is firstly proven in Duan (2020b) based on a controllability form of system (2.62). A frequency-domain version of the proof is given in Duan and Zhou (2022). It was revealed in Duan and Zhou (2022) that the integers μ_k, $k = 1, 2, \ldots, \eta$, are actually the controllability indices of system (2.62). A discrete-time version of the above result can be found in Duan (2022f).

The above result is a fundamental contribution to linear system theory. It is important because of the following two aspects:

1. It actually reveals that the FAS is really a canonical controllability form of a general multi-variable linear system, which cannot be derived within the state-space framework;

2. It provides a natural, extremely easy and robust way to realize control of linear systems by means of pole assignment.

To show clearly the above second point, let us consider control of LTI systems based on the derived FAS model.

2.4.2 Eigenstructure Assignment

As a matter of fact, based on the FAS model (2.86), we can immediately propose the completely decoupled controller

$$\begin{cases} u = \hat{B}^{-1}\left(\begin{bmatrix} u_1^* \\ u_2^* \\ \vdots \\ u_\eta^* \end{bmatrix} - \hat{L}\, z_k^{(0\sim\mu_k-1)}\Big|_{k=1\sim\eta}\right) \\ u_k^* = -A_{k,0\sim\mu_k-1} z_k^{(0\sim\mu_k-1)},\ k = 1, 2, \ldots, \eta, \end{cases}$$

which results in the following set of LTI closed-loop systems:

$$z_k^{(\mu_k)} + A_{k,0\sim\mu_k-1} z_k^{(0\sim\mu_k-1)} = 0,\ k = 1, 2, \ldots, \eta, \tag{2.89}$$

where $A_{k,0\sim\mu_k-1}$, $k = 1, 2, \ldots, \eta$, are a series of row vectors which make the systems in (2.89) stable.

Once the sub-vectors $z_k^{(0\sim\mu_k-1)}$, $k = 1, 2, \ldots, \eta$, or, equivalently, the overall state vector $z_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}$, are derived by solving the set of LTI systems in (2.89), the original state vector of system x can be given through the state transformation $\hat{x} = Px$

and (2.85) as

$$\begin{cases} x = P^{-1}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \vdots \\ \hat{x}_r \end{bmatrix}, \ \hat{x}_i \in \mathbb{R}^{\mu_i} \\ \hat{x}_i = \begin{bmatrix} \hat{x}_{i1} & \hat{x}_{i2} & \cdots & \hat{x}_{i\mu_i} \end{bmatrix}^{\mathrm{T}} \\ \hat{x}_{i,1\sim\mu_i} = z_i^{(0\sim\mu_i-1)}, \ i = 1, 2, \ldots, r. \end{cases} \tag{2.90}$$

2.4.3 Example

Consider the following linear system (see the Problem 8.13 in Chen (1995)):

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 1 & 2 & 3 \\ 2 & 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 2 \\ 0 & 2 \end{bmatrix} u. \tag{2.91}$$

After confirming the controllability of system (2.91), it is obtained that the controllability indices are $\mu_1 = 3$ and $\mu_2 = 1$. Carrying out the first two steps in Algorithm 2.1, we obtain

$$\begin{aligned} \bar{Q}_c &= \begin{bmatrix} b_1 & Ab_1 & A^2b_1 & b_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 2 & 5 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \\ \bar{Q}_c^{-1} &= \begin{bmatrix} 0 & -2 & 1 & -1 \\ -2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \bar{q}_{10}^{\mathrm{T}} \\ \bar{q}_{11}^{\mathrm{T}} \\ \bar{q}_{12}^{\mathrm{T}} \\ \bar{q}_{20}^{\mathrm{T}} \end{bmatrix}. \end{aligned}$$

Fulfilling the last step in Algorithm 2.1, give the transformation matrix as

$$P = \begin{bmatrix} \bar{q}_{12}^{\mathrm{T}} \\ \bar{q}_{12}^{\mathrm{T}} A \\ \bar{q}_{12}^{\mathrm{T}} A^2 \\ \bar{q}_{20}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}. \tag{2.92}$$

Therefore, with the state transformation $\hat{x} = Px$, system (2.91) can be equivalently converted into the following controllability canonical form:

$$\dot{\hat{x}} = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 6 \\ \hline 1 & 0 & 0 & 0 \end{array}\right] \hat{x} + \left[\begin{array}{c|c} 0 & 0 \\ 0 & 0 \\ 1 & 2 \\ \hline 0 & 1 \end{array}\right] u,$$

Applying Theorem 2.2, we obtain the FAS model of system (2.91) as

$$z_k^{(\mu_k)}\Big|_{k=1\sim 2} = L_z\, z_k^{(0\sim\mu_k-1)}\Big|_{k=1\sim 2} + B_z u,$$

or

$$\begin{bmatrix} \dddot{z}_1 \\ \dot{z}_2 \end{bmatrix} = L_z \begin{bmatrix} z_1^{(0\sim 2)} \\ z_2 \end{bmatrix} + B_z u,$$

where

$$\begin{bmatrix} z_1^{(0\sim 2)} \\ z_2 \end{bmatrix} = \begin{bmatrix} \hat{x}_{1\sim 3} \\ \hat{x}_4 \end{bmatrix}, \tag{2.93}$$

and

$$L_z = \begin{bmatrix} 0 & 1 & 2 & 6 \\ 1 & 0 & 0 & 0 \end{bmatrix},\ B_z = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Then, the controller can be readily designed as

$$\begin{aligned} u &= -B_z^{-1}\left(L_z + \begin{bmatrix} a_0 & a_1 & a_2 & 0 \\ 0 & 0 & 0 & b_0 \end{bmatrix}\right)\begin{bmatrix} z_1^{(0\sim 2)} \\ z_2 \end{bmatrix} \\ &= -B_z^{-1}\left(L_z + \begin{bmatrix} a_0 & a_1 & a_2 & 0 \\ 0 & 0 & 0 & b_0 \end{bmatrix}\right)\begin{bmatrix} \hat{x}_{1\sim 3} \\ \hat{x}_4 \end{bmatrix}, \end{aligned}$$

which produces the following set of closed-loop systems:

$$\begin{aligned} \dddot{z}_1 + a_{0\sim 2} z_1^{(0\sim 2)} &= 0, \\ \dot{z}_2 + b_0 z_2 &= 0, \end{aligned}$$

where a_0, a_1, a_2 and b_0 are a set of arbitrary numbers selected to make the above closed-loop systems stable.

Once $z_1^{(0\sim 2)}$ and z_2 are obtained, $x_{1\sim 4}$ can be obtained via transformations $x = P^{-1}\hat{x}$ and (2.93).

2.4.4 Alternative Solutions

Frequency solution

The treatment in the above of this section is done in the time-domain. As a matter of fact, a parallel treatment has also been presented in the frequency domain by Duan and Zhou (2022).

Consider another internal description of a linear system in the frequency domain, namely, the polynomial matrix description (PMD), which is in the form of

$$P(s)Z(s) = Q(s)U(s), \tag{2.94}$$

where $Z(s)$ is the Laplace transformation of $z(t) \in \mathbb{R}^q$, which is referred to as the generalized state vector, and $P(s) \in \mathbb{R}^{q\times q}(s)$ and $Q(s) \in \mathbb{R}^{q\times r}(s)$ are some polynomial matrices. In general, we have $q \geq r$. We first give the following definition.

Definition 2.1 *PMD (2.94) is said to be fully actuated if $q = r$ and $Q(s) = Q_0$ is a nonsingular constant matrix, and is said to be underactuated if either $q > r$ or $q = r$ but $Q(s)$ is not a nonsingular constant matrix.*

Therefore, a FAS in the frequency domain appears as

$$P(s)Z(s) = Q_0U(s), \ \det Q_0 \neq 0. \tag{2.95}$$

If the state feedback is applied on PMD (2.94), the closed-loop system is $(P(s) - Q(s)K(s))\,Z(s) = 0$, whose characteristic equation is given by

$$\gamma_{\mathrm{c}}(s) \triangleq \det(P(s) - Q(s)K(s)) = 0.$$

Once we get a fully actuated PMD in the form of (2.95), we can design the state feedback controller

$$U(s) = Q_0^{-1}\,(P(s) - P_{\mathrm{c}}(s))\,Z(s), \tag{2.96}$$

where $P_{\mathrm{c}}(s)$ is any polynomial matrix with desired roots. Obviously, applying controller (2.96) to fully actuated PMD (2.95), gives the following closed-loop system:

$$P_{\mathrm{c}}(s)Z(s) = 0.$$

Thus the the characteristic polynomial of the closed-loop system

$$\gamma_{\mathrm{c}}(s) = \det P_{\mathrm{c}}(s)$$

can be arbitrarily assigned. Eventually, for a fully actuated PMD, pole assignment turns to be extremely easy.

By taking the Laplace transformation of the state-space model (2.62) with zero initial condition, we obtain

$$(sI_n - A)\,X(s) = BU(s), \tag{2.97}$$

which is exactly in the form of PMD (2.95) with

$$(P(s), Q_0) = (sI_n - A, B)\,. \tag{2.98}$$

Similarly, by taking the Laplace transformation of the high-order FAS model (2.78) with zero initial condition, we have immediately the following interpretation of Theorem 2.2.

Corollary 2.1 *The linear system (2.62) is controllable if and only if there exists a nonsingular matrix $T \in \mathbb{R}^{n\times n}$ such that $(Z(s), U(s))$ with $z \in \mathbb{R}^r$ given by (2.79), satisfies the fully actuated PMD*

$$P(s)Z(s) = B_0U(s), \tag{2.99}$$

where $P(s) \in \mathbb{R}^{r\times r}(s)$ is some polynomial matrix, and $B_0 \in \mathbb{R}^{r\times r}$ is a square upper triangular matrix with diagonal elements all being 1.

In general, PMD (2.97) is underactuated since $Q(s) = B$ is not square in general. Parallelly, we are interested in the following problem.

Problem 2.1 *Let the underactuated PMD (2.97) be controllable. Find a linear transformation*

$$X(s) = R(s)Z(s),\ R(s) \in \mathbb{R}^{n\times r}(s), \tag{2.100}$$

under which the underactuated PMD (2.97) is equivalently transformed into a fully actuated PMD in the form of (2.99) with $\det B_0 \neq 0$.

The above problem has been completely solved by Duan and Zhou (2022). Specifically, for a controllable first-order linear state-space system model, by using the right coprime factorization (RCF) of its transfer function matrix, under the condition that the denominator matrix in the RCF is column reduced, it is equivalently transformed into a fully actuated PMD model, whose time-domain expression is just a high-order FAS model. This frequency version is a supplement to the time-domain version, and reveals a connection between the controllability of the first-order linear state-space system model and the full-actuation of its PMD model. Both continuous-time and discrete-time linear systems are considered. Some numerical examples are also worked out to illustrate the effectiveness of the proposed approaches.

Solution via Hessenberg forms

An LTI system in upper Hessenberg form appears as follows Duan (2001):

$$\dot{x} = H_A x + H_B u, \tag{2.101}$$

where $u \in \mathbb{R}^r$ is the control vector,

$$x = \begin{bmatrix} x_\mu^{\mathrm{T}} & x_{\mu-1}^{\mathrm{T}} & \cdots & x_1^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^n$$

is the state vector, with $x_i \in \mathbb{R}^{n_i}$, $i = 1, 2, \ldots, \mu$, being a set of vectors. Further,

$$H_A = \begin{bmatrix} H_{\mu,\mu} & H_{\mu,\mu-1} & \cdots & H_{\mu,2} & H_{\mu,1} \\ H_{\mu-1} & H_{\mu-1,\mu-1} & \cdots & H_{\mu-1,2} & H_{\mu-1,1} \\ 0 & H_{\mu-2} & \cdots & H_{\mu-2,2} & H_{\mu-2,1} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & H_1 & H_{1,1} \end{bmatrix} \in \mathbb{R}^{n\times n}, \tag{2.102}$$

$$H_B = \begin{bmatrix} H_\mu^{\mathrm{T}} & 0 & 0 & \cdots & 0 \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{n\times r}, \tag{2.103}$$

where $H_{i,j} \in \mathbb{R}^{n_i\times n_j}$, $i, j = 1, 2, \ldots, \mu$, are a set of matrices, with n_i, $i = 1, 2, \ldots, \mu$, being a set of integers satisfying

$$n_1 \leq n_2 \leq \cdots \leq n_\mu = \mathrm{rank}\,(H_B) = r, \tag{2.104}$$

$$n_1 + n_2 + \cdots + n_\mu = n, \tag{2.105}$$

and $H_i \in \mathbb{R}^{n_i\times n_{i+1}}$, $i = 1, 2, \ldots, \mu-1$, and $H_\mu \in \mathbb{R}^{n_\mu\times r}$ are full-row rank upper triangular matrices.

As is well known, any controllable LTI system (2.62) can be transformed by some orthogonal similarity transformation into a system Hessenberg form. It is also well

agreed that this transformation procedure is numerically reliable Duan (2001); Laub and Linnemann (1986). Therefore, in order to derive a numerically stable FAS realization of a controllable LTI system (2.62), we can carry out the following two steps:

Step 1 transform the controllable LTI system (2.62) into an upper Hessenberg form;

Step 2 obtain an equivalent FAS model for the transformed upper Hessenberg form in a direct and numerically reliable way.

It is well-known that the upper Hessenberg form is a canonical controllability form of controllable LTI systems. Of course, an equivalent FAS model of an LTI system in an upper Hessenberg form can be obtained by applying the above Theorem 2.2. However, for some large scale systems, this is not recommended due to numerical considerations. Fortunately, a very efficient numerically reliable FAS realization of an LTI system in upper Hessenberg form has been recently proposed by Zhang and Duan (2025), which gives a direct explicit solution.

2.5 Linear Time-varying FASs

The FAS model of a dynamical system is closely related to its certain controllability, which also corresponds to a certain controllability canonical form. For the LTV system, the so-called phase-variable canonical, also known as Frobenius canonical form, has attracted a great deal of attention in the control community (Mutoh and Nakada (2009); Lee and Choi (2005); Nguyen (1987, 1986); Mutoh and Hara (2011); Mutoh (2011)). As a pioneering work, Silverman has proved that a single-input LTV system can be converted into a phase-variable form if and only if it is uniformly controllable, which is almost parallel to that derived by Luenberger for LTI cases (Luenberger (1967)). However, unlike the single-input case, it is more difficult for multi-input LTV systems due to the lack of uniqueness. Specifically, the controllability indices of a multi-input LTV system may not be fixed, even though it is uniformly controllable. It results in that the structure of the canonical form is changing with time. This forces the designer to determine the best form from the several possibilities. Most directly, provided that the LTV system is uniformly controllable with fixed controllability indices, namely, lexicography-fixedly controllable, then the corresponding canonical form can be established successfully (Nguyen (1987, 1986); Chai et al. (1991); Valášek and Olgac (1995)). As for the non-lexicography-fixed system, the key idea is to construct an auxiliary system such that the augmented system is lexicography-fixedly controllable, then the treatment is the same as the lexicography-fixed case (Valášek and Olgaç (1999); Mutoh and Kimura (2011); Mutoh (2013)). It should be emphasized that we can only obtain the phase-variable canonical form of the augmented system rather than the original one.

Consider the LTV system

$$\dot{x} = A(t)x + B(t)u,\ t \geq t_0, \tag{2.106}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^r$ are the state and the input, respectively, t_0 is the initial time, and $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times r}$ are matrix functions.

For LTI systems, the system equivalence relation is simple. For LTV systems, system equivalence is defined as follows (Chai et al. (1991)).

Definition 2.2 *The system*

$$\dot{\bar{x}} = \bar{A}(t)\,\bar{x} + \bar{B}(t)\,u, \tag{2.107}$$

is said to be equivalent to system (2.106) in the time interval of interest $t \in [t_0, \infty)$ if there exists a transformation $\bar{x} = P(t)\,x$ with $P(t) \in \mathbb{R}^{n\times n}$ being nonsingular and continuously differentiable, such that

$$\begin{cases} \bar{A}(t) = \left(P(t)\,A(t) + \dot{P}(t)\right) P^{-1}(t) \\ \bar{B}(t) = P(t)\,B(t), \end{cases}$$

and $P(t)$, $P^{-1}(t)$ and $\dot{P}(t)$ are continuous and bounded matrices in the time interval of interest $t \in [t_0, \infty)$.

2.5.1 Preliminaries

For LTV systems, the issue of controllability is much more complicated than that of LTI systems. Let us first introduce the following well-known Silverman criterion for checking controllability of system (2.106) (see the Theorem 3 in Silverman and Meadows (1967)).

Lemma 2.1 *Let $A(t) \in \mathbb{R}^{n\times n}$ and $B(t) \in \mathbb{R}^{n\times r}$ be at least n-times continuously differentiable on the time interval of interest J. For $t_0 \in J$, if there exists $t_0 < t_1 \in J$ such that* $\operatorname{rank} Q(t_1) = n$*, where the controllability matrix $Q(t)$ is defined by*

$$Q(t) = \left[\begin{array}{cccc} Q_1(t) & Q_2(t) & \cdots & Q_n(t) \end{array}\right], \tag{2.108}$$

with

$$\begin{aligned} Q_1(t) &= B(t), \\ Q_{i+1}(t) &= A(t)\,Q_i(t) - \dot{Q}_i(t),\ i = 1, 2, \ldots, n-1, \end{aligned}$$

then system (2.106) is controllable on the interval J. Furthermore, if $\operatorname{rank} Q(t_1) = n$ *holds for any $t_1 \in [t_0, \infty)$, system (2.106) is said to be uniformly controllable on the interval J.*

In the following, we consider the uniformly controllable case only. Similar to the LTI case, the controllability matrix $Q(t)$ given by (2.108) can be reordered as

$$\bar{Q}(t) = \left[\begin{array}{cccc} \bar{Q}_1(t) & \bar{Q}_2(t) & \cdots & \bar{Q}_r(t) \end{array}\right], \tag{2.109}$$

$$\bar{Q}_i(t) = \left[\begin{array}{cccc} \bar{q}_1^i & \bar{q}_2^i & \cdots & \bar{q}_{\mu_i}^i \end{array}\right],\ i = 1, 2, \ldots, r, \tag{2.110}$$

where

$$\begin{aligned} \bar{q}_1^i &= b_i, \\ \bar{q}_{j+1}^i &= A(t)\,\bar{q}_j^i - \dot{\bar{q}}_j^i,\ j = 1, 2, \ldots, \mu_i - 1, \end{aligned}$$

with b_i being the i-th column of the input matrix $B(t)$ and μ_i being the controllability index corresponding to b_i. Note that different from the LTI systems, controllability indices μ_i, $i = 1, 2, \ldots, r$, are dependent on the time t, that is, they may not be fixed for all $t \geq t_0$. However, there must be a maximum finite number $\mu_{i\max}$ over $t \in [t_0, \infty)$, namely,

$$\mu_{i\max} = \max_{t \geq t_0} \mu_i(t). \tag{2.111}$$

Regarding controllability indices of the LTV system (2.106), we have the following definition.

Definition 2.3 *(Nguyen (1986)) If controllability indices μ_i, $i = 1, 2, \ldots, r$, of system (2.106) are fixed on the time interval $[t_0, \infty)$ and satisfy*

$$\sum_{i=1}^{r} \mu_i = n,$$

then system (2.106) is said to be lexicography-fixedly controllable on $[t_0, \infty)$.

Clearly, for the case of $\sum_{i=1}^{\eta} \mu_{i\max} \geq n$, system (2.106) is called non-lexicography-fixedly controllable on $[t_0, \infty)$. To establish the solvability conditions, the following assumptions on system (2.106) are made.

Assumption 2.5 *The LTV system (2.106) is lexicography-fixedly controllable on $t \in [t_0, \infty)$.*

Assumption 2.6 *For $\forall t \geq t_0$, all elements of $A(t)$ and $B(t)$ are at least n-times continuously differentiable.*

Problem 2.2 *Find a state transformation described by $z_k^{(0\sim\mu_k-1)}\Big|_{k=1\sim\eta} = P(t)x$, with $P(t)$ being nonsingular for $\forall t \geq t_0$, such that system (2.106) under Assumptions 2.5 and 2.6 is converted into a FAS in the form of*

$$z_k^{(\mu_k)}\Big|_{k=1\sim\eta} = L(t)\, z_k^{(0\sim\mu_k-1)}\Big|_{k=1\sim\eta} + B_z(t)\, u, \tag{2.112}$$

where $L(t) \in \mathbb{R}^{r\times n}$ and $B_z(t) \in \mathbb{R}^{r\times r}$ are two linear matrix functions, with $B_z(t)$ being nonsingular for all $t \geq t_0$.

Problem 2.3 *Design a controller based on the FAS model (2.112) such that the resulting closed-loop system of the original LTV system (2.106) is exponentially stable.*

2.5.2 Linear Time-varying FASs

Lexicography-fixed Case

Supposing that Assumption 2.5 holds, we immediately know that $\bar{Q}(t)$ is nonsingular on $t \in [t_0, \infty)$, then its inverse can be partitioned into

$$\bar{Q}^{-1}(t) = \left[\begin{array}{cccccccc} \bar{q}_{10} & \cdots & \bar{q}_{1,\mu_1-1} & \cdots & \bar{q}_{r0} & \cdots & \bar{q}_{r,\mu_r-1} \end{array}\right]^{\mathrm{T}}, \tag{2.113}$$

with $\bar{q}_{ij} \in \mathbb{R}^n$, $i = 1, 2, \ldots, r$, $j = 0, 1, \ldots, \mu_i - 1$. Denoting $\sigma_k = \sum_{i=1}^{k} \mu_k$, we take the σ_k-th row of (2.113) and construct the state transformation matrix as follows:

$$P(t) = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_r \end{bmatrix}, \; P_i = \begin{bmatrix} p_1^i & p_2^i & \cdots & p_{\mu_i}^i \end{bmatrix}^{\mathrm{T}}, \; i = 1, 2, \ldots, r, \tag{2.114}$$

with

$$p_1^i = \bar{q}_{i,\mu_i - 1}, \; i = 1, 2, \ldots, r, \tag{2.115}$$

$$p_{j+1}^i = p_j^i A(t) + \dot{p}_j^i, \; j = 1, 2, \ldots, \mu_i - 1. \tag{2.116}$$

Based on the above, we can obtain the following theorem for the lexicography-fixed case.

Theorem 2.3 *Let the LTV system (2.106) satisfy Assumptions 2.5 and 2.6. Under the state transformation*

$$z_k^{(0 \sim \mu_k - 1)}\Big|_{k=1 \sim \eta} = P(t)\, x,$$

with $P(t)$ given by (2.114), it can be equivalently converted into a FAS in the form of

$$z_k^{(\mu_k)}\Big|_{k=1 \sim \eta} = L(t)\, z_k^{(0 \sim \mu_k - 1)}\Big|_{k=1 \sim \eta} + B_z(t)\, u, \tag{2.117}$$

where $L(t) \in \mathbb{R}^{r \times n}$ and $B_z(t) \in \mathbb{R}^{r \times r}$ are two linear matrix functions, with $B_z(t)$ being nonsingular for all $t \geq t_0$.

For the proof of Theorem 2.3, please refer to the Appendix in detail.

Non-lexicographically-fixed Case

Compared to the lexicography-fixed case, the significant difference is that the reordered controllability matrix $\tilde{Q}(t)$ given by (2.109) becomes

$$\begin{aligned} \tilde{Q}(t) &= \begin{bmatrix} \tilde{Q}_1(t) & \tilde{Q}_2(t) & \cdots & \tilde{Q}_r(t) \end{bmatrix}, \\ \tilde{Q}_i(t) &= \begin{bmatrix} \tilde{q}_1^i & \tilde{q}_2^i & \cdots & \tilde{q}_{\mu_i \max}^i \end{bmatrix}, \; i = 1, 2, \ldots, r, \end{aligned} \tag{2.118}$$

where

$$\begin{aligned} \tilde{q}_1^i &= b_i, \; i = 1, 2, \ldots, r, \\ \tilde{q}_{j+1}^i &= A(t)\, \tilde{q}_j^i - \dot{\tilde{q}}_j^i, \; j = 1, 2, \ldots, \mu_{i\max} - 1, \end{aligned}$$

with $\mu_{i\max}$, $i = 1, 2, \ldots, r$, given by (2.111). Note that $\tilde{Q}(t)$ is clearly not square, making the transformation problem more difficult. However, following the idea in Valášek and Olgaç (1999); Mutoh and Kimura (2011), we could construct an auxiliary system such that the augmented system is lexicography-fixedly controllable.

Denote

$$n_e = \sum_{i=1}^{r} \mu_{i\max} - n.$$

Accordingly, we replace Assumption 2.5 with the following one.

Assumption 2.7 *The LTV system (2.106) is uniformly controllable but not lexicography-fixedly controllable on $t \in [t_0, \infty)$ and the maximum values $\mu_{i\max}$, $i = 1, 2, \ldots, r$, of all controllability indices are known.*

The following augmented system instead of (2.106) is considered

$$\dot{\tilde{x}} = \tilde{A}(t)\,\tilde{x} + \tilde{B}(t)\,u, \tag{2.119}$$

where

$$\tilde{A}(t) = \begin{bmatrix} A(t) & 0_{n\times n_e} \\ A_{e1}(t) & A_{e2}(t) \end{bmatrix},\ \tilde{B}(t) = \begin{bmatrix} B(t) \\ B_e(t) \end{bmatrix}.$$

with $\tilde{x} = \begin{bmatrix} x^{\mathrm{T}} & x_e^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{n+n_e}$ being the extended state vector, $A_{e1}(t) \in \mathbb{R}^{n_e\times n}$, $A_{e2}(t) \in \mathbb{R}^{n_e\times n_e}$ and $B_e(t) \in \mathbb{R}^{n_e\times r}$ being the auxiliary system matrices to be determined. Regarding the augmented system, Valasek and Olgac provide the following result (Valášek and Olgaç (1999)).

Lemma 2.2 *Consider the LTV system (2.106) under Assumptions 2.6 and 2.7. There exists an auxiliary system in the form of*

$$\dot{x}_e = A_{e1}(t)\,x + A_{e2}(t)\,x_e + B_e(t)\,u,$$

such that the augmented system (2.119) is lexicography-fixedly controllable over $t \in [t_0, \infty)$.

The specific expressions for $A_{e1}(t)$, $A_{e2}(t)$, and $B_e(t)$ can be found in Valášek and Olgaç (1999). Applying Lemma 2.2 and Theorem 2.3, it is not difficult to directly obtain the following corollary.

Corollary 2.2 *For the LTV system (2.106) satisfying Assumptions 2.6 and 2.7, there still exists a FAS model of the augmented system (2.119) in the form of*

$$z_k^{(\mu_k)}\Big|_{k=1\sim\eta} = L(t)\,z_k^{(0\sim\mu_k-1)}\Big|_{k=1\sim\eta} + B_z(t)\,u, \tag{2.120}$$

where $L(t) \in \mathbb{R}^{r\times(n+n_e)}$ and $B_z(t) \in \mathbb{R}^{r\times r}$ are two linear matrix functions, with $B_z(t)$ being nonsingular for all $t \geq t_0$, μ_k, $k = 1, 2, \ldots, \eta$, are a set of distinct integers, and $z_k \in \mathbb{R}^{r_k}$ with r_k satisfying

$$r_1 + r_2 + \cdots + r_\eta = r.$$

2.5.3 Controller Design

For the control of FAS (2.117), a controller can be readily designed as

$$u=-B_z^{-1}(t)\left(L(t)\,z_k^{(0\sim\mu_k-1)}\Big|_{k=1\sim\eta}+\begin{bmatrix}[A_1]_{0\sim\mu_1-1}\,z_1^{(0\sim\mu_1-1)}\\ [A_2]_{0\sim\mu_2-1}\,z_2^{(0\sim\mu_2-1)}\\ \vdots\\ [A_\eta]_{0\sim\mu_\eta-1}\,z_\eta^{(0\sim\mu_\eta-1)}\end{bmatrix}\right),\tag{2.121}$$

which produces the following set of constant linear closed-loop systems:

$$z_k^{(\mu_k)}+[A_k]_{0\sim\mu_k-1}\,z_k^{(0\sim\mu_k-1)}=0,\ k=1,2,\ldots,\eta,\tag{2.122}$$

where $[A_k]_{0\sim\mu_k-1}\in\mathbb{R}^{r_k\times\mu_k r_k}$, $k=1,2,\ldots,\eta$, are a series of arbitrarily given matrices. Particularly, the closed-loop eigenvalues can be arbitrarily assigned by properly selecting these matrices. Such a property of FASs totally reveals the meaning of controllability of control systems in the sense that a controller can be found such that the closed-loop systems are constant linear ones with arbitrarily desired eigenvalues.

The main result of this section is as below.

Theorem 2.4 *Let Assumptions 2.5 and 2.6 be satisfied, $[A_k]_{0\sim\mu_k-1}$, $k=1,2,\ldots,\eta$, be a series of given matrices, and the state transformation*

$$z_k^{(0\sim\mu_k-1)}\Big|_{k=1\sim\eta}=P(t)\,x\tag{2.123}$$

be defined as in (2.114). Controller (2.121) based on the FAS model (2.117) such that the eigenvalues of the closed-loop subsystems (2.122) are arbitrarily assigned and the state response of the original LTV system (2.106) is determined by

$$x(t)=P^{-1}(t)\exp(\Phi_0 t)\,Z_0,\tag{2.124}$$

with $Z_0=z_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}(0)$ and

$$\Phi_0=\operatorname{diag}\left(\Phi\left([A_i]_{0\sim\mu_i-1}\,z_i^{(0\sim\mu_i-1)}\right),\ i=1,2,\ldots,\eta\right).$$

Remark 2.5 According to Definition 2.2, system (2.106) under controller (2.121) is exponentially stable as long as translation (2.123) is in the sense of Lyapunov, that is, $P(t)$, $P^{-1}(t)$ and $\dot{P}(t)$ are continuous and bounded matrices. Then, we can readily conclude that controller (2.121) is effective and efficient, which eliminates the time-varying elements in the system matrix and arbitrarily assigns the eigenvalues of the closed-loop subsystems (2.122) via state feedback. Problem 2.3 is handled properly. On the other hand, $P(t)$ might not necessarily be a Lyapunov matrix. Theoretically, if $z_k^{(0\sim\mu_k-1)}\Big|_{k=1\sim\eta}$ converges to zero fast enough, the original state trajectories driven by controller (2.121) could still converge to zero under nonzero initial conditions. Additionally, designing a finite-time or prescribed-time stabilizing controller (see, e.g., Zhou (2020); Zhou and Shi (2021)) could be a possible way to resolve such an issue for the FAS model (2.117).

2.5.4 Example

Example 2.4 Let us consider the following example in Mutoh and Nakada (2009):

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ \cos t & \sin t & 1+\mathrm{e}^{-t} \\ 1 & \sin t & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 1+0.2\sin t \\ 0 & 1 \\ 1 & 0 \end{bmatrix} u,\ t \geq 0, \tag{2.125}$$

whose controllability matrix is given by

$$Q(t) = \begin{bmatrix} 0 & 0.2\sin t+1 & 0 & 1-0.5\cos t & \mathrm{e}^{-t}+1 & \alpha_3 \\ 0 & 1 & \mathrm{e}^{-t}+1 & \alpha_1 & \alpha_2 & \alpha_4 \\ 1 & 0 & 1 & 1.2\sin t+1 & \left(\mathrm{e}^{-t}+1\right)\sin t+1 & \alpha_5 \end{bmatrix}, \tag{2.126}$$

where

$$\begin{aligned} \alpha_1 &= \sin t + (0.2\sin t+1)\cos t, \\ \alpha_2 &= 2\mathrm{e}^{-t} + \left(\mathrm{e}^{-t}+1\right)\sin t + 1, \\ \alpha_3 &= 0.8\sin t + (0.2\sin t+1)\cos t, \\ \alpha_4 &= \mathrm{e}^{-t} + 0.2\cos t + 2.2\sin t + \sin t\cos t \\ &\quad -1.6\cos^2 t - 0.2\cos^3 t + 1.2\mathrm{e}^{-t}\sin t + 2.2, \\ \alpha_5 &= 1.2\sin t - 1.4\cos t + \sin t\,(\sin t + (0.2\sin t+1)\cos t) + 2. \end{aligned}$$

According to the controllability matrix (2.126), one can derive that system (2.125) is uniformly controllable with the fixed controllability indices being $\mu_1 = 2$ and $\mu_2 = 1$. Then, $Q(t)$ can be reordered as

$$\begin{aligned} \bar{Q}(t) &= \begin{bmatrix} b_1 & A(t)b_1 - \dot{b}_1 & b_2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0.2\sin t+1 \\ 0 & \mathrm{e}^{-t}+1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \end{aligned}$$

which is nonsingular for all $t \geq 0$. The inverse of $\bar{Q}(t)$ is given by

$$\bar{Q}^{-1}(t) = \frac{1}{\varepsilon}\begin{bmatrix} 5\mathrm{e}^t & -\mathrm{e}^t(\sin t+5) & \varepsilon \\ -5\mathrm{e}^t & \mathrm{e}^t(\sin t+5) & 0 \\ 5\left(\mathrm{e}^t+1\right) & 0 & 0 \end{bmatrix},$$

with $\varepsilon = \left(\mathrm{e}^t+1\right)(\sin t+5)$.

Based on Theorem 2.3, the following state transformation matrix is constructed:

$$P(t) = \frac{1}{\varepsilon}\begin{bmatrix} -5\mathrm{e}^t & \mathrm{e}^t(\sin t+5) & 0 \\ \frac{\beta_1}{\varepsilon} & \beta_2 & \varepsilon \\ 5\left(\mathrm{e}^t+1\right) & 0 & 0 \end{bmatrix}, \tag{2.127}$$

where

$$\begin{aligned}\beta_1 &= \mathrm{e}^t\left(5\sin 2t+31\cos t-5\sin t+31\mathrm{e}^t\cos t-\left(1+\mathrm{e}^t\right)\cos^3 t+5\mathrm{e}^t\sin 2t-25\right),\\ \beta_2 &= \frac{\mathrm{e}^t}{\mathrm{e}^t+1}\left(6\sin t-5\mathrm{e}^t+5\mathrm{e}^t\sin t+\sin^2 t+\mathrm{e}^t\sin^2 t\right).\end{aligned}$$

After confirming that $P(t)$ given by (2.127) is a Lyapunov matrix, we obtain the FAS model of system (2.125) as follows:

$$z_k^{(\mu_k)}\Big|_{k=1\sim 2} = L(t)\, z_k^{(0\sim\mu_k-1)}\Big|_{k=1\sim 2} + B_z(t)\,u,$$

where

$$L(t)=\begin{bmatrix} l_{10}^1 & l_{11}^1 & l_{20}^1 \\ l_{10}^2 & 0 & l_{20}^2 \end{bmatrix},\ B_z(t)=\begin{bmatrix} 1 & \frac{X_{12}}{5\varepsilon} \\ 0 & 1 \end{bmatrix},$$

with

$$\begin{aligned} l_{10}^2 &= \frac{5\left(\mathrm{e}^{-t}+1\right)}{\sin t+5},\\ l_{11}^1 &= \frac{1}{\varepsilon}\left(7\sin t+6\mathrm{e}^t\sin t+\left(1+\mathrm{e}^t\right)\sin^2 t+5\right),\\ l_{10}^1 &= -\frac{\mathrm{e}^{-t}}{\varepsilon^2}\hat{l}_{10}^1,\\ \hat{l}_{10}^1 &= \left(5+12\mathrm{e}^t\right)\cos 2t+41\mathrm{e}^{2t}-2.5\mathrm{e}^{3t}+13\mathrm{e}^t\\ &\quad+\left(0.25+0.5\mathrm{e}^t+0.25\mathrm{e}^{2t}\right)\sin 3t+0.5\mathrm{e}^t\cos 3t\\ &\quad-\left(25.75+71.5\mathrm{e}^t+65.75\mathrm{e}^{2t}+30\mathrm{e}^{3t}\right)\sin t\\ &\quad-\left(60.5\mathrm{e}^t+121\mathrm{e}^{2t}+60.5\mathrm{e}^{3t}\right)\cos t\\ &\quad-\left(10\mathrm{e}^t+20\mathrm{e}^{2t}+10\mathrm{e}^{3t}\right)\sin 2t+\left(9\mathrm{e}^{2t}+2.5\mathrm{e}^{3t}\right)\cos 2t\\ &\quad+\left(\mathrm{e}^{2t}+0.5\mathrm{e}^{3t}\right)\cos 3t-5,\\ l_{20}^1 &= \frac{1}{40\varepsilon^2}\hat{l}_{20}^1,\\ \hat{l}_{20}^1 &= 797\mathrm{e}^{2t}+1257\mathrm{e}^t+\left(1636+2982\mathrm{e}^t+516\mathrm{e}^{2t}\right)\sin t\\ &\quad+\left(2610\mathrm{e}^t+1380\mathrm{e}^{2t}\right)\cos t\\ &\quad+\left(6\mathrm{e}^t+28\mathrm{e}^{2t}-12\right)\sin 3t+\left(124\mathrm{e}^{2t}-260-336\mathrm{e}^t\right)\cos 2t\\ &\quad+\left(20\mathrm{e}^{2t}-10\mathrm{e}^t\right)\cos 3t+\mathrm{e}^{2t}\sin 4t-\left(\mathrm{e}^t+\mathrm{e}^{2t}\right)\cos 4t\\ &\quad+\left(300\mathrm{e}^t-22\mathrm{e}^{2t}\right)\sin 2t)+1260,\\ l_{20}^2 &= \frac{5-\cos t}{\sin t+5},\\ X_{12} &= \mathrm{e}^t\left(31\cos t+25\sin t+10\sin t\cos t-5\cos^2 t-\cos^3 t-20\right).\end{aligned}$$

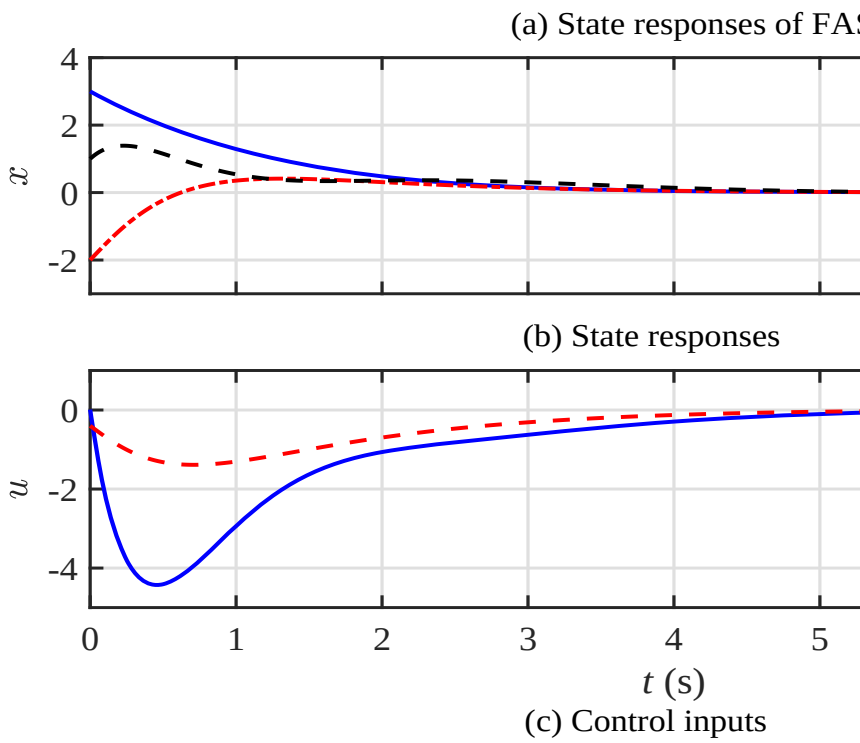


Figure 2.1: Simulation results for Example 2.4

Using Theorem 2.4, we design a controller in the form of

$$u = -B_z^{-1}(t)\left(L(t)\, z_k^{(0\sim\mu_k-1)}\Big|_{k=1\sim 2} + \begin{bmatrix} [A_1]_{0\sim\mu_1-1}\, z_1^{(0\sim\mu_1-1)} \\ [A_2]_{0\sim\mu_2-1}\, z_2^{(0\sim\mu_2-1)} \end{bmatrix} \right). \tag{2.128}$$

Choose the initial condition

$$x_0 = \begin{bmatrix} 3 & -2 & 1 \end{bmatrix}^{\mathrm{T}},$$

and the design parameters

$$[A_1]_{0\sim\mu_1-1} = \begin{bmatrix} 6 & 5 \end{bmatrix},\ [A_2]_{0\sim\mu_2-1} = 1.$$

The corresponding simulation results are shown in Fig. 2.1. It is seen from this figure that the strict equivalence in the Lyapunov sense is achieved. The established controller ensures exponential stability, which is in agreement with Theorem 2.4. Thanks to the framework of the FAS theories, the closed-loop system becomes an LTI system whose eigenvalues are -2, -3 and -1.

2.6 Feedback Linearizable Systems

2.6.1 Affine Systems

Consider the affine nonlinear system

$$\dot{x} = f(x) + g(x)\, u, \tag{2.129}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^r$ are the state vector and the input vector, respectively, and $f(x) \in \mathbb{R}^n$ and $g(x) \in \mathbb{R}^{n\times r}$ are the sufficiently smooth functions.

We are interested in the problem when system (2.129) can be equivalently turned into a controllable linear system by a state feedback. Concretely, the definition of feedback linearization can be given as follows (see the Definition 13.1 in Khalil (2002b)).

Definition 2.4 *The nonlinear system (2.129) is said to be feedback linearizable if there exists a differentiable homeomorphism $\hat{x} = T(x)$ defined on $\mathbb{R}^n$, and a control transformation*

$$u = \alpha(x) + \beta(x)\upsilon, \tag{2.130}$$

with $\alpha(x) \in \mathbb{R}^r$, and $\beta(x) \in \mathbb{R}^{r\times r}$ being nonsingular for all $x \in \mathbb{R}^n$, with which system (2.129) can be converted into the form of

$$\dot{\hat{x}} = A\hat{x} + B\upsilon, \tag{2.131}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$, with (A, B) controllable.

The following result indicates that feedback linearizable systems can be equivalently transformed into global FASs.

Theorem 2.5 *Let system (2.129) be feedback linearizable via (2.130). Then system (2.129) is equivalent to the following global FAS:*

$$z_k^{(\mu_k)}\Big|_{k=1\sim\eta} = f_z\left(z_k^{(0\sim\mu_k-1)}\Big|_{k=1\sim\eta}\right) + B_z\left(z_k^{(0\sim\mu_k-1)}\Big|_{k=1\sim\eta}\right)u, \tag{2.132}$$

where

$$B_z\left(z_k^{(0\sim\mu_k-1)}\Big|_{k=1\sim\eta}\right) = \hat{B}\beta^{-1}\left[T\left(P^{-1}\, z_k^{(0\sim\mu_k-1)}\Big|_{k=1\sim\eta}\right)\right], \tag{2.133}$$

μ_k, $k = 1, 2, \ldots, \eta$ $(1 \le \eta \le r)$, are a set of distinct integers, $z_k \in \mathbb{R}^{r_k}$, $k = 1, 2, \ldots, \eta$, are a set of state vectors, with r_k, $k = 1, 2, \ldots, \eta$, being a set of distinct integers satisfying (2.88), $f_z(\cdot) \in \mathbb{R}^r$ is a function, and $\hat{B} \in \mathbb{R}^{r\times r}$ is a nonsingular matrix.

Proof. Suppose that, under $\hat{x} = T(x)$ and (2.130), system (2.129) can be converted into the constant linear system (2.131) with (A, B) controllable. Then, by Theorem 2.2, we further know that there exists a state homeomorphism transformation of the following form:

$$z_k^{(0\sim\mu_k-1)}\Big|_{k=1\sim\eta} = P\hat{x}, \tag{2.134}$$

where P is an invertible matrix, under which the controllable LTI system (2.131) can be equivalently converted into a global FAS in the form of

$$z_k^{(\mu_k)}\Big|_{k=1\sim\eta} = \hat{L}\, z_k^{(0\sim\mu_k-1)}\Big|_{k=1\sim\eta} + \hat{B}\upsilon, \tag{2.135}$$

where $\hat{L} \in \mathbb{R}^{r\times n}$ and $\hat{B} \in \mathbb{R}^{r\times r}$ are two constant matrices, but with $\hat{B}$ being nonsingular.

Note

$$z_k^{(0\sim\mu_k-1)}\Big|_{k=1\sim\eta} = PT^{-1}(x), \tag{2.136}$$

we have

$$x = T\left(P^{-1}\, z_k^{(0\sim\mu_k-1)}\Big|_{k=1\sim\eta}\right),$$

and

$$\begin{aligned} v &= \beta^{-1}(x)\,u - \beta^{-1}(x)\,\alpha(x) \\ &= \beta^{-1}\left(T\left(P^{-1}\, z_k^{(0\sim\mu_k-1)}\Big|_{k=1\sim\eta}\right)\right) \\ &\quad \times\left[u-\alpha\left(T\left(P^{-1}\, z_k^{(0\sim\mu_k-1)}\Big|_{k=1\sim\eta}\right)\right)\right]. \end{aligned} \tag{2.137}$$

Therefore, system (2.129) can be equivalently converted by (2.130) and the variable transformation (2.136) into (2.132), with

$$\begin{aligned} & f_z\left(z_k^{(0\sim\mu_k-1)}\Big|_{k=1\sim\eta}\right) \\ = \;& \hat{L}\, z_k^{(0\sim\mu_k-1)}\Big|_{k=1\sim\eta} \\ & -\beta^{-1}\left(T\left(P^{-1}\, z_k^{(0\sim\mu_k-1)}\Big|_{k=1\sim\eta}\right)\right)\alpha\left(T\left(P^{-1}\, z_k^{(0\sim\mu_k-1)}\Big|_{k=1\sim\eta}\right)\right), \end{aligned}$$

and $B_z(\cdot)$ being given by (2.133).

Finally, noting the one-to-one property of $T(\cdot)$ and $\beta(\cdot)$, and the nonsingularity of P, we can immediately conclude that $B_z(x)$ is invertible for all $x \in \mathbb{R}^n$. Thus (2.132) is a global FAS. The proof is completed. ■

Please note that the above result is in fact a special case of the Corollary # in the second volume of the book Duan (2025g).

2.6.2 Nonaffine Systems

Consider the nonlinear system

$$\dot{x} = f(x,u), \tag{2.138}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^r$ are the state vector and input vector, respectively, $f(\cdot) \in \mathbb{R}^n$ is a sufficiently smooth function satisfying the initial condition $f(x_0,u_0) = \dot{x}_0$. Regarding when system (2.138) is locally equivalent to a controllable linear system, we introduce the following definition (Van Der Schaft (1984)).

Definition 2.5 *The nonlinear system (2.138) is called (globally) feedback linearizable via the differentiable feedback*

$$u = \xi(x,v),\ \xi(0,0) = 0, \tag{2.139}$$

satisfying

$$\operatorname{rank}\frac{\partial}{\partial v}\xi(x,v)\Big|_{v=0} = r,\ \forall x \in \mathbb{R}^n, \tag{2.140}$$

if there exists a sufficiently differentiable transformation of coordinates in the state space

$$x = T\left(\hat{x}\right),\ T\left(0\right) = 0,\ \text{rank}\ \frac{\partial}{\partial \hat{x}} T\left(\hat{x}\right)\Big|_{\hat{x}=0} = n, \tag{2.141}$$

such that system (2.138) is equivalently converted, under the control transformation (2.139) and the state transformation (2.141), into the form of

$$\dot{\hat{x}} = A\hat{x} + Bv, \tag{2.142}$$

where $A \in \mathbb{R}^{nxn}$ and $B \in \mathbb{R}^{n\times r}$ with (A, B) controllable.

The following result indicates that the feedback linearizable system can be equivalently transformed into a FAS, which is actually a corollary of the Theorem # in the second volume of the book Duan (2025g).

Theorem 2.6 *If system (2.138) is globally feedback linearizable, then it is equivalent to a globally nonaffine FAS in the form of*

$$z_k^{(\mu_k)}\Big|_{k=1\sim\eta} = f_z\left(z_k^{(0\sim\mu_k-1)}\Big|_{k=1\sim\eta}\right) + g_z\left(z_k^{(0\sim\mu_k-1)}\Big|_{k=1\sim\eta}, u\right), \tag{2.143}$$

where $f_z(\cdot) \in \mathbb{R}^r$ is a function, and $g_z(z,u)$ is one-to-one with respect to u for all $z \in \mathbb{R}^n$ and $u \in \mathbb{R}^r$.

Again, we mention that essence of the conversion from a feedback linearizable system to a FAS is finding a homeomorphism under which the system can be transformed into the FAS and vice versa. This actually can be seen from the above proof of Theorem 2.5, and also the proof of the above Theorem 2.6 given in Duan (2025g).

Remark 2.6 Some readers who are not yet familiar with the FAS approach might be mistaken that the FAS approach is nothing but feedback linearization. The above result can only tell that feedback linearization is a special case of the FAS approach, but not vice versa. On the one hand, the FAS approach also treats systems not in the state-space form, and on the other hand, it has been shown that the FAS approach can solve the stabilization of systems which have already been proven to be not feedback linearizable. Indeed, as we can find out in this book volume, the FAS approach really does tremendously more than the feedback linearization technique.

2.7 Notes and References

To finish this chapter, let us further comment on the two types of models for dynamical systems.

2.7.1 The State-space Models

State-space models are not the original forms used to describe dynamical control systems. The original models for physical dynamical systems come in the form of a set of basic equations, which can be differential, difference, or mixed. They are established based on the system's mechanism or certain physical laws, which frequently manifest as high-order differential equations. Renowned first-order state-space models are derived from these basic equations, primarily through the method of variable extension. State-space models are extreme ones: they feature the lowest order of differentials but encompass the maximum number of scalar equations.

State-space models prove efficient for linear systems, yet struggle considerably when dealing with nonlinearities, time-varying properties, time delays, system constraints, and issues of controllability of nonlinear systems. As early as 1982, it was described in a survey paper by Casti (1982) that "seeking a completely general theory of nonlinear systems is somewhat akin to the search for the Holy Grail: a relatively harmless activity full of many pleasant surprises and mild disappointments, but ultimately unrewarding". It was also claimed by Alberto Isidori at an open talk in Korea that "from the mid 1990s, the study of problems of feedback design for MIMO nonlinear systems came to a (almost complete) stall". He then asked: "Why was this the case? Was the study of these problems irrelevant? Or was it too difficult?" Eventually, these observations have become widely acknowledged in the last few decades, and many control theorists have turned to application topics.

Why is this the case? Isidori's question is certainly worth serious consideration.

As a matter of fact, the reason lies in that state-space models, despite their dominance in modern control theory for centuries, are not necessarily the best suited for control tasks. This reason can be difficult to accept, especially for those who frequently utilize state-space models. Indeed, the state-space models serve well in tackling problems directly tied to the states. This is evident from the application of linear state-space models, firstly proposed by Euler in early 1750 to solve high-order linear differential equations (giving the state responses), and of nonlinear models, used by Lyapunov in 1892 to analyze the limit behavior of states. They were not originally proposed to solve control problems. State-space models do have found success in controlling linear systems, but largely because linear systems are straightforward to handle.

If state-space models are not the ideal choice, could there be a better model for managing control tasks?

2.7.2 FAS Models

There is no doubt that a model which permits the control vector to be explicitly solved will provide great convenience for its controller design, and will simultaneously optimize control performance. Do such models exist? Indeed, they are prevalent but often overlooked, overshadowed by the existence of state-space models and our substantial reliance on the state-space approach.

In certain practical applications, there are physical systems referred to as FASs. Frequently modeled as Lagrangian systems (a type of second-order differential equations), these systems are prevalent in mechanical, electrical, and aerospace domains,

which allow the control vectors to be explicitly solved, and are therefore inherently more straightforward to be controlled. Often, time-invariant linear closed-loop systems can be derived in control system designs based on such models. In fact, many systems originally modeled by physical laws, such as Newton's Law or Lagrangian Equations, fall under this category of FASs.

However, the range of physical FASs is very limited. What about control of the numerous UASs and those which are neither FASs nor UASs? An alternative approach is originated by Duan (2020a, 2021a,g), which converts the model of a dynamical system into a mathematically generalized FAS, rather than a state-space one. Similar to a physical FAS, the controller can also be easily designed for this converted mathematical model, and a linear closed-loop system can also be achieved with an arbitrarily assignable eigenstructure.

The idea of proposing FAS models as general models for control of dynamical systems is firstly introduced in Duan (2020a,b) and Duan (2021a,b,g). Specifically, the single-order FAS models are firstly introduced in Duan (2020a, 2021a,b), while the multi-order FAS models are firstly introduced in Duan (2020b, 2021g). Later, the discrete-time FAS models are proposed in Duan (2022f), and those for continuous-time FASs with time delays and those for discrete-time FASs with time delays are proposed in Duan (2023b,c) and Duan (2022b,c), respectively. FAS models for both time-varying systems with time delays are proposed in Duan (2023b,c, 2022b,c, 2023d,e).

In this chapter, it is shown that both controllable LTI and LTV systems, as well as feedback linearizable systems, can be transformed equivalently into FASs. Let us emphasize again that FAS models are general models for control of dynamical systems. As we will see in the second volume Duan (2025g), the well-known techniques of deriving a at system and the theory for relative degrees of dynamical systems can be actually both viewed as ways to derive a FAS model of a nonlinear system.

The FAS approach, owing to its distinct advantages, has attracted widespread and rapid-increasing attention ever since its emergence. By now, numerous publications in the field have been produced (see the overview sections 1.3 and 1.4).

Indeed, the FAS approach has been instrumental in resolving many nonlinear control problems, encompassing areas such as robust control, adaptive control, disturbance rejection, optimal control, and tracking control. Several issues, insurmountable or inadequately addressed with the state-space approach, have been effectively and satisfactorily resolved by using the FAS approach. Frequently, the closed-loop systems designed through the FAS approach have been rendered linear and stable, or globally exponentially stable or sub-stable. Furthermore, this methodology has been extended to diverse domains including discrete-time systems, time-delay systems, stochastic systems and impulsive systems.

Chapter 3

Control of FASs

Different from the state-space models, the FAS models possess a huge advantage, i.e., their full-actuation feature allows us to arbitrarily replace the open-loop dynamics. In this chapter, this feature is firstly explained, and as an example the problem of finite-time stabilization is examined, which is, in fact, a problem of replacing the open-loop dynamics with one that permits the finite stability of the closed-loop system.

However, a much more common and preferred problem than finite-time stabilization is to replace the nonlinear open-loop dynamics with desired linear ones. Such a problem is studied in this chapter in some detail. Firstly, it is shown that, using such a feature of the FAS models, many nonlinear control problems can be converted into corresponding linear ones. Secondly, standard linearizing controllers for single- and multi-order FASs are presented. Furthermore, parametric solutions of both the decoupled design and the coupled design are provided. Finally, a general design scheme is provided and demonstrated with examples.

3.1 Arbitrary Assignment of Closed-loop Dynamics

A huge advantage of the FAS models of dynamical systems is that their full actuation property allows us to replace completely the open-loop dynamics with new dynamics in the closed-loop. In other words, by employing the full actuation feature, nonlinear terms in the open-loop system can be completely canceled, while desired terms, either linear or nonlinear, can be assigned to the closed-loop system. Such a fact is demonstrated in this section in some detail.

3.1.1 Single-order FASs

For single-order FASs, the dynamics is expressed as:

$$x^{(n)} = f(x^{(0\sim n-1)}, t) + B(x^{(0\sim n-1)}, t)u, \tag{3.1}$$

where $n \geq 1$ is an integer, $x \in \mathbb{R}^r$ is the system state vector, $u \in \mathbb{R}^r$ is the control input vector, $f(x^{(0\sim n-1)}, t) \in \mathbb{R}^r$ is a piece-wise continuous vector function, and

$B(x^{(0\sim n-1)},t)\in\mathbb{R}^{r\times r}$ is the control distribution matrix satisfying Assumption 2.1.

In general, the dynamical properties of the open-loop system (3.1) may not be satisfactory. Our goal is to design a controller such that the closed-loop system is given by

$$x^{(n)}=f_0^*(x^{(0\sim n-1)},t)+v, \tag{3.2}$$

where $f_0^*(x^{(0\sim n-1)},t)\in\mathbb{R}^r$ is a desired function in the sense that it guarantees certain desired properties of system (3.2). Due to Assumption 2.1, such a controller can be immediately designed as follows:

$$u=-B^{-1}(x^{(0\sim n-1)},t)\Big[f(x^{(0\sim n-1)})-f_0^*(x^{(0\sim n-1)},t)-v\Big]. \tag{3.3}$$

For the nonaffine single-order FAS:

$$x^{(n)}=f(x^{(0\sim n-1)},t)+g(x^{(0\sim n-1)},u,t), \tag{3.4}$$

where $g(x^{(0\sim n-1)},u,t)\in\mathbb{R}^r$ is a piece-wise continuous function satisfying Assumption 2.2. To achieve the same goal, based on Assumption 2.2, we can design the following controller:

$$\begin{cases} u=-g^{-1}(x^{(0\sim n-1)},u',t)\\ u'=f(x^{(0\sim n-1)})-f_0^*(x^{(0\sim n-1)},t)-v. \end{cases} \tag{3.5}$$

If $f_0^*(x^{(0\sim n-1)},t)$ is linear, that is,

$$f_0^*(x^{(0\sim n-1)},t)=-A_{0\sim n-1}x^{(0\sim n-1)},$$

controllers (3.3) and (3.5) reduce to

$$u=-B^{-1}(x^{(0\sim n-1)},t)\Big[f(x^{(0\sim n-1)})+A_{0\sim n-1}x^{(0\sim n-1)}-v\Big]. \tag{3.6}$$

and

$$\begin{cases} u=-g^{-1}(x^{(0\sim n-1)},u',t)\\ u'=f(x^{(0\sim n-1)})+A_{0\sim n-1}x^{(0\sim n-1)}-v, \end{cases} \tag{3.7}$$

respectively, and system (3.2) becomes

$$x^{(n)}+A_{0\sim n-1}x^{(0\sim n-1)}=v. \tag{3.8}$$

In this case, we call the corresponding controller (3.3), or (3.7), a linearizing one.

3.1.2 Multi-order FASs

Consider the multi-order FAS presented by

$$\begin{bmatrix} x_1^{(\mu_1)}\\ x_2^{(\mu_2)}\\ \vdots\\ x_\eta^{(\mu_\eta)} \end{bmatrix}=\begin{bmatrix} f_1\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta},t\right)\\ f_2\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta},t\right)\\ \vdots\\ f_\eta\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta},t\right) \end{bmatrix}+B\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta},t\right)u, \tag{3.9}$$

or, more compactly, by

$$x_k^{(\mu_k)}|_{k=1\sim\eta} = f\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, t\right) + B\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, t\right)u,$$

where the variables are as stated in Chapter 2, with $B(\cdot) \in \mathbb{R}^{r\times r}$ being a piece-wise continuous matrix function satisfying Assumption 2.3.

Similarly, by applying the FAS approach, we can design the controller

$$u = -B^{-1}(\cdot)\left[f(\cdot) - f^*\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, t\right) - v\right], \tag{3.10}$$

such that the system dynamics becomes

$$x_k^{(\mu_k)}|_{k=1\sim\eta} = f^*\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, t\right) + v, \tag{3.11}$$

where $f^*\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, t\right)$ is a desired nonlinear vector function.

For the nonaffine multi-order FAS

$$x_k^{(\mu_k)}|_{k=1\sim\eta} = f\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, t\right) + g\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, u, t\right), \tag{3.12}$$

where the variables are as stated in Chapter 2, with $g(\cdot) \in \mathbb{R}^r$ being a piece-wise continuous vector function satisfying Assumption 2.4, the controller can be designed as

$$\begin{cases} u = -g^{-1}\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, u', t\right) \\ u' = f(\cdot) - f^*\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, t\right) - v, \end{cases} \tag{3.13}$$

which produces the same system (3.11), i.e.,

$$x_k^{(\mu_k)}|_{k=1\sim\eta} = f^*\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, t\right) + v. \tag{3.14}$$

In the case of

$$f^*\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, t\right) = \begin{bmatrix} f_1^*\left(x_1^{(0\sim\mu_1-1)}, t\right) \\ f_2^*\left(x_2^{(0\sim\mu_2-1)}, t\right) \\ \vdots \\ f_\eta^*\left(x_\eta^{(0\sim\mu_\eta-1)}, t\right) \end{bmatrix},\ v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_\eta \end{bmatrix},$$

then controller (3.10) produces a set of independent desired systems:

$$x_k^{(\mu_k)} = f_k^*\left(x_k^{(0\sim\mu_k-1)}, t\right) + v_k,\ k = 1, 2, \ldots, \eta. \tag{3.15}$$

Through the FAS approach, the nonlinear components of a system can be exibly configured, rather than being limited to linear systems. This general concept provides greater design freedom, allowing the most suitable nonlinear function $f^*(\cdot)$ to be selected based on specific application scenarios and performance requirements.

Particularly, the case of $f_k^*\left(x_k^{(0\sim\mu_k-1)}, t\right)$, $k = 1, 2, \ldots, \eta$, being linear will be treated in the section after the next.

3.1.3 Finite-time Stabilization

As it was pointed out in the previous section, a huge advantage of the FAS approach is that the full-actuation property of the FAS model allows us to cancel any unwanted system dynamics, and assign any desired system dynamics to the closed-loop system directly. As is commonly done, often it is the desired linear dynamics that is assigned to the closed-loop. However, in certain cases, nonlinear dynamics can also be assigned to the closed-loop system in order to get a nonlinear closed-loop system with certain desired properties that a linear system is unable to provide.

In this section, we demonstrate this point of assigning proper nonlinear dynamics to the closed-loop systems instead of linear ones, to achieve finite-time stabilization. To make the presentation simpler, let us consider the control of the single-order FAS (3.1) only. Please note that the process can be easily generalized to multi-order FASs.

With our notations, the above closed-loop system in (3.2) is equivalent to the following state-space form:

$$\dot{x}^{(0\sim n-1)} = \Phi\left(0_{0\sim n-1}\right) x^{(0\sim n-1)} + B_c f^*\left(x^{(0\sim n-1)}, t\right), \tag{3.16}$$

where B_c is given by (3.30). When expended, the above system (3.16) can be equivalently converted into the following one:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \quad \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = f^*\left(x_{1\sim n}, t\right), \end{cases} \tag{3.17}$$

under the differential homeomorphism

$$x_i = x^{(i-1)},\ i = 1, 2, \ldots, n,$$

which permits the relation

$$f^*\left(x^{(0\sim n-1)}, t\right) = f^*\left(x_{1\sim n}, t\right).$$

In the rest of this subsection, let us concentrate on the specific problem of finite-time stabilization (see Song et al. (2017), and the review paper Song et al. (2023)). For convenience of presentation, let us introduce the following definition.

Definition 3.1 *A function $f\left(x_{1\sim n}, t\right) \in \mathbb{R}^r$, with $x_{1\sim n} \in \mathbb{R}^{nr}$, is called a function permitting finite-time stability if it makes system (3.17) finite-time stable in certain sense.*

It thus follows from the above that, in order to realize certain finite-time stabilization, it suffices only to design a controller in the form of (3.3) for the affine system (3.1), or in the form of (3.5) for the nonaffine system (3.4), with the term $f^*\left(x^{(0\sim n-1)}, t\right)$ being such that the corresponding $f^*\left(x_{1\sim n}, t\right)$ is a certain function permitting finite-time stability.

In the literature, finite-time stabilization, including prescribed-time stabilization, has been extensively studied, and a number of different types of typical functions permitting finite-time stability have been proposed (Song et al. (2023)). Examples in the case of $r = 1$ include the following three types:

1. Functions permitting finite-time stability (Bhat and Bernstein (2005)):

$$f^*(x) = -\sum_{i=1}^{n} k_i x_i^{\alpha_i},$$

where k_i, $i = 1, 2, \ldots, n$, make the polynomial $k_1 + k_2 s + \cdots + k_n s^{n-1} + s^n$ Hurwitz, and α_i, $i = 1, 2, \ldots, n$, are given by an iteration formula.

2. Functions permitting stability with prescribed time t_f (Ye and Song (2022)):

$$f^*(x) = -\sum_{i=1}^{n} k_i \gamma^{n+1-i} x_i, \ \gamma = \frac{b}{t_f - t},$$

where $t_f > 0$ denotes the desired convergence time, and b is a constant design parameter.

3. Functions permitting stability with prescribed time t_f (Zhou and Shi (2021)):

$$f^*(x) = -\frac{1}{2} B^{\mathrm{T}} P(\gamma) x(t), \ \gamma = \frac{b}{t_f - t},$$

where t_f and b are as stated above, $P(\gamma)$ is the unique positive definite solution to the following parametric Lyapunov equation:

$$\Phi(0_{0\sim n-1})^{\mathrm{T}} P(\gamma) + P(\gamma)\Phi(0_{0\sim n-1}) - P(\gamma) B_c B_c^{\mathrm{T}} P(\gamma) = -\gamma P(\gamma).$$

To sum up, controller (3.3) is a finite-time stabilizing controller for FAS (3.1) if $f^*(x_{1\sim n}, t)$ is a function permitting finite-time stability. In this case, the corresponding closed-loop system given by (3.2) is indeed finite-time stable.

To end this subsection, let us point out that, following the ideas in this subsection, we can consider control of the single-order FAS (3.1), or the multi-order FAS (3.9), with other properties by similarly defining functions permitting certain other properties.

3.2 Converting Nonlinear Problems into Linear Ones

Another huge advantage of the FAS models is that it helps convert many nonlinear control problems into linear ones. In this section, let us lay stress on this feature of the FAS approach with several dominant control problems except the basic stabilization problem which is to be examined in the next section.

3.2.1 Deterministic Systems

For convenience, let us only demonstrate this point with single-order FASs. Firstly, let us consider the deterministic single-order FAS (3.1).

Asymptotical tracking

For FAS (3.1), let us introduce the output equation

$$y = Cx^{(0\sim n-1)}, \tag{3.18}$$

and a given reference signal $r(t)$. The problem is to let $y(t)$ track $r(t)$.

Let us look upon controller (3.6) as a control transformation, with v being the new control vector. Under this control transformation, the tracking problem is turned into the following linear one:

$$\begin{cases} x^{(n)} + A_{0\sim n-1}x^{(0\sim n-1)} = v \\ y = Cx^{(0\sim n-1)} \\ \lim\limits_{t\to\infty}(y(t) - r(t)) = 0. \end{cases} \tag{3.19}$$

Now, the variety of solutions for tracking control in linear systems can be applied. The reference signal $r(t)$ may be a constant one, a time-varying one or one generated by a reference model. In the last case, the problem is in fact a model reference tracking problem (Duan (2022d)).

This problem is treated in Duan (2025g) in some detail, for the case that FAS (3.1) is a sub-FAS.

Optimal control

Nonlinear optimal control has always been a challenging problem in the literature. The well-known Pontryagin Maximum Principle gives general guidance, but is very difficult to be realized in applications except for some very simple systems.

Here let us consider a particular type of index for system (3.1) in the following form:

$$J = \int_0^\infty \left(\left\| x^{(0\sim n-1)} \right\|_Q^2 + \left\| x^{(n)} \right\|_R^2 \right) \mathrm{d}t, \tag{3.20}$$

where $Q \in \mathbb{R}^{nr\times nr}$ and $R \in \mathbb{R}^{r\times r}$ are symmetric weighting matrices, with R being positive definite.

The meaning of the above index (3.20) is to let the derivatives of the state be as small as possible. This has many practical implications. For example, in stationary control and turning elimination control of a satellite, such an index may be required. After all, when all the derivatives of the state of a system are minimized, the control effort is naturally small since the whole system is not acting violently, but working in an energy-saving mode.

Again, under the control transformation (3.6), the nonlinear optimal control problem is turned into the following well-known linear quadratic control problem:

$$\begin{cases} x^{(n)} + A_{0\sim n-1}x^{(0\sim n-1)} = v \\ \min \int_0^\infty \left(\left\| x^{(0\sim n-1)} \right\|_Q^2 + \left\| x^{(n)} \right\|_R^2 \right) \mathrm{d}t. \end{cases}$$

This problem is again treated in Duan (2025g) in some detail, for a multi-order sub-FAS.

Observer-based control

Consider FAS (3.1) with the output equation (3.18). Recall that, under the control transformation (3.6), the linear system (3.8) is derived. At this stage, we can design, instead of an observer for the original nonlinear system, an observer for the linear system (3.8). Therefore, an observer-based controller can then be given by

$$\begin{cases} u = -B^{-1}\left(f(\cdot) + A_{0\sim n-1}\hat{x}^{(0\sim n-1)} - v\right) \\ \dot{\hat{x}}^{(0\sim n-1)} = \Phi\left(A_{0\sim n-1}\right)\hat{x}^{(0\sim n-1)} + B_c v - L\left(y - C\hat{x}^{(0\sim n-1)}\right). \end{cases} \tag{3.21}$$

Obviously, in the case of

$$B\left(x^{(0\sim n-1)}, t\right) = B\left(y(t), y(t-\tau), t\right),$$

the above dynamical compensator (3.21) is realizable. When the function $f\left(x^{(0\sim n-1)}, t\right)$ is subject to some proper assumptions, e.g., the Lipschitz condition, asymptotical stability of the closed-loop system can be established (Zhao et al. (2023a)).

This problem is considered in Chapter 11 in some detail.

3.2.2 Uncertain Systems

In this subsection, let us turn to consider the following uncertain system:

$$x^{(n)} = f\left(x^{(0\sim n-1)}, t\right) + B\left(x^{(0\sim n-1)}, t\right)u + \mathrm{UCT}, \tag{3.22}$$

where UCT stands for an "uncertain term", which may be an external disturbance or a system dynamical uncertainty. With the control transformation (3.6), the system becomes the following linear uncertain one:

$$x^{(n)} + A_{0\sim n-1}x^{(0\sim n-1)} = v + \mathrm{UCT}. \tag{3.23}$$

When a proper controller with a property of reducing the effect of the uncertain term is designed using an existing or newly developed method for uncertain linear systems, as

$$v = v_d\left(x^{(0\sim n-1)}, [\mathrm{UCT}]\right), \tag{3.24}$$

where [UCT] represents some possible rough information of the uncertain term, a controller with the same property for the original nonlinear system (3.22) can be given as follows:

$$\begin{cases} u = -B^{-1}\left(f(\cdot) + A_{0\sim n-1}x^{(0\sim n-1)} - v\right) \\ v = v_d\left(x^{(0\sim n-1)}, [\mathrm{UCT}]\right). \end{cases}$$

Disturbance attenuation and decoupling

This corresponds to

$$\mathrm{UCT} = d(t), \tag{3.25}$$

where $d(t)$ is an external disturbance which can be of many types. In Duan (2021f), the cases of $d(t)$ being a direct external disturbance signal, or a signal generated by a dynamical model are effectively solved.

It is worth pointing out that, in the case that $d(t)$ is a stochastic white noise signal, we can possibly apply the well-known Kalman filter to the linear system (3.23) to solve the filtering problem in the original linear system, and based on this idea, certain optimal control problems for stochastic systems can then be further solved.

This problem is treated in Chapter 4 in some detail.

Robust control

This corresponds to

$$\text{UCT} = \Delta f(x^{(0\sim n-1)}, t),$$

where Δf is the dynamical uncertainty. A detailed treatment of this problem can be found in Duan (2021c). Again, this problem is treated in the first section of Chapter 5 in some detail.

Adaptive control

This corresponds to

$$\text{UCT} = H^{\text{T}}(x^{(0\sim n-1)}, t)\theta,$$

where θ is an unknown parameter vector, and $H^{\text{T}}(\cdot, t)$ is a matrix function of appropriate dimensions. This problem is treated in Duan (2021d). For time-varying parameter θ, ultimate boundedness of the closed-loop system is achieved. When the parameter vector is restricted to be constant, global asymptotical stability of the closed-loop system is obtained.

This problem is treated in the second section of Chapter 5 in some detail.

Robust adaptive control

This corresponds to

$$\text{UCT} = \Delta f(x^{(0\sim n-1)}, t) + H^{\text{T}}(x^{(0\sim n-1)}, t)\theta.$$

A detailed treatment of this problem can be found in Duan (2021e). Again, this problem is treated in the third section of Chapter 5 in some detail.

Remark 3.1 For convenience, in this section, we have only mentioned the case that the considered deterministic system (3.1) and the uncertain system (3.22) are global FASs. As a matter of fact, all the treatments can be extended to the case where the considered deterministic system (3.1) and the uncertain system (3.22) are sub-FASs. What is more, it is the latter case that really reveals the advantage and power of the FAS approach.

Remark 3.2 Let us mention that the problem of disturbance decoupling and attenuation is studied in Chapter 4, the problems of robust control, adaptive control and robust adaptive control are investigated in Chapter 5, and the problem of observer-based control of FASs is treated in Chapter 11. Please note that the problems of asymptotical tracking and sub-optimal control are proposed for sub-FASs and are discussed in the second volume of this book Duan (2025g).

3.3 Feedback Linearization

3.3.1 Single-order FASs

The following important fact about control of FAS (3.1) can be found in Duan (2020a, 2021a).

Theorem 3.1 *Let $A_{0\sim n-1} \in \mathbb{R}^{r\times nr}$ be a set of arbitrarily given matrices. Then the following controller*

$$\begin{cases} u = -g^{-1}\left(x^{(0\sim n-1)}, u', t\right) \\ u' = f\left(x^{(0\sim n-1)}, t\right) + A_{0\sim n-1}x^{(0\sim n-1)} - v, \end{cases} \tag{3.26}$$

for system (3.4), with $v \in \mathbb{R}^r$ being an external signal, produces the linear time-invariant (LTI) closed-loop system

$$x^{(n)} + A_{0\sim n-1}x^{(0\sim n-1)} = v. \tag{3.27}$$

Please note that, for the affine single-order FAS (3.1),

$$g\left(x^{(0\sim n-1)}, u, t\right) = B\left(x^{(0\sim n-1)}, t\right)u,$$

the above controller (3.26) reduces to

$$\begin{cases} u = -B^{-1}\left(x^{(0\sim n-1)}, t\right)\left(A_{0\sim n-1}x^{(0\sim n-1)} + u^*\right) \\ u^* = f\left(x^{(0\sim n-1)}, t\right) - v, \end{cases} \tag{3.28}$$

It should be noted that the linear system (3.27) is equivalent to the following state-space form:

$$\dot{x}^{(0\sim n-1)} = \Phi\left(A_{0\sim n-1}\right)x^{(0\sim n-1)} + B_c v, \tag{3.29}$$

where

$$B_c = \begin{bmatrix} 0 & \cdots & 0 & I_r \end{bmatrix}^{\mathrm{T}}. \tag{3.30}$$

Note that the coefficient matrix $A_{0\sim n-1}$ in (3.27) is at the designer's choice. The above result indicates that a controller for a FAS can be easily designed, which results in an LTI closed-loop system with an arbitrarily desired eigenstructure.

Furthermore, it can be easily observed that, when all A_i, $i = 0, 1, \ldots, n-1$, are chosen to be diagonal, r number of decoupled systems can be immediately obtained. This fact states that system decoupling, as a hard problem in the nonlinear control field, can also be easily solved with the FAS approach. Also, we remark that the above comments also apply to the case of sub-FAS control discussed in the second volume of the book Duan (2025g).

3.3.2 Decoupled Design for Multi-order FASs

As stated before, there are two designs for control of a multi-order FAS, namely, the coupled design and the decoupled design (Duan (2021g)). The former produces an overall linear system of full dimension, while the latter produces η number of decoupled linear systems with smaller dimensions. Here, let us only present the decoupled design as shown in the following result.

Theorem 3.2 *Let $A_{k,0\sim\mu_k-1}\in\mathbb{R}^{r_k\times\mu_k r_k}$, $k=1,2,\ldots,\eta$, be a set of arbitrarily given matrices. Then the following controller*

$$\begin{cases} u=-g^{-1}\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta},u',t\right)\\ u'=f\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta},t\right)+\begin{bmatrix} A_{1,0\sim\mu_1-1}x_1^{(0\sim\mu_1-1)}\\ A_{2,0\sim\mu_2-1}x_2^{(0\sim\mu_2-1)}\\ \vdots\\ A_{\eta,0\sim\mu_\eta-1}x_\eta^{(0\sim\mu_\eta-1)}\end{bmatrix}-v, \end{cases} \tag{3.31}$$

for the nonaffine FAS (3.12), with

$$v=\begin{bmatrix} v_1^{\mathrm{T}} & v_2^{\mathrm{T}} & \cdots & v_\eta^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}},\quad v_k\in\mathbb{R}^{r_k}, \tag{3.32}$$

being an external signal, produces the following set of decoupled LTI closed-loop systems:

$$x_k^{(\mu_k)}+A_{k,0\sim\mu_k-1}x_k^{(0\sim\mu_k-1)}=v_k,\quad k=1,2,\ldots,\eta. \tag{3.33}$$

In the case of

$$g\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta},u,t\right)=B\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta},t\right)u, \tag{3.34}$$

the nonaffine FAS (3.12) becomes affine FAS (3.9), and controller (3.31) reduces to

$$\begin{cases} u=-B^{-1}\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta},t\right)\left(\begin{bmatrix} A_{1,0\sim\mu_1-1}x_1^{(0\sim\mu_1-1)}\\ A_{2,0\sim\mu_2-1}x_2^{(0\sim\mu_2-1)}\\ \vdots\\ A_{\eta,0\sim\mu_\eta-1}x_\eta^{(0\sim\mu_\eta-1)}\end{bmatrix}+u^*\right)\\ u^*=f\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta},t\right)-v. \end{cases} \tag{3.35}$$

Please note that the state-space form of the closed-loop systems in (3.33) is

$$\dot{x}_k^{(0\sim\mu_k-1)}=\Phi\left(A_{k,0\sim\mu_k-1}\right)x_k^{(0\sim\mu_k-1)}+B_{ck}v_k,\quad k=1,2,\ldots,\eta, \tag{3.36}$$

where, again, by our notations,

$$\begin{aligned}\Phi\left(A_{k,0\sim\mu_k-1}\right) &= \begin{bmatrix} 0 & I_{r_k} & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \cdots & I_{r_k} \\ -A_{k,0} & -A_{k,1} & \cdots & -A_{k,\mu_k-1} \end{bmatrix}, \quad B_{ck} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_{r_k} \end{bmatrix} \\ k &= 1,2,\ldots,\eta.\end{aligned} \tag{3.37}$$

The stability and performance of the closed-loop system can be achieved by properly selecting the coefficients $A_{k,0\sim\mu_k-1}, \quad k=1,2,\ldots,\eta$, (Duan (1992a, 1993a), see also Subsection 3.4.1).

3.3.3 Coupled Design for Multi-order FASs

For control of a multi-order FAS, there are two designs, namely, the coupled design and the decoupled design (Duan (2021g)). The former produces an overall linear system of full dimension, while the latter produces η number of decoupled linear systems with smaller dimensions. Here, let us only present the more complicated coupled design as shown in the following result.

Theorem 3.3 *Let $A_E \in \mathbb{R}^{r\times x_0}$ be an arbitrarily given matrix. Then the following controller*

$$\begin{cases} u = -g^{-1}\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, u', t\right) \\ u' = f\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, t\right) - A_E x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta} - \upsilon, \end{cases} \tag{3.38}$$

for system (3.12), with $\upsilon \in \mathbb{R}^r$ being an external signal, produces the following closed-loop system:

$$x_k^{(\mu_k)}|_{k=1\sim\eta} = A_E x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta} + \upsilon. \tag{3.39}$$

When (3.34) holds, the above controller (3.38) reduces to the following one for the affine FAS (3.9):

$$\begin{cases} u = -B^{-1}\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, t\right)\left(-A_E x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta} + u^*\right) \\ u^* = f\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, t\right) - \upsilon. \end{cases} \tag{3.40}$$

3.4 Parametric Solutions of Controller Parameters

In this section, let us further present parametric solutions of the parameters involved in the feedback linearizing controllers. To save space, here only the multi-order case is treated.

3.4.1 Decoupled Design

In Subsection 3.3.2, the proposed decoupled controller (3.31) for nonaffine FAS (3.12), or (3.35) for affine FAS (3.9), involves a series of parameter matrices $A_{k,0\sim\mu_k-1}, \quad k=1,2,\ldots,\eta$. Here we give a parametric solution for these parameters.

Recall that the matrices $\Phi\left(A_{k,0\sim\mu_k-1}\right), \quad k=1,2,\ldots,\eta$, in the closed-loop system (3.36) are given by (3.37). Our purpose here is to let these matrices possess desired eigenstructure.

Directly applying Proposition 1.1, we immediately have the following result.

Theorem 3.4 *Let $k\in\{1,2,\cdots,\eta\}$. Then, for an arbitrary matrix $F_k\in\mathbb{R}^{\mu_k r_k\times\mu_k r_k}$, all the matrices $A_{k,0\sim\mu_k-1}$ and $V_k\in\mathbb{R}^{\mu_k r_k\times\mu_k r_k}$, satisfying $\det V_k\neq 0$ and*

$$\Phi_k\left(A_{k,0\sim\mu_k-1}\right)=V_kF_kV_k^{-1}, \tag{3.41}$$

are given by

$$A_{k,0\sim\mu_k-1}=-Z_kF_k^{\mu_k}V_k^{-1}\left(Z_k,F_k\right), \tag{3.42}$$

and

$$V_k=V_k\left(Z_k,F_k\right)=\begin{bmatrix} Z_k \\ Z_kF_k \\ \vdots \\ Z_kF_k^{\mu_k-1}\end{bmatrix}, \tag{3.43}$$

with $Z_k\in\mathbb{R}^{r_k\times\mu_k r_k}$ being a parameter matrix satisfying

$$\det V_k\left(Z_k,F_k\right)\neq 0. \tag{3.44}$$

The above theorem clearly establishes a general parametric solution of the controller parameters $A_{k,0\sim\mu_k-1}$, $k=1,2,\cdots,\eta$, which arbitrarily assign the desired eigenstructure of the closed-loop system (3.36). Further, let

$$\begin{aligned} F &= \operatorname{diag}\left(F_1,F_2,\cdots,F_\eta\right), \\ V &= \operatorname{diag}\left(V_1,V_2,\cdots,V_\eta\right), \end{aligned}$$

the whole closed-loop system can be obtained as

$$\dot{x}_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}=\left(VFV^{-1}\right)x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, \tag{3.45}$$

which has a clear arbitrarily assignable eigenstructure. Particularly, the closed-loop eigenvalues (or poles) are given by the eigenvalues of the matrix F, or equivalently, given by the eigenvalues of F_k, $k=1,2,\ldots,\eta$. The design degrees of freedom (DOFs) provided in the whole design are composed of two parts, the first part lies in the choice of the matrix F, and the second part is given by $Z_k\in\mathbb{R}^{r_k\times\mu_k r_k}$, $k=1,2,\ldots,\eta$, which accounts

$$d_{\text{freedom}}^{\text{d}}=\mu_1r_1^2+\mu_2r_2^2+\cdots+\mu_\eta r_\eta^2. \tag{3.46}$$

These DOFs can be properly utilized to achieve additional system performance (see, e.g., Duan (1992a, 1993a); Duan et al. (2002); Duan (2024a,b, 2025a)).

3.4.2 Coupled Design

In Subsection 3.3.3, the proposed coupled controller (3.38) for nonaffine FAS (3.12), or (3.40) for affine FAS (3.9), involves a parameter matrix A_E. Here we give a parametric solution for this parameter matrix.

To realize this, let us equivalently rewrite the coupled controller (3.38) for nonaffine FAS (3.12), and controller (3.40) for affine FAS (3.9), respectively as

$$\begin{cases} u = -g^{-1}\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, u^*, t\right) \\ u^* = f\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, t\right) - u' \\ u' = A_E x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta} + v, \end{cases} \tag{3.47}$$

and

$$\begin{cases} u = -B^{-1}\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, t\right)\left(f\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, t\right) - u'\right) \\ u' = A_E x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta} + v. \end{cases} \tag{3.48}$$

Applying the above controller (3.47) to the nonaffine FAS (3.12), and controller (3.48) and the affine FAS (3.9), respectively, yields, firstly, the linear system

$$x_k^{(\mu_k)}|_{k=1\sim\eta} = u', \tag{3.49}$$

which can be decomposed as

$$x_k^{(\mu_k)} = u'_k,\ k = 1, 2, \ldots, \eta, \tag{3.50}$$

where

$$u' = \begin{bmatrix} u'_1 \\ u'_2 \\ \vdots \\ u'_\eta \end{bmatrix}, u'_k \in \mathbb{R}^{r_k}.$$

Next, applying the linear feedback controller

$$u' = A_E x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta} + v \tag{3.51}$$

to system (3.49), gives the same closed-loop system as in (3.39).

Clearly, system (3.50) can be written into the state-space form:

$$\dot{x}_k^{(0\sim\mu_k-1)} = \Phi\left(0_{0\sim\mu_k-1}\right) x_k^{(0\sim\mu_k-1)} + B_{kc} u'_k,\ k = 1, 2, \ldots, \eta, \tag{3.52}$$

where

$$\Phi\left(0_{0\sim\mu_k-1}\right) = \begin{bmatrix} 0 & I_{r_k} & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \cdots & I_{r_k} \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad k = 1, 2, \ldots, \eta.$$

Define

$$\begin{aligned} A &= \operatorname{diag}\left(\Phi\left(0_{0\sim\mu_k-1}\right),\ k=1,2,\ldots,\eta\right), \\ B &= \operatorname{diag}\left(B_{kc},\ k=1,2,\ldots,\eta\right), \end{aligned}$$

then the set of systems in (3.52) can be more compactly written as

$$\dot{x}_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta} = Ax_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta} + Bu'. \tag{3.53}$$

Equivalently, we can now apply controller (3.51) to the above linear system (3.53). Based on Theorem 1.1, a design procedure can be immediately provided as follows.

Step 1 Choose

$$N_k(s) = \begin{bmatrix} I_{r_k} \\ I_{r_k}s \\ \vdots \\ I_{r_k}s^{\mu_k-1} \end{bmatrix}, \quad D_k(s) = I_{r_k}s^{\mu_k}. \tag{3.54}$$

It is clear that $N_k(s)$ and $D_k(s)$ are right coprime, and satisfy the right coprime factorization (RCF)

$$\begin{aligned} \left[sI_{\mu_k r_k} - \Phi_k\left(0_{0\sim\mu_k-1}\right)\right]^{-1} B_{kc} &= N_k(s)\, D_k^{-1}(s), \\ k &= 1,2,\ldots,\eta. \end{aligned}$$

Step 2 Define

$$\begin{cases} N(s) = \operatorname{blockdiag}\left(N_k(s),\quad k=1,2,\ldots,\eta\right) \\ D(s) = \operatorname{blockdiag}\left(D_k(s),\quad k=1,2,\ldots,\eta\right). \end{cases} \tag{3.55}$$

It can be verified that $N(s)$ and $D(s)$ are right coprime and satisfy RCF (1.14).

Step 3 Solve, according to Theorem 1.1, the general parametric expression of the parameter matrix $A_E \in \mathbb{R}^{r\times\varkappa_0}$ by

$$\begin{cases} A_E = WV^{-1} \\ V = N_0Z + N_1ZF + \cdots + N_\omega ZF^\omega \\ W = D_0Z + D_1ZF + \cdots + D_\omega ZF^\omega, \end{cases} \tag{3.56}$$

where the parameter matrix $Z \in \mathbb{R}^{r\times\varkappa_0}$ needs to satisfy a constraint in the form of (1.17).

Despite the freedom existing in the choice of the matrix F, the DOFs existing in the above design are represented by the parameter matrix $Z \in \mathbb{R}^{r\times\varkappa_0}$. In view of the dimension of the parameter matrix Z, the DOFs provided by the coupled design are

$$d_{\text{freedom}}^{\text{c}} = \sum_{k=1}^{\eta} r_k \sum_{i=1}^{\eta} r_i\mu_i.$$

Remark 3.3 The decoupled control design is obviously very simple, while the advantage of the coupled design is that it provides much more DOFs. This can be clearly seen from

$$\begin{aligned} d^{\mathrm{c}}_{\mathrm{freedom}} - d^{\mathrm{d}}_{\mathrm{freedom}} &= \sum_{k=1}^{\eta}\left(\sum_{i=1}^{\eta} r_k r_i \mu_i - r_k^2 \mu_k\right) \\ &= \sum_{k=1}^{\eta}\sum_{\substack{i=1 \\ i\neq k}}^{\eta} r_k r_i \mu_i . \end{aligned}$$

When properly utilized, these extra DOFs can make the system perform much better.

3.4.3 Elementary Problem

It is easy to see from the above that the control gain $A_{k,0\sim\mu_k-1}$ can be designed separately for each subsystem. Therefore, control of FASs is finally reduced to a problem of finding a parameter matrix $A_{0\sim n-1}$ such that the system

$$x^{(n)} + A_{0\sim n-1} x^{(0\sim n-1)} = 0 \tag{3.57}$$

is asymptotically stable.

To solve this problem, we rewrite system (3.57) into the state-space form

$$\dot{x}^{(0\sim n-1)} = \Phi\left(A_{0\sim n-1}\right) x^{(0\sim n-1)}. \tag{3.58}$$

Thus we only need to find a proper parameter matrix $A_{0\sim n-1}$ such that $\Phi\left(A_{0\sim n-1}\right)$ is Hurwitz. Since a matrix is Hurwitz if and only if it is similar to a Hurwitz matrix of the same size, and

$$\Phi\left(A_{0\sim n-1}\right) = \Phi\left(0_{0\sim n-1}\right) - B_c A_{0\sim n-1}, \tag{3.59}$$

the problem is finally transformed into the following one.

Problem 3.1 *(The elementary problem) For an arbitrarily chosen Hurwitz matrix $F \in \mathbb{R}^{nr\times nr}$, find the parameter matrix $A_{0\sim n-1}$ such that the following equation holds:*

$$\Phi\left(0_{0\sim n-1}\right) - B_c A_{0\sim n-1} = V F V^{-1}, \tag{3.60}$$

where $V \in \mathbb{R}^{nr\times nr}$ is some nonsingular matrix.

It is noted that there is always a solution (see Proposition 1.1) to the above elementary problem, because it is well-known that $\left(\Phi\left(0_{0\sim n-1}\right), B_c\right)$ is controllable. A thorough study of this problem is presented in Duan et al. (2024), where a complete and explicit parametric solution for this general problem is proposed. Besides, complete and explicit parametric solutions for the most commonly used special cases, determined only by variation of the pair of integers n and r, are also proposed. The set of these solutions can be regarded as a solution database in applications.

Remark 3.4 With the above preparations, it is easily understood that control of a continuous-time system, whether linear or nonlinear, whether time-varying or time-invariant, whether possessing time delays or not, is ultimately reduced to the solution to the elementary problem, i.e., Problem 3.1. For the case of discrete-time systems (see Duan (2022f)), a similar conclusion can be obtained, in which the matrix F needs to be Schur instead of Hurwitz.

Remark 3.5 It is often seen that one class of problems is converted into another one in scientific research, but the above particular conversion is special in two aspects: 1) the scope of the problems that can be converted is astonishingly wide, as we have seen; 2) the finally converted problem turns out to be not a class, but a single one, namely, the eigenstructure assignment of a specific pair of matrices $(\Phi(0_{0\sim n-1}), B_c)$, which is determined by only two parameters n and r.

3.5 Illustrative Examples

Before giving the examples, let us summarize a general procedure for the global FAS approach.

3.5.1 Basic Procedure for Global FAS Approach

Having seen the advantages of the FAS models in stabilization in the above subsections, we can now give a basic procedure for the FAS approach (see Fig. 3.1).

Suppose that we have got a so-called original system whose model is derived by certain physical laws. To realize the stabilization of this system, we carry out the following steps:

Step 1 find a homeomorphism φ under which the original physical system is converted into a global FAS;

Step 2 design the standard controller in the form of (3.26) or (3.31) for the obtained FAS, and obtain a linear closed-loop system;

Step 3 based on the controller obtained in Step 2 and the homeomorphism φ^{-1}, obtain the controller for the original system;

Step 4 for the linear closed-loop system obtained in Step 2, obtain its analytical expression for its response; and

Step 5 based on the response obtained in Step 4 and the homeomorphism φ^{-1}, obtain and analyze the response of the closed-loop system corresponding to the original system.

Regarding the above procedure, we have the following statements.

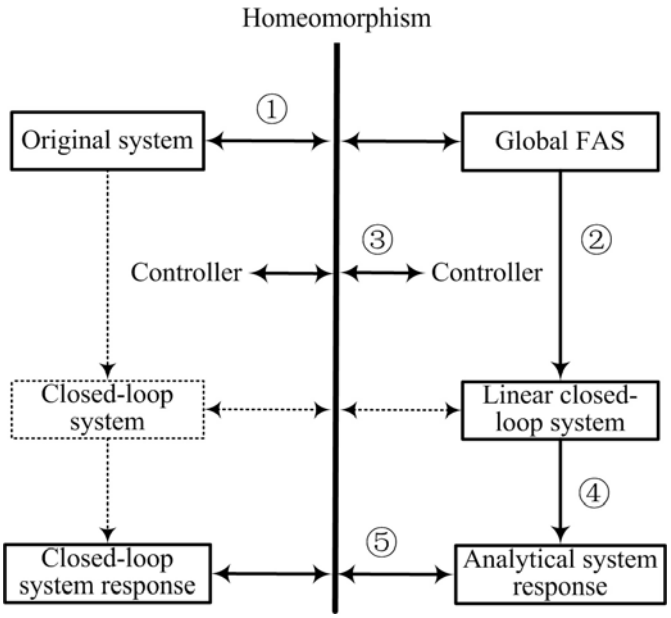


Figure 3.1: General procedure for global FAS approach

- The original physical system is not necessarily in the state-space form, but may be in any form as originally modeled by physical laws. It may be as complicated as possessing time-varying features and time delays. However, this basic procedure is not directly applicable to those original systems with dynamical uncertainties or unknown parameters. To cope with systems with dynamical uncertainties, unknown parameters, disturbances and reference signals, the procedure needs to be modified in certain ways.

- In certain complicated applications, the above procedure may be applied repeatedly to some subsystems, but not the whole original physical system (See Chapter 12).

- Assured by theory, the closed-loop response obtained in Step 4 always converges to zero exponentially. However, this may not be the case for the closed-loop response corresponding to the original system. It also depends on the homeomorphism φ^{-1}. Usually, convergence to zero is guaranteed, but may not be exponential.

An illustration of the above procedure is given in Fig. 3.1. The dotted line with a single arrow represents a route which is hard to complete, and the dotted line with two arrows represents a route which is unnecessary to be carried out.

3.5.2 Example 1: A Single-order FAS

Let us consider the following system (Isidori (1995), p.358):

$$\begin{cases} \dot{x}_1 = x_1 + x_1 x_4 + x_3 u_1 + u_2 \\ \dot{x}_2 = x_2 \mathrm{e}^{x_3} + u_1 \\ \dot{x}_3 = x_2 + x_3^2 \\ \dot{x}_4 = x_1 + x_2 - x_4 + x_1 x_4 + (1 + x_3) u_1 + u_2, \end{cases} \tag{3.61}$$

where x_i, $i = 1, 2, 3, 4$, are the states, and u_i, $i = 1, 2$, are the inputs.

An equivalent state-space form

Write system (3.61) into the form of

$$\dot{x}_{1\sim 4} = f(x_{1\sim 4}) + B(x_{1\sim 4}) u_{1\sim 2},$$

with

$$f(x_{1\sim 4}) = \begin{bmatrix} x_1 + x_1 x_4 \\ x_2 \mathrm{e}^{x_3} \\ x_2 + x_3^2 \\ x_1 + x_2 - x_4 + x_1 x_4 \end{bmatrix}, \; B(x_{1\sim 4}) = \begin{bmatrix} x_3 & 1 \\ 1 & 0 \\ 0 & 0 \\ 1 + x_3 & 1 \end{bmatrix}.$$

Let

$$Q = \begin{bmatrix} -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \tag{3.62}$$

then it can be easily verified that

$$QB(x_{1\sim 4}) = \begin{bmatrix} 0_{2\times 2} \\ G(x_3) \end{bmatrix},$$

where

$$G(x_3) = \begin{bmatrix} 1 & 0 \\ x_3 & 1 \end{bmatrix}.$$

From the transformation

$$y_{1\sim 4} = Q x_{1\sim 4}, \tag{3.63}$$

that is,

$$\begin{cases} y_1 = -x_1 - x_2 + x_4 \\ y_2 = x_3 \\ y_3 = x_2 \\ y_4 = x_1, \end{cases}$$

we can obtain

$$\begin{aligned}
\dot{y}_1 &= -\dot{x}_1 - \dot{x}_2 + \dot{x}_4 \\
&= -x_1 - x_1x_4 - x_3u_1 - u_2 - x_2e^{x_3} - u_1 \\
&\quad + x_1 + x_2 - x_4 + x_1x_4 + (1+x_3)u_1 + u_2 \\
&= x_2 - x_4 - x_2e^{x_3} \\
&= -y_1 - y_4 - y_3e^{y_2}, \\
\dot{y}_2 &= \dot{x}_3 \\
&= x_2 + x_3^2 \\
&= y_3 + y_2^2, \\
\dot{y}_3 &= \dot{x}_2 \\
&= x_2e^{x_3} + u_1 \\
&= y_3e^{y_2} + u_1, \\
\dot{y}_4 &= \dot{x}_1 \\
&= x_1 + x_1x_4 + x_3u_1 + u_2 \\
&= y_4 + y_4(y_1 + y_3 + y_4) + y_2u_1 + u_2.
\end{aligned}$$

From these, system (3.61) can be equivalently written into the following two subsystems:

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} -y_1 - y_4 - y_3e^{y_2} \\ y_2^2 + y_3 \end{bmatrix}, \tag{3.64}$$

$$\begin{bmatrix} \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} = \begin{bmatrix} y_3e^{y_2} \\ y_4 + y_4^2 + y_1y_4 + y_3y_4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ y_2 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{3.65}$$

The FAS model

From (3.64), we can obtain

$$\begin{aligned}
y_3 &= \dot{y}_2 - y_2^2, \\
y_4 &= -y_1 - \dot{y}_1 - y_3e^{y_2} \\
&= -y_1 - \dot{y}_1 - \left(\dot{y}_2 - y_2^2\right)e^{y_2}.
\end{aligned}$$

Based on these relations, let us introduce the following differential homeomorphism:

$$\begin{cases} z_1 = y_1 \\ z_2 = y_2 \\ \dot{z}_1 = -y_1 - y_4 - y_3e^{y_2} \\ \dot{z}_2 = y_3 + y_2^2, \end{cases} \tag{3.66}$$

whose inverse is

$$\begin{cases} y_1 = z_1 \\ y_2 = z_2 \\ y_3 = \dot{z}_2 - z_2^2 \\ y_4 = -z_1 - \dot{z}_1 - \left(\dot{z}_2 - z_2^2\right)e^{z_2}. \end{cases} \tag{3.67}$$

Taking differentials on system (3.64), gives

$$\begin{aligned}\begin{bmatrix}\ddot{y}_1\\ \ddot{y}_2\end{bmatrix} &= \begin{bmatrix}-\dot{y}_1-\dot{y}_4-\dot{y}_3e^{y_2}-\dot{y}_2y_3e^{y_2}\\ 2y_2\dot{y}_2+\dot{y}_3\end{bmatrix}\\ &= \begin{bmatrix}-\dot{y}_1-\dot{y}_2y_3e^{y_2}\\ 2y_2\dot{y}_2\end{bmatrix}+\begin{bmatrix}-e^{y_2} & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\dot{y}_3\\ \dot{y}_4\end{bmatrix}.\end{aligned} \tag{3.68}$$

Further, substituting (3.65) into the above, yields

$$\begin{aligned}\begin{bmatrix}\ddot{y}_1\\ \ddot{y}_2\end{bmatrix} &= \begin{bmatrix}-\dot{y}_1-\dot{y}_2y_3e^{y_2}-y_4-y_4^2-y_1y_4-y_3y_4-y_3e^{2y_2}\\ y_3e^{y_2}+2y_2\dot{y}_2\end{bmatrix}\\ &\quad+\begin{bmatrix}-y_2-e^{y_2} & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}.\end{aligned} \tag{3.69}$$

Next, using the above (3.69), and transformations (3.66) and (3.67), we can obtain

$$\begin{aligned}\ddot{z}_1 &= \ddot{y}_1\\ &= -\dot{y}_1-\dot{y}_2y_3e^{y_2}-y_3e^{2y_2}\\ &\quad -(1+y_1+y_3+y_4)y_4-y_2u_1-e^{y_2}u_1-u_2\\ &= -\dot{z}_1-\dot{z}_2(\dot{z}_2-z_2^2)e^{z_2}-(\dot{z}_2-z_2^2)e^{2z_2}-z_2u_1-e^{z_2}u_1-u_2\\ &\quad +(1+z_1+\dot{z}_2-z_2^2-z_1-\dot{z}_1-(\dot{z}_2-z_2^2)e^{z_2})\\ &\quad \times(z_1+\dot{z}_1+(\dot{z}_2-z_2^2)e^{z_2})\\ &= -\dot{z}_1-(\dot{z}_2-z_2^2)(\dot{z}_2e^{z_2}+e^{2z_2})-z_2u_1-e^{z_2}u_1-u_2\\ &\quad +(1+\dot{z}_2-z_2^2-\dot{z}_1-(\dot{z}_2-z_2^2)e^{z_2})(z_1+\dot{z}_1+(\dot{z}_2-z_2^2)e^{z_2}),\\ \ddot{z}_2 &= \ddot{y}_2\\ &= 2y_2\dot{y}_2+y_3e^{y_2}+u_1\\ &= 2z_2\dot{z}_2+(\dot{z}_2-z_2^2)e^{z_2}+u_1,\end{aligned}$$

which gives the following FAS model for the system:

$$\begin{bmatrix}\ddot{z}_1\\ \ddot{z}_2\end{bmatrix} = h\left(z_{1\sim 2}^{(0\sim 1)}\right)+L(z_2)u, \tag{3.70}$$

where

$$L(z_2) = \begin{bmatrix}-z_2-e^{z_2} & -1\\ 1 & 0\end{bmatrix}, \tag{3.71}$$

$$h\left(z_{1\sim 2}^{(0\sim 1)}\right) = \begin{bmatrix}-\dot{z}_1+\left(1-\zeta+\dot{z}_2-z_2^2\right)(z_1+\zeta)-\eta\\ \left(\dot{z}_2-z_2^2\right)e^{z_2}+2z_2\dot{z}_2\end{bmatrix}, \tag{3.72}$$

with

$$\zeta = \dot{z}_1+\left(\dot{z}_2-z_2^2\right)e^{z_2}, \tag{3.73}$$

$$\eta = \left(\dot{z}_2-z_2^2\right)\left(e^{2z_2}+\dot{z}_2e^{z_2}\right). \tag{3.74}$$

Decoupled design

It follows from previous Theorem 3.2 about decoupled design that the controller for the system is then given by

$$u = -L^{-1}(z_2)\left(h\left(z_{1\sim 2}^{(0\sim 1)}\right) + \begin{bmatrix} a_1\dot{z}_1 + a_0 z_1 \\ b_1\dot{z}_2 + b_0 z_2 \end{bmatrix}\right), \tag{3.75}$$

where a_i and b_i, $i = 0, 1$, are all positive real scalars. By using transformation (3.66), this controller can be immediately converted into a feedback form of variables $y_{1\sim 4}$, and into a feedback form of the original states $x_{1\sim 4}$ by further using transformation $x = Q^{-1}y$.

Obviously, applying controller (3.75) to the obtained FAS (3.70) gives the following closed-loop system:

$$\begin{cases} \ddot{z}_1 + a_1\dot{z}_1 + a_0 z_1 = 0 \\ \ddot{z}_2 + b_1\dot{z}_2 + b_0 z_2 = 0. \end{cases} \tag{3.76}$$

In this simulation, the controller parameters are chosen as

$$a_1 = a_0 = 2,\ b_1 = 6,\ b_0 = 5,$$

which give the closed-loop eigenvalues $-1 \pm \mathrm{i}$, and -2, -3. Once $z_{1\sim 2}^{(0\sim 1)}$ is obtained from the closed-loop system (3.76), $y_{1\sim 4}$ can be obtained via transformation (3.66), and $x_{1\sim 4}$ can be obtained by further using transformation $x_{1\sim 4} = Q^{-1}y_{1\sim 4}$.

When the initial values of the closed-loop system, namely, $z_{1\sim 2}^{(0\sim 1)}(0)$ is chosen as

$$z_1(0) = z_2(0) = 0.1,\ \dot{z}_1(0) = 0.2,\ \dot{z}_2(0) = 0.2,$$

the simulations are carried out and the results are shown in Figs. 3.2 and 3.3.

Coupled design

A coupled controller for FAS (3.70)-(3.74) is given by

$$u = -L^{-1}(z_2)\left(A_{0\sim 1} z_{1\sim 2}^{(0\sim 1)} + h\left(z_{1\sim 2}^{(0\sim 1)}\right) - \upsilon\right), \tag{3.77}$$

which results in the following closed-loop system:

$$\ddot{z}_{1\sim 2} + A_{0\sim 1} z_{1\sim 2}^{(0\sim 1)} = \upsilon, \tag{3.78}$$

where υ is an external vector, and $A_{0\sim 1} \in \mathbb{R}^{2\times 4}$ is a parameter matrix. Again, by using transformation (3.66), this controller can be immediately converted into a feedback form of variables $y_{1\sim 4}$, and into a feedback form of the original states $x_{1\sim 4}$ by further using transformation $x = Q^{-1}y$. In addition, once $z_{1\sim 2}^{(0\sim 1)}$ is obtained from the closed-loop system (3.78), $y_{1\sim 4}$ can be obtained via transformation (3.66), and $x_{1\sim 4}$ can be obtained by further using transformation $x_{1\sim 4} = Q^{-1}y_{1\sim 4}$.

By our convention, the state-space form of the closed-loop system (3.78) is

$$\dot{z}_{1\sim 2}^{(0\sim 1)} = \Phi(A_{0\sim 1})\, z_{1\sim 2}^{(0\sim 1)} + B_c\upsilon.$$

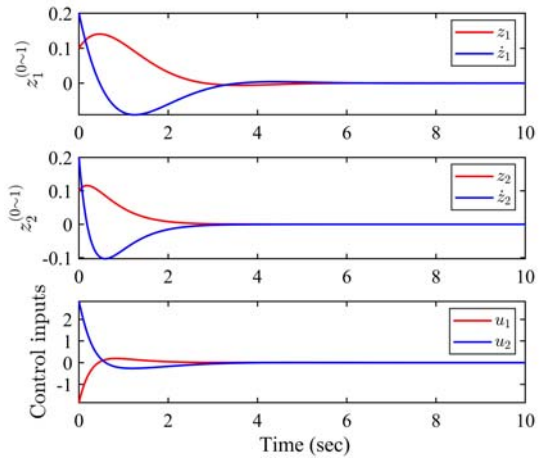


Figure 3.2: Example 1: State responses and control signals, decoupled design

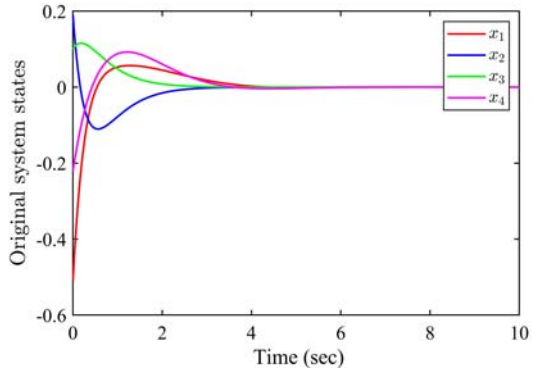


Figure 3.3: Example 1: Original state responses, decoupled design

In view of the relation $\Phi(A_{0\sim 1}) = \Phi(0) + B_c A_{0\sim 1}$, and Proposition 1.1 (see also Duan et al. (2024)), the controller parameter $A_{0\sim 1}$ can be given by

$$A_{0\sim 1} = ZF^2V^{-1},\ V = \begin{bmatrix} Z \\ ZF \end{bmatrix},$$

where $F \in \mathbb{R}^{4\times 4}$ is an arbitrary stable (Hurwitz) matrix, and $Z \in \mathbb{R}^{2\times 4}$ is an arbitrary parameter matrix making the matrix V nonsingular. For this example system, a typical choice of the matrix F may be

$$F = \begin{bmatrix} -a & -b & 0 & 0 \\ b & -a & 0 & 0 \\ 0 & 0 & -c & 0 \\ 0 & 0 & 0 & -d \end{bmatrix},$$

where a, b, c and d are some positive scalars. This corresponds to the case that the set of closed-loop eigenvalues is

$$\{-a \pm b\,\mathrm{i},\ -c,\ -d\}.$$

Specifically choose

$$a = b = 1,\ c = 2,\ d = 3.$$

We obtain

$$A_{0\sim 1} = \begin{bmatrix} 8/3 & 2 & 10/3 & 2/3 \\ -10/3 & 2 & -5/3 & 11/3 \end{bmatrix}.$$

As in the decoupled design above, when the initial values of the closed-loop system are chosen as

$$z_1(0) = z_2(0) = 0.1,\ \dot{z}_1(0) = 0.2,\ \dot{z}_2(0) = 0.2,$$

the simulation results are carried out and are shown in Figs. 3.4 and 3.5.

Remark 3.6 According to Isidori (1995), (p.358), this example system has, corresponding to a selected output, a zero dynamics defined on the sub-manifold

$$Z^* = \left\{x \in \mathbb{R}^4 : x_1 = x_2 = 0\right\},$$

with the representation, expressed in the (x_3, x_4) coordinates of Z^*, as

$$\begin{cases} \dot{x}_3 = x_3^2 \\ \dot{x}_4 = -x_4. \end{cases}$$

Therefore, the approach of non-interacting control fails. Even with a modified design, a stable zero dynamics still exists. While in our above design, globally exponential stabilization of the system is realized. What is more, a constant linear closed-loop system is obtained with all its four poles arbitrarily assignable.

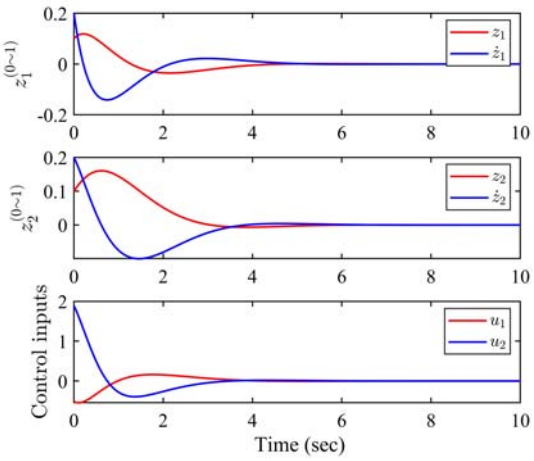


Figure 3.4: Example 1: State responses and control signals, coupled design

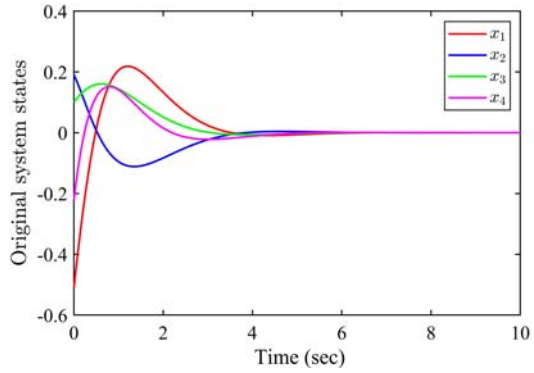


Figure 3.5: Example 1: Original state responses, coupled design

3.5.3 Example 2: A Multi-order FAS

Consider the following dynamical system:

$$\begin{cases} \dot{x}_1 = x_2 + x_3^2 \\ \dot{x}_2 = -x_1 + x_4 + x_4^2 + u_1 \\ \dot{x}_3 = x_1 + x_4 \\ \dot{x}_4 = -x_1^3 + x_5 \\ \dot{x}_5 = -x_3^2 + \left(1 + x_1^2\right) u_2, \end{cases} \tag{3.79}$$

where x_1, x_2, x_3, x_4 and x_5 are the states of the system, u_1 and u_2 are the control inputs. This example system is similar to that in Duan (2023c), except that the time-delay terms are neglected.

The FAS

First of all, based on the system equations in (3.79), we introduce the following differential homeomorphism:

$$\begin{cases} x_1 = y_1 \\ x_2 = \dot{y}_1 - y_2^2 \\ x_3 = y_2 \\ x_4 = \dot{y}_2 - y_1 \\ x_5 = \ddot{y}_2 - \dot{y}_1 + y_1^3, \end{cases} \tag{3.80}$$

whose inverse can be obtained as

$$\begin{cases} y_1 = x_1 \\ \dot{y}_1 = x_2 + y_2^2 = x_2 + x_3^2 \\ y_2 = x_3 \\ \dot{y}_2 = x_4 + y_1 = x_1 + x_4 \\ \ddot{y}_2 = x_5 + \dot{y}_1 - y_1^3 = x_5 + x_2 + x_3^2 - x_1^3, \end{cases} \tag{3.81}$$

that is,

$$y_k^{(0\sim\mu_k-1)}|_{k=1\sim 2} = \begin{bmatrix} x_1 \\ x_2 + x_3^2 \\ x_3 \\ x_1 + x_4 \\ x_5 + x_2 + x_3^2 - x_1^3 \end{bmatrix}, \tag{3.82}$$

where $\mu_1 = 2$, and $\mu_2 = 3$.

By taking the derivative of the first equation in (3.79), we can obtain

$$\ddot{x}_1 = \dot{x}_2 + 2x_3\dot{x}_3.$$

Substituting the second and the third equation in (3.79) into the above equation, gives

$$\begin{aligned} \ddot{x}_1 &= -x_1 + x_4 + x_4^2 + u_1 + 2x_3\left(x_1 + x_4\right) \\ &= -x_1 + x_4 + x_4^2 + 2x_1x_3 + 2x_3x_4 + u_1. \end{aligned} \tag{3.83}$$

Further, by transformation (3.80), the above equation can be written as

$$\ddot{y}_1 = f_1\left(y_1, y_2^{(0\sim 2)}\right) + u_1, \tag{3.84}$$

where

$$f_1(\cdot) = -2y_1 + \dot{y}_2 + (y_1 - \dot{y}_2)^2 + 2y_2\dot{y}_2. \tag{3.85}$$

By taking the derivative of the third equation in (3.79) and using the first and fourth equations in (3.79), we can obtain

$$\begin{aligned} \ddot{x}_3 &= \dot{x}_4 + \dot{x}_1 \\ &= -x_1^3 + x_5 + x_2 + x_3^2. \end{aligned} \tag{3.86}$$

Further, taking the derivatives of the above equation (3.86), gives

$$\dddot{x}_3 = -3x_1^2\dot{x}_1 + \dot{x}_5 + \dot{x}_2 + 2x_3\dot{x}_3,$$

which can be further converted, by using (3.79) again, into the following:

$$\begin{aligned} \dddot{x}_3 &= -x_1 + x_4 - x_3^2 + x_4^2 \\ &\quad +2x_3(x_1 + x_4) - 3x_1^2\left(x_2 + x_3^2\right) + u_1 + \left(1 + x_1^2\right)u_2. \end{aligned} \tag{3.87}$$

Next, using transformation (3.80) again, the above equation (3.87) can be further converted into

$$\begin{aligned} \dddot{y}_2 &= \dddot{x}_3 \\ &= -y_1 + \dot{y}_2 - y_1 - y_2^2 + (\dot{y}_2 - y_1)^2 + 2y_2(y_1 + \dot{y}_2 - y_1) \\ &\quad -3y_1^2(\dot{y}_1 - y_2^2 + y_2^2) + u_1 + (1 + y_1^2)u_2, \end{aligned}$$

which can be further arranged into the following FAS:

$$\dddot{y}_2 = f_2\left(y_1^{(0\sim 1)}, y_2^{(0\sim 2)}\right) + u_1 + \left(1 + y_1^2\right)u_2, \tag{3.88}$$

where

$$f_2(\cdot) = -2y_1 + \dot{y}_2 - y_2^2 + (y_1 - \dot{y}_2)^2 + 2y_2\dot{y}_2 - 3y_1^2\dot{y}_1. \tag{3.89}$$

Finally, combining (3.84) and (3.88), yields the following equivalent FAS model for system (3.79):

$$\begin{bmatrix} \ddot{y}_1 \\ \dddot{y}_2 \end{bmatrix} = f\left(y_1^{(0\sim 1)}, y_2^{(0\sim 2)}\right) + B(y_1)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{3.90}$$

where

$$f\left(y_1^{(0\sim 1)}, y_2^{(0\sim 2)}\right) = \begin{bmatrix} f_1\left(y_1, y_2^{(0\sim 2)}\right) \\ f_2\left(y_1^{(0\sim 1)}, y_2^{(0\sim 2)}\right) \end{bmatrix},\ B(y_1) = \begin{bmatrix} 1 & 0 \\ 1 & 1 + y_1^2 \end{bmatrix}. \tag{3.91}$$

Obviously, we have

$$\det B(y_1) = 1 + y_1^2 > 0.$$

Thus the above FAS (3.90)-(3.91) is a global one.

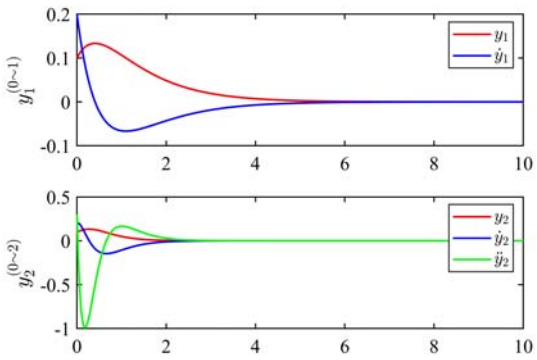


Figure 3.6: Example 2: State responses, decoupled design

Decoupled design

It follows from previous Theorem 3.2 about decoupled design that the controller for the system is given by

$$u = -B^{-1}(y_1)\left(\begin{bmatrix} f_1\left(y_1, y_2^{(0\sim2)}\right) \\ f_2\left(y_1^{(0\sim1)}, y_2^{(0\sim2)}\right) \end{bmatrix} + \begin{bmatrix} A_{1,0\sim1} y_1^{(0\sim1)} \\ A_{2,0\sim2} y_2^{(0\sim2)} \end{bmatrix}\right), \tag{3.92}$$

where $A_{1,0\sim1}$ and $A_{2,0\sim2}$ are real parameters to be designed. By substituting transformation (3.82) into the above, controller (3.92) in the original coordinates, $u = u(x_{1\sim5})$, can be easily obtained.

The closed-loop system under controller (3.92) is clearly given by

$$\begin{cases} \ddot{y}_1 + A_{1,0\sim1} y_1^{(0\sim1)} = 0 \\ \dddot{y}_2 + A_{2,0\sim2} y_2^{(0\sim2)} = 0. \end{cases} \tag{3.93}$$

In the case that the eigenvalues of the first subsystem in (3.93) are set as -1, -2, and those of the second subsystem in (3.93) are set as -3, -4, -5, the corresponding controller parameters are

$$A_{1,0\sim1} = \begin{bmatrix} 2 & 3 \end{bmatrix},\ A_{2,0\sim2} = \begin{bmatrix} 60 & 47 & 12 \end{bmatrix}.$$

To carry out the simulation, the initial values of the closed-loop system, namely, $y_1^{(0\sim1)}(0)$ and $y_2^{(0\sim2)}(0)$, are taken as

$$y_1(0) = y_2(0) = 0.1,\ \dot{y}_1(0) = \dot{y}_2(0) = 0.2,\ \ddot{y}_2(0) = 0.3, \tag{3.94}$$

the corresponding simulation results are shown in Figs. 3.6, 3.7 and 3.8.

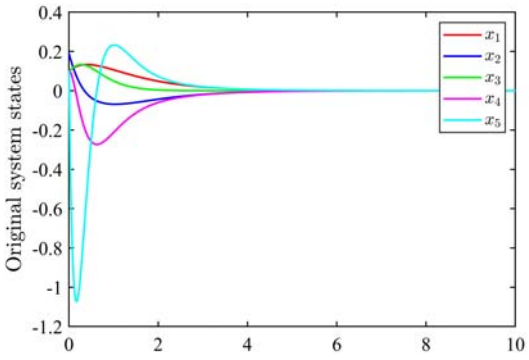


Figure 3.7: Example 2: Original state responses, decoupled design

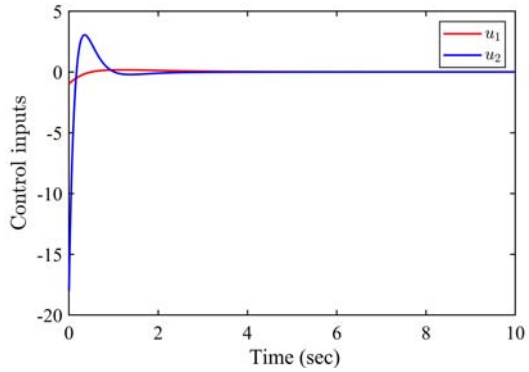


Figure 3.8: Example 2: Control signals, decoupled design

Coupled design

It follows from Theorem 3.3 about a coupled design that the controller for the system is given by

$$u = -B^{-1}(y_1)\left(A_E y_k^{(0\sim\mu_k-1)}|_{k=1\sim2} + f\left(y_1^{(0\sim1)}, y_2^{(0\sim2)}\right)\right), \tag{3.95}$$

where $A_E \in \mathbb{R}^{2\times5}$ is a parameter matrix. Again, by substituting transformation (3.82) into the above (3.95), the controller in the original coordinates, $u = u(x_{1\sim5})$, can be readily obtained.

The closed-loop system under controller (3.95) obviously is given by

$$y_k^{(\mu_k)}|_{k=1\sim2} = A_E y_k^{(0\sim\mu_k-1)}|_{k=1\sim2},$$

which can be rewritten in the following state-space form:

$$\dot{y}_k^{(0\sim\mu_k-1)}|_{k=1\sim2} = (\Phi_E + B_{cE}A_E)\, y_k^{(0\sim\mu_k-1)}|_{k=1\sim2}, \tag{3.96}$$

where

$$\Phi_E = \operatorname{diag}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right), \tag{3.97}$$

$$B_{cE} = \operatorname{diag}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right). \tag{3.98}$$

Select the set of desired closed-loop eigenvalues as $\{-1,-2,-3,-4,-5\}$, that is,

$$\operatorname{eig}(\Phi_E + B_{cE}A_E) = \{-1,-2,-3,-4,-5\}.$$

Then, by applying the complete parametric solutions in Duan et al. (2024) to the above equation, we can obtain

$$A_E = \begin{bmatrix} -13.4929 & -7.5428 & 2.1753 & 2.7877 & 0.5713 \\ 4.9263 & 1.5856 & -9.6878 & -16.1647 & -7.4572 \end{bmatrix}.$$

When the initial values of the closed-loop system, namely, $y_1^{(0\sim1)}(0)$ and $y_2^{(0\sim2)}(0)$ are taken the same as in (3.94), the simulation of the system is carried out and the results are shown in Figs. 3.9, 3.10 and 3.11.

3.6 Notes and References

The FAS approach provides a general effective method for nonlinear control problems. As long as a high-order FAS model is derived, a controller can be immediately designed by directly cancelling the nonlinearity of the system, and as a result, a closed-loop system with desired properties can be immediately derived. Particularly, a linear

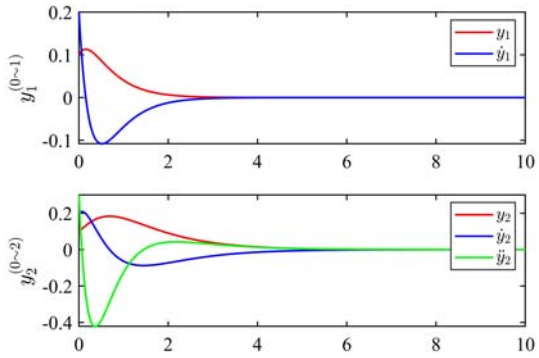


Figure 3.9: Example 2: State responses, coupled design

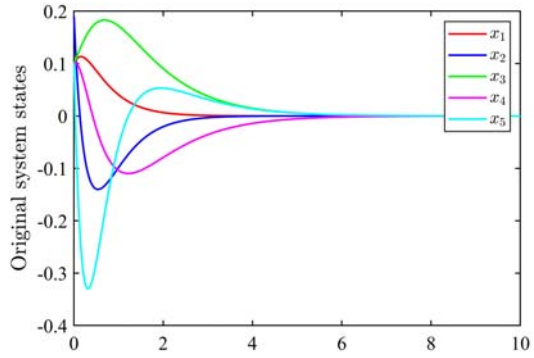


Figure 3.10: Example 2: Original state responses, coupled design

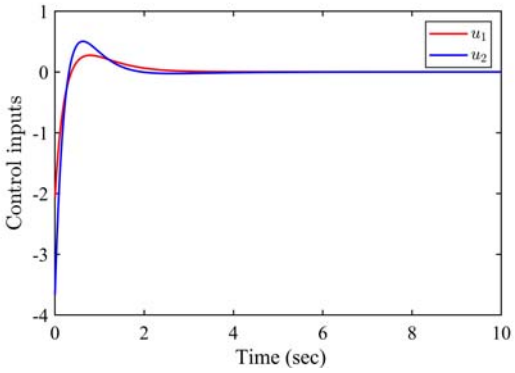


Figure 3.11: Example 2: Control signals, coupled design

time-invariant closed-loop system with arbitrarily assignable eigenstructures can be obtained. This advantage provides great convenience for the control of systems and greatly promotes the development of modern control theories. Consequently, the FAS approach has been proven to be very simple and effective in dealing with the robust control Duan (2021c), adaptive control Duan (2021d), optimal control Duan (2022e), tracking control Duan (2022d), disturbance attenuation and decoupling Duan (2021f) of general nonlinear dynamical systems.

This approach has also been generalized to the time-varying and time-delay systems cases Duan (2023b,c, 2022b,c). Furthermore, some long-standing problems are well addressed with the FAS approach. The well-known Brockett's first example system and the Brockett's second example system are both successfully treated in Duan (2022a) and Duan (2023a), respectively, and are both generalized and solved with the FAS approach in Duan (2024c,d, 2025e). For a benchmark example system firstly proposed by Aeyels (1985), a state feedback stabilizing controller is also designed in Duan (2022g) by using the FAS approach. These researches related to the so-called nonholonomic systems are discussed in the second volume of the book Duan (2025g).

As shown in this chapter (see also Duan (2021a)), after completely canceling the nonlinearity of the system, the stabilization control problem of the system is immediately transformed into a specific eigenstructure assignment (ESA) problem of a certain matrix pair, regardless of whether the open-loop system is linear or not, and no matter how complex the structure of the open-loop system is. In addition, it has been proven that this specific ESA problem, termed as "fundamental problem", also plays an important role in many other control problems when the FAS approach is employed, such as adaptive control Duan (2021d), robust control Duan (2021c), disturbance attenuation and decoupling Duan (2021f), tracking control Duan (2022d), and optimal control Duan (2022e).

Let us mention again that the FAS approach was proposed in 2020-2021 as a general framework for control system analysis and design based on a newly discovered general type of fully-actuated models for dynamical systems. Due to its great advan-

tages and power in dealing with complicated nonlinear time-varying and time-delay systems with possibly nonholonomic features, it has attracted much attention in the control community immediately since its birth. By now, numerous results have been produced for analysis and control of various types of complicated systems, which cover the topics of adaptive control, robust control, predictive control and fault-tolerant control, and involve time-varying and time-delay systems, discrete-time systems, stochastic systems and even impulsive systems (see the overview Section 1.3). Meanwhile, a large number of applications have also been carried out. These include spacecraft control, aircraft and quadrotor control, robot control, power electronic system control and servo systems (see the overview Section 1.4).

Part II

Uncertain FASs

Chapter 4

Disturbance Attenuation and Decoupling

In this chapter, two types of high-order FAS models subject to external disturbances are firstly introduced. For the type of FASs with deterministic disturbances, the problem of disturbance attenuation via state feedback is treated, while for the type of FASs with dynamical disturbances, the problem of asymptotic disturbance decoupling via output feedback is considered. By utilizing the full-actuation feature of the FASs, disturbance attenuation and decoupling controllers for the corresponding systems are conveniently designed such that constant linear closed-loop systems with designed disturbance rejection properties are resulted in. Parametric designs for both controllers are provided, and disturbance attenuation is achieved by establishing a parametric form of the closed-loop transfer function from the disturbance to the output, while the parametric form of the disturbance decoupling controller is derived based on a complete parametric solution to a type of generalized Sylvester equations (GSEs). As a consequence of the parametric approaches, additional performance requirements on the closed-loop systems can be also easily handled. An illustrative example demonstrates the effect of the proposed approach.

4.1 FAS Models with Disturbances

In this section, two types of FAS models for dynamical systems with disturbances are proposed on the basis of a type of uncertain FAS models introduced in the Part II of the series (Duan (2021b)).

4.1.1 Uncertain FAS Models

Consider the following FAS model introduced in Duan (2021b):

$$x^{(n)} = f\left(x^{(0\sim n-1)}, \zeta, t\right) + \Delta f\left(x^{(0\sim n-1)}, \zeta, t\right) + B\left(x^{(0\sim n-1)}, \zeta, t\right) u, \qquad (4.1)$$

where $n \geq 1$ is an integer, x, $u \in \mathbb{R}^r$ are the state vector and the control input vector, respectively, $\zeta \in \mathbb{R}^p$ may represent a parameter vector, an external variable vector, a time-delayed state vector, an unmodeled dynamic state vector, etc.; $f\left(x^{(0\sim n-1)}, \zeta, t\right) \in \mathbb{R}^r$ is a known sufficiently smooth vector function, while $\Delta f\left(x^{(0\sim n-1)}, \zeta, t\right) \in \mathbb{R}^r$ is an unknown term, $B\left(x^{(0\sim n-1)}, \zeta, t\right) \in \mathbb{R}^{r\times r}$ is a sufficiently smooth matrix function satisfying the following full-actuation condition:

Assumption 4.1 $\det B(X, \zeta, t) \neq 0,\ \forall X \in \mathbb{R}^{nr},\ \zeta \in \mathbb{R}^p$, *and* $t \geq 0$.

From appearance, the above Assumption 4.1 seems to be strict, while as a matter of fact, the above FAS (4.1) represents a quite general model for "completely controllable" systems, and can either be obtained through physical modelling, or through conversion from other types of models (Duan (2020a,b, 2021a,b,c,d,e)).

As a demonstration, let us restate that the following uncertain strict feedback system (SFS) can be converted into the form of (4.1):

$$\begin{cases} \dot{x}_1 = f_1(x_1, \zeta, t) + \Delta f_1(x_1, \zeta, t) + G_1 x_2 \\ \dot{x}_2 = f_2(x_1, x_2, \zeta, t) + \Delta f_2(x_1, x_2, \zeta, t) + G_2 x_3 \\ \quad \vdots \\ \dot{x}_{n-1} = f_{n-1}(x_{1\sim n-1}, \zeta, t) + \Delta f_{n-1}(x_{1\sim n-1}, \zeta, t) + G_{n-1} x_n \\ \dot{x}_n = f_n(x_{1\sim n}, \zeta, t) + \Delta f_n(x_{1\sim n}, \zeta, t) + G_n u, \end{cases} \tag{4.2}$$

where $x_i \in \mathbb{R}^r$, $i = 1, 2, \ldots, n$, are the state vectors, $u \in \mathbb{R}^r$ is the control input, $f_i(x_{1\sim i}, \zeta, t)$ and $\Delta f_i(x_{1\sim i}, \zeta, t) \in \mathbb{R}^r$, $i = 1, 2, \ldots, n$, are sufficiently smooth vector functions, and $G_i \in \mathbb{R}^{r\times r}$, $i = 1, 2, \ldots, n$, are a set of nonsingular matrices. According to the Theorem 3.1 or 3.6 in Duan (2021b), we have the following simplified result.

Proposition 4.1 *The above SFS (4.2), with G_i, $i = 1, 2, \ldots, n$, being nonsingular, can be transformed into the form of*

$$z^{(n)} = f\left(z^{(0\sim n-1)}, \zeta, t\right) + \Delta f\left(z^{(0\sim n-1)}, \zeta, t\right) + L_n u, \tag{4.3}$$

with $z = x_1$, and

$$L_k = G_1 G_2 \cdots G_k,\ k = 1, 2, \ldots, n, \tag{4.4}$$

$$f(z^{(0\sim n-1)}, \zeta, t) = L_{0\sim n-1} \begin{bmatrix} f_1^{(n-1)}(z, \zeta, t) \\ f_2^{(n-2)}(z, x_2, \zeta, t) \\ \vdots \\ \dot{f}_{n-1}(x_{1\sim n-1}, \zeta, t) \\ f_n(x_{1\sim n}, \zeta, t) \end{bmatrix}, \tag{4.5}$$

$$\Delta f(z^{(0\sim n-1)}, \zeta, t) = L_{0\sim n-1} \begin{bmatrix} \Delta f_1^{(n-1)}(z, \zeta, t) \\ \Delta f_2^{(n-2)}(z, x_2, \zeta, t) \\ \vdots \\ \Delta \dot{f}_{n-1}(x_{1\sim n-1}, \zeta, t) \\ \Delta f_n(x_{1\sim n}, \zeta, t) \end{bmatrix}, \tag{4.6}$$

where $L_0 = I$, and in (4.5)-(4.6) the x_k, $k = 2, 3, \ldots, n$, and their derivatives are all given or determined by certain transformation (see the Theorem 3.1 in Duan (2021b).

Similarly, it can be shown via the Theorems 3.2, 3.3 and 3.6 in Duan (2021b) that the second- and high-order SFSs proposed in Duan (2021b), with uncertainties properly added, can be also converted into the form of FAS (4.1).

4.1.2 FAS Models with Deterministic Disturbances

Now let us consider the case where $\Delta f\left(x^{(0\sim n-1)}, \zeta, t\right)$ is a deterministic external disturbance. In such a case we may assume

$$\Delta f\left(x^{(0\sim n-1)}, \zeta, t\right) = Ed\left(t\right), \tag{4.7}$$

where $d\left(t\right) \in \mathbb{R}^p$ is an external disturbance, depending only on time t, $E \in \mathbb{R}^{r\times p}$ is a known constant distribution matrix. Therefore, the uncertain system (4.1) becomes the following one with a deterministic disturbance:

$$x^{(n)} = f\left(x^{(0\sim n-1)}, \zeta, t\right) + B\left(x^{(0\sim n-1)}, \zeta, t\right)u + Ed\left(t\right). \tag{4.8}$$

Again, the above FAS (4.8) can be either obtained through physical modelling (see the two examples in Duan (2021b), and also the example in the Section 5 of Duan (2021f)), or through conversion from other types of models (Duan (2020a,b, 2021a,b)).

Parallelly, taking

$$\Delta f_i\left(x_{1\sim i}, \zeta, t\right) = E_i d_i\left(t\right), \quad i = 1, 2, \ldots, n,$$

in (4.2), where $d_i \in \mathbb{R}^p$ and $E_i \in \mathbb{R}^{r\times p}$ are the external disturbances and the constant distribution matrices, respectively, gives the following SFS with disturbances:

$$\begin{cases} \dot{x}_1 = f_1\left(x_1, \zeta, t\right) + G_1 x_2 + E_1 d_1 \\ \dot{x}_2 = f_2\left(x_1, x_2, \zeta, t\right) + G_2 x_3 + E_2 d_2 \\ \quad \vdots \\ \dot{x}_{n-1} = f_{n-1}\left(x_{1\sim n-1}, \zeta, t\right) + G_{n-1} x_n + E_{n-1} d_{n-1} \\ \dot{x}_n = f_n\left(x_{1\sim n}, \zeta, t\right) + G_n u + E_n d_n. \end{cases} \tag{4.9}$$

Following Proposition 4.1, we can convert the above SFS into a FAS in the form of

$$z^{(n)} = f\left(z^{(0\sim n-1)}, \zeta, t\right) + L_n u + d, \tag{4.10}$$

with L_n and $f\left(z^{(0\sim n-1)}, \zeta, t\right)$ being given by (4.4) and (4.5), respectively, and $d\left(t\right)$ being determined by (4.6) as

$$\begin{aligned} d\left(t\right) &= L_{0\sim n-1}\begin{bmatrix} E_1 d_1^{(n-1)}(t) \\ E_2 d_2^{(n-2)}(t) \\ \vdots \\ E_{n-1} d_{n-1}(t) \\ E_n d_n(t) \end{bmatrix} \\ &= L_0 E_1 d_1^{(n-1)} + L_1 E_2 d_2^{(n-2)} + \cdots + L_{n-1} E_n d_n. \end{aligned} \tag{4.11}$$

Similarly, it can be shown via the Theorems 3.2, 3.3 and 3.6 in Duan (2021b) that the second- and high-order SFSs proposed in Duan (2021b), with disturbances added, can be also converted into the form of (4.8).

Remark 4.1 It is clearly seen from (4.11) that some constant disturbances, slope disturbances, etc., may automatically disappear when the FAS method is used for design. Therefore, the FAS approach itself has certain "natural" anti-disturbance characteristics. In certain situations, the control of a system subject to disturbances can be turned into a control problem free of disturbances (see the examples in Duan (2021b)).

4.1.3 FAS Models with Dynamical Disturbances

Inspired by the form of (4.11), we introduce the following FAS model with a disturbance:

$$x^{(n)} = f\left(x^{(0\sim n-1)}, \zeta, t\right) + B\left(x^{(0\sim n-1)}, \zeta, t\right)u + Ed^{(0\sim k-1)}(t), \tag{4.12}$$

where the disturbance $d \in \mathbb{R}^p$ is generated by some exogenous system

$$d^{(k)} + S_{0\sim k-1}d^{(0\sim k-1)} = 0, \tag{4.13}$$

which can be also expressed into the following state-space form:

$$\dot{d}^{(0\sim k-1)} = \Phi(S_{0\sim k-1})\, d^{(0\sim k-1)}. \tag{4.14}$$

Here we mention that $\Phi(S_{0\sim k-1})$ is generally assumed to be not Hurwitz, since otherwise the disturbance $d^{(0\sim k-1)}$ will eventually die out automatically.

Example 4.1 Let us consider the disturbance

$$d(t) = c\sin\omega t,$$

where c and ω are two positive scalars. Note that

$$\dot{d}(t) = c\omega\cos\omega t, \quad \ddot{d}(t) = -c\omega^2\sin\omega t,$$

this particular disturbance can be modelled by

$$\ddot{d}(t) = -\omega^2 d(t),$$

which can be expressed as

$$\ddot{d}(t) = S_{0\sim 1}d^{(0\sim 1)}(t), \tag{4.15}$$

with

$$S_{0\sim 1} = \begin{bmatrix} -\omega^2 & 0 \end{bmatrix}. \tag{4.16}$$

Model (4.15)-(4.16) can be also equivalently expressed in the following state-space form:

$$\dot{d}^{(0\sim 1)}(t) = \Phi(S_{0\sim 1})\, d^{(0\sim 1)}(t), \tag{4.17}$$

with

$$\Phi(S_{0\sim 1}) = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}.$$

Please note that, since $d(t)$ is a disturbance signal, the initial values of the dynamical model (4.15), or (4.17), are generally assumed to be unknown. In certain circumstances, even the parameter ω is also assumed to be unknown.

Remark 4.2 For the FAS model (4.8) with a deterministic disturbance and the FAS model (4.12)-(4.13) with a dynamical disturbance, many practical examples exist at least for the second-order system case. For the FAS model (4.8) with a deterministic disturbance, we aim to consider the problem of disturbance attenuation since no further information about the disturbance is known. For the FAS model (4.12)-(4.13) with a dynamical disturbance, we can realize asymptotical disturbance decoupling based on the dynamical model of the disturbance.

4.2 Disturbance Attenuation

Adding an output equation to system (4.8), gives the following FAS model:

$$\begin{cases} x^{(n)} = f\left(x^{(0\sim n-1)}, \zeta, t\right) + B\left(x^{(0\sim n-1)}, \zeta, t\right)u + Ed(t) \\ y = Cx^{(0\sim n-1)}, \end{cases} \tag{4.18}$$

where $y \in \mathbb{R}^m$ is the system output, $C \in \mathbb{R}^{m\times nr}$ is a constant measurement matrix.

In this section, we will consider the disturbance attenuation problem in the above system (4.18).

4.2.1 Problem Statement

For the above high-order system (4.18) with a disturbance $d(t)$, if we design the following control law:

$$\begin{cases} u = -B^{-1}\left(x^{(0\sim n-1)}, \zeta, t\right)\left[f\left(x^{(0\sim n-1)}, \zeta, t\right) + u_0\right] \\ u_0 = A_{0\sim n-1}x^{(0\sim n-1)} - \upsilon, \end{cases} \tag{4.19}$$

then it can be easily verified that the closed-loop system is

$$\begin{cases} x^{(n)} + A_{n-1}x^{(n-1)} + \cdots + A_1\dot{x} + A_0 x = \upsilon + Ed \\ y = Cx^{(0\sim n-1)}, \end{cases} \tag{4.20}$$

which can be also rewritten, more compactly, as

$$\begin{cases} x^{(n)} + A_{0\sim n-1}x^{(0\sim n-1)} = \upsilon + Ed \\ y = Cx^{(0\sim n-1)}. \end{cases} \tag{4.21}$$

Further denote

$$B_c = \begin{bmatrix} 0 \\ I_r \end{bmatrix} \in \mathbb{R}^{nr\times r}. \tag{4.22}$$

Then the above equation (4.21) can be also equivalently transformed into the following state-space form:

$$\begin{cases} \dot{x}^{(0\sim n-1)} = \Phi\left(A_{0\sim n-1}\right) x^{(0\sim n-1)} + B_c \upsilon + B_c E d \\ y = C x^{(0\sim n-1)}. \end{cases} \tag{4.23}$$

The above deduction states a very important fact: once a nonlinear system is represented in a FAS model, a controller for the system can be easily designed using the full-actuation feature of the FAS model such that the closed-loop system becomes a constant linear one subject to the same disturbance. Hence the disturbance attenuation problem for a nonlinear FAS is turned into one for a linear system.

With the above preparation, the disturbance attenuation problem to be considered in this section is to design the matrix $A_{0\sim n-1}$ such that the above linear system (4.23) has an anti-disturbance property. This can be stated as follows.

Problem 4.1 *For a given FAS (4.18), determine the coefficient matrix $A_{0\sim n-1}$ in control law (4.19) such that*

$$J_* = \left\| G_{dy}(s) \right\|_*, \ * = 2 \text{ or } \infty, \tag{4.24}$$

is minimized, where $G_{dy}(s)$ is the transfer function from d to y defined by

$$G_{dy}(s) = C\left(sI - \Phi\left(A_{0\sim n-1}\right)\right)^{-1} B_c E. \tag{4.25}$$

Since the problem determined by (4.24)-(4.25) is a standard H_∞ or H_2 problem for linear systems, many existing methods can be readily applied, e.g., the Riccati equation approach (Zhou et al. (1994); Doyle et al. (1994)), and the linear matrix inequality (LMI) approach (see, e.g., Duan and Yu (2013)). In this section, we will present a complete parametric approach for the H_2 problem. The idea is to first establish a complete parametric solution of $A_{0\sim n-1}$, and then to give a parametric expression of $\left\| G_{dy}(s) \right\|_2$ based on the parameters Z and F existing in the general parametric solution of $A_{0\sim n-1}$.

4.2.2 Preliminaries

To derive our main results, three preliminary results are needed. The first one is about the solution to the following GSE:

$$\Phi\left(0_{0\sim n-1}\right) V + B_c W = V F. \tag{4.26}$$

Lemma 4.1 *For an arbitrarily selected matrix $F \in \mathbb{R}^{q\times q}$, all the matrices $W \in \mathbb{R}^{r\times q}$ and $V \in \mathbb{R}^{nr\times q}$ satisfying the above equation (4.26) are given by*

$$\begin{cases} V = V(Z,F) = \begin{bmatrix} Z \\ ZF \\ \vdots \\ ZF^{n-1} \end{bmatrix} \\ W = W(Z,F) = ZF^n, \end{cases} \tag{4.27}$$

where $Z \in \mathbb{R}^{r\times q}$ is an arbitrary parameter matrix.

The above result can be easily proven by using the Theorem 4.3 in Duan (2015) (p.120), about the parametric solution to a GSE.

The second preliminary result gives an analytical expression of the H_2 norm of a transfer function in terms of solutions to Lyapunov matrix equations (see, e.g., the Lemma 5.1 in Duan and Yu (2013), p.140).

Lemma 4.2 *Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$, $C \in \mathbb{R}^{m\times n}$ and A be Hurwitz. Then*

$$\left\| C\left(sI_n - A\right)^{-1} B \right\|_2^2 = \text{trace}\left(CP_1C^{\mathrm{T}}\right) = \text{trace}\left(B^{\mathrm{T}}P_2B\right),$$

where P_1 and P_2 are the unique positive definite solutions to the Lyapunov equations

$$AP_1 + P_1A^{\mathrm{T}} = -BB^{\mathrm{T}},$$

and

$$A^{\mathrm{T}}P_2 + P_2A = -C^{\mathrm{T}}C,$$

respectively.

Introduce the following notations related to a square matrix $\Phi \in \mathbb{R}^{n\times n}$:

$$\det\left(sI_n + \Phi\right) \triangleq \sum_{i=0}^{n} c_i^{\Phi} s^i, \ \text{adj}\left(sI_n + \Phi^{\mathrm{T}}\right) \triangleq \sum_{i=0}^{n-1} C_i^{\Phi} s^i.$$

Then we have the following result which is simplified from the Theorem 9.2 in Duan (2015) (p.356).

Lemma 4.3 *If $\Phi \in \mathbb{R}^{n\times n}$ is Hurwitz, then, for an arbitrary positive definite matrix $Q \in \mathbb{R}^{n\times n}$, the Lyapunov equation*

$$\Phi^{\mathrm{T}}P + P\Phi = -Q$$

has a unique solution given by

$$P = -\sum_{i=0}^{n-1} C_i^{\Phi} Q P_0^{-1} \Phi^i, \tag{4.28}$$

with

$$P_0 = \sum_{i=0}^{n} c_i^{\Phi} \Phi^i. \tag{4.29}$$

4.2.3 Parametric Solution

Now let us turn to consider the solution to the H_2 problem.

Explicit solution of $A_{0\sim n-1}$

To derive a parametric solution of the coefficient matrix $A_{0\sim n-1}$ in controller (4.19), Proposition 1.1 can be readily used, which can be brie y restated as below.

Theorem 4.1 *For an arbitrarily selected matrix $F \in \mathbb{R}^{nr\times nr}$, all the matrices $A_{0\sim n-1}$ and $V \in \mathbb{R}^{nr\times nr}$ satisfying $\det V \neq 0$ and*

$$\Phi(A_{0\sim n-1}) = VFV^{-1} \tag{4.30}$$

are given by (4.27) and

$$A_{0\sim n-1} = -ZF^nV^{-1}(Z,F), \tag{4.31}$$

with $Z \in \mathbb{R}^{r\times nr}$ being a parameter matrix satisfying

$$\det V(Z,F) \neq 0. \tag{4.32}$$

Proof. Write equation (4.30) as

$$\Phi(A_{0\sim n-1})V = VF. \tag{4.33}$$

Further, introducing the variable

$$W = -A_{0\sim n-1}V, \tag{4.34}$$

and in view of

$$\Phi(A_{0\sim n-1}) = \Phi(0_{0\sim n-1}) - B_cA_{0\sim n-1}, \tag{4.35}$$

we can convert equivalently the above equation (4.33) into the form of (4.26). Therefore, it follows from Lemma 4.1 that all the matrices V and W are given by (4.27), and equation (4.31) can then be further solved from (4.34). ■

Remark 4.3 The above theorem clearly establishes a general parametric solution of controller (4.19), which arbitrarily assigns the desired eigenstructure of the closed-loop system (4.23). It is actually a result on eigenstructure assignment (ESA) in the linear system $\left[\Phi(0_{0\sim n-1}), B_c\right]$, which is termed as the "basic problem" in Duan (2020a) (see, also the Corollary 1 in Duan (2020b), or refer to Duan (2021b) and Duan (2021c)),

Parametric expression of $\|G_{dy}(s)\|_2$

Based on Lemmas 4.2 and 4.3, and Theorem 4.1, the following result can be obtained, which gives a parametric expression of the H_2 norm of the transfer function (4.25).

Theorem 4.2 *Let $F \in \mathbb{R}^{nr\times nr}$ be a given Hurwitz matrix, and $f_i \in \mathbb{R}$ and $F_i \in \mathbb{R}^{nr\times nr}$, $i = 0, 1, 2, \ldots, nr$, are determined by*

$$\det(sI + F) = \sum_{i=0}^{nr} f_i s^i, \tag{4.36}$$

and

$$\operatorname{adj}(sI+F)=\sum_{i=0}^{nr-1}F_i s^i, \tag{4.37}$$

respectively. If the control law for FAS (4.18) is taken as (4.19), with $A_{0\sim n-1}$ being given by (4.31), then we have

$$\begin{aligned}\left\|G_{dy}(s)\right\|_2^2 &= \operatorname{trace}\left((CV)P_1(CV)^{\mathrm{T}}\right)\\ &= \operatorname{trace}\left(\left(V^{-1}B_cE\right)^{\mathrm{T}}P_2\left(V^{-1}B_cE\right)\right),\end{aligned} \tag{4.38}$$

where V is given by (4.27), and

$$\begin{cases} P_1=-\sum_{i=0}^{nr-1}F_i\left(V^{-1}B_cE\right)\left(V^{-1}B_cE\right)^{\mathrm{T}}P_{10}^{-1}\left(F^{\mathrm{T}}\right)^i\\ P_2=-\sum_{i=0}^{nr-1}F_i^{\mathrm{T}}(CV)^{\mathrm{T}}(CV)P_{20}^{-1}F^i, \end{cases} \tag{4.39}$$

with

$$P_{10}=\sum_{i=0}^{nr}f_i\left(F^{\mathrm{T}}\right)^i,\quad P_{20}=\sum_{i=0}^{nr}f_iF^i. \tag{4.40}$$

Proof. When control law (4.19), with $A_{0\sim n-1}$ chosen as in (4.31), is applied to system (4.18), the closed-loop system can be obtained and rewritten in the state-space form (4.23). It follows from Theorem 4.1 that (4.30) holds. Thus, substituting (4.30) into (4.25), yields

$$G_{dy}(s)=CV(sI-F)^{-1}V^{-1}B_cE. \tag{4.41}$$

Considering that the matrix F is Hurwitz, there exist unique positive definite solutions P_1 and P_2 satisfying the Lyapunov equations

$$FP_1+P_1F^{\mathrm{T}}=-V^{-1}B_cEE^{\mathrm{T}}B_c^{\mathrm{T}}V^{-\mathrm{T}}, \tag{4.42}$$

and

$$F^{\mathrm{T}}P_2+P_2F=-V^{\mathrm{T}}C^{\mathrm{T}}CV, \tag{4.43}$$

respectively. According to Lemma 4.3, the matrices P_1 and P_2 given by (4.39)-(4.40) are respectively the solutions to the Lyapunov equations (4.42) and (4.43). Therefore, it follows from Lemma 4.2 that $\left\|G_{dy}(s)\right\|_2^2$ is given by (4.38)-(4.40). The proof is then completed. ■

With the help of the above two theorems, the solution to the disturbance attenuation problem in system (4.18) via controller (4.19) can then be founded by optimizing the design parameters F and Z to minimize $\left\|G_{dy}(s)\right\|_2^2$. Clearly, an advantage of the parametric approach is that it is easy to have additional indices to be comprehensively optimized together with this H_2 index.

In applications, it suffices to choose F to be a diagonal matrix with negative diagonal elements. In the case that complex eigenvalues are expected, we may have such a block

$$\begin{bmatrix} -a & -b \\ b & -a \end{bmatrix}, \tag{4.44}$$

with a and b being two positive scalars, included as a diagonal block in F.

Clearly, there are quite some degrees of freedom (DOFs) in the selection of F, while on the other hand, we also have the parameter matrix Z, which provides $r^2 n$ DOFs. All these DOFs can be further utilized to achieve additional performance of the system (Duan (1992a, 1993a); Duan et al. (2000); Duan and Zhao (2020)).

4.3 Asymptotic Disturbance Decoupling

Adding two output equations to system (4.12), and replacing the state $x^{(0\sim n-1)}$ in the system with the measured output y_m, give the following FAS:

$$\begin{cases} x^{(n)} = f\left(y_m, \zeta, t\right) + B\left(y_m, \zeta, t\right) u + E d^{(0\sim k-1)}\left(t\right) \\ y_m = C_m x^{(0\sim n-1)} \\ y_r = C_r x^{(0\sim n-1)} + E_r d^{(0\sim k-1)}\left(t\right), \end{cases} \tag{4.45}$$

where $y_m \in \mathbb{R}^{m_1}$ is the measured output, $y_r \in \mathbb{R}^{m_2}$ is the regulated output, C_m, C_r and E_r are the coefficient matrices of appropriate dimensions, and the disturbance $d \in \mathbb{R}^p$ is generated by the exogenous system (4.13), or equivalently, the state-space system (4.14).

In this section, asymptotic disturbance decoupling in the above system (4.45) is considered via dynamical output feedback. The aim is to find a control law, employing the feedback of the measured output y_m, for the given FAS (4.45) with the disturbance d generated by the exogenous system (4.13), such that the following asymptotic output regulation requirement

$$\lim_{t\to\infty} y_r\left(t\right) = 0 \tag{4.46}$$

is met for arbitrary initial values $x^{(0\sim n-1)}\left(0\right)$, $\zeta\left(0\right)$ and $d^{(0\sim k-1)}\left(0\right)$.

To solve this problem, we also need some preliminary results.

4.3.1 Preliminary Results

Let us first state a result about disturbance decoupling in linear systems.

Disturbance decoupling in linear systems

Consider the following linear system:

$$\begin{cases} \dot{x} = Ax + Bu + E_w w \\ y = C_y x \\ e = C_e x + D_{ew} w, \end{cases} \tag{4.47}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^r$ are the state and the control input, respectively; $y \in \mathbb{R}^{m_1}$ and $e \in \mathbb{R}^{m_2}$ are the measured output and the regulated output, respectively; $w \in \mathbb{R}^p$ is the disturbance generated by the exogenous system

$$\dot{w} = Sw, \tag{4.48}$$

where the matrix S is usually not Hurwitz since otherwise the disturbance w dies out itself. Furthermore, the system coefficient matrices are assumed to satisfy the following conditions.

Assumption 4.2 *The matrix pair (A, B) is stabilizable.*

Assumption 4.3 *The matrix pair $(\tilde{A}, \tilde{C})$ is detectable, where*

$$\tilde{A} = \begin{bmatrix} A & E_w \\ 0 & S \end{bmatrix}, \ \tilde{C} = \begin{bmatrix} C_y & 0 \end{bmatrix}.$$

The following lemma is a simplified version of the Theorem 2.4.1 in Saberi et al. (2012) (p.25) (see, also the Theorem 2.6 in Duan (2015) (p.51)), which gives a basic output regulation result for the linear system (4.47).

Lemma 4.4 *Suppose that system (4.47) satisfies Conditions 4.2-4.3. Let the control law for the system be designed as*

$$\begin{cases} \begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{w}} \end{bmatrix} = \begin{bmatrix} A & E_w \\ 0 & S \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} K_A \\ K_S \end{bmatrix} \left(C_y \hat{x} - y \right) \\ u = \begin{bmatrix} K & \Gamma - K\Pi \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix}, \end{cases} \tag{4.49}$$

where K, K_A and K_S are arbitrary matrices making

$$A + BK \text{ and } \begin{bmatrix} A + K_A C_y & E_w \\ K_S C_y & S \end{bmatrix} \tag{4.50}$$

both Hurwitz; Γ and Π are the solutions to the following matrix equations:

$$A\Pi + B\Gamma + E_w = \Pi S, \tag{4.51}$$

$$C_e \Pi + D_{ew} = 0. \tag{4.52}$$

Then,

1. *the closed-loop system*

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \\ \dot{\hat{w}} \end{bmatrix} = \begin{bmatrix} A & BK & A_2 \\ -K_A C_y & A_1 & E_w + A_2 \\ -K_S C_y & K_S C_y & S \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \\ \hat{w} \end{bmatrix}, \tag{4.53}$$

is internally asymptotically stable, where

$$A_1 = A + K_A C_y + BK,$$

and

$$A_2 = B(\Gamma - K\Pi);$$

2. *the asymptotic output regulation requirement*

$$\lim_{t\to\infty} e(t) = 0 \tag{4.54}$$

is met for arbitrary initial values $x(0), \hat{x}(0)$ *and* $w(0)$.

Solution to a generalized Sylvester equation

In this subsection, let us give a general parametric solution to the following nonhomogeneous GSE:

$$\Phi(0_{0\sim n-1})\Pi + B_c\Gamma = \Pi F + B_c R. \tag{4.55}$$

Lemma 4.5 *Let* $F \in \mathbb{C}^{q\times q}$, $R \in \mathbb{R}^{r\times q}$. *Then all the matrices* $\Pi \in \mathbb{C}^{nr\times q}$ *and* $\Gamma \in \mathbb{C}^{r\times q}$ *satisfying equation (4.55) are given by*

$$\Pi = \begin{bmatrix} Z \\ ZF \\ \vdots \\ ZF^{n-1} \end{bmatrix}, \quad \Gamma = R + ZF^n, \tag{4.56}$$

where $Z \in \mathbb{R}^{r\times q}$ *is an arbitrary parameter matrix.*

Proof. Obviously, the homogeneous equation of the above GSE (4.55) is equation (4.26). Let (V, W) be a general solution to the homogeneous GSE (4.26), and (Π_0, Γ_0) be a particular solution to the above GSE (4.55), then it is easily known that the general solution (Π, Γ) to the above nonhomogeneous equation (4.55) is given by

$$\Pi = \Pi_0 + V, \quad \Gamma = \Gamma_0 + W. \tag{4.57}$$

It follows from Lemma 4.1 that a general solution to the homogeneous GSE (4.26) is given by (4.27). Further, it can be easily verified that a particular solution to the above nonhomogeneous GSE (4.55) is given by

$$\Pi_0 = 0, \quad \Gamma_0 = R. \tag{4.58}$$

Therefore, the general solution (4.56) immediately follows from (4.27), (4.57) and (4.58). ■

4.3.2 Parametric Solution

It is obvious that $(\Phi(0_{0\sim n-1}), B_c)$ is controllable. Further denote

$$\Phi_E = \begin{bmatrix} \Phi(0_{0\sim n-1}) & B_c E \\ 0 & \Phi(S_{0\sim k-1}) \end{bmatrix}, \; C_E = \begin{bmatrix} C_m & 0 \end{bmatrix}, \tag{4.59}$$

then we also need the following assumption:

Assumption 4.4 *The matrix pair* (Φ_E, C_E) *defined by (4.59) is detectable.*

Let us introduce the following control law for FAS (4.45):

$$\begin{cases} \begin{bmatrix} \dot{\hat{x}}^{(0\sim n-1)} \\ \dot{\hat{d}}^{(0\sim k-1)} \end{bmatrix} = \begin{bmatrix} \Phi(0_{0\sim n-1}) & B_c E \\ 0 & \Phi(S_{0\sim k-1}) \end{bmatrix} \begin{bmatrix} \hat{x}^{(0\sim n-1)} \\ \hat{d}^{(0\sim k-1)} \end{bmatrix} + \begin{bmatrix} B_c \\ 0 \end{bmatrix} \upsilon \\ \qquad\qquad + \begin{bmatrix} L_x \\ L_d \end{bmatrix} \left(C_m \hat{x}^{(0\sim n-1)} - y_m \right) \\ \upsilon = \begin{bmatrix} -K_{0\sim n-1} & \Gamma + K_{0\sim n-1}\Pi \end{bmatrix} \begin{bmatrix} \hat{x}^{(0\sim n-1)} \\ \hat{d}^{(0\sim k-1)} \end{bmatrix} \\ u = -B^{-1}(y_m, \zeta, t)\left(f(y_m, \zeta, t) - \upsilon \right), \end{cases} \tag{4.60}$$

where

$$\begin{cases} K_{0\sim n-1} = -Z_1 F_1^n V^{-1}(F_1, Z_1) \\ V(F_1, Z_1) = \begin{bmatrix} Z_1 \\ Z_1 F_1 \\ \vdots \\ Z_1 F_1^{n-1} \end{bmatrix}, \end{cases} \tag{4.61}$$

and

$$\begin{cases} \Pi = \begin{bmatrix} Z_2 \\ Z_2 \Phi(S_{0\sim k-1}) \\ \vdots \\ Z_2 \Phi^{k-1}(S_{0\sim k-1}) \end{bmatrix} \\ \Gamma = -E + Z_2 \Phi^k(S_{0\sim k-1}), \end{cases} \tag{4.62}$$

with $F_1 \in \mathbb{C}^{nr\times nr}$, $Z_1 \in \mathbb{R}^{r\times nr}$ and $Z_2 \in \mathbb{R}^{r\times kp}$ being some parameter matrices. Applying Lemmas 4.1, 4.4 and 4.5, we can obtain the following result.

Theorem 4.3 *Consider FAS (4.45), with the disturbance $d(t)$ generated by (4.13), and let Assumptions 4.1 and 4.4 be met. If the control law is taken as (4.60)-(4.62), with the parameter matrices L_x, L_d, F_1, Z_1 and Z_2 satisfying the following conditions:*

1. *L_x and L_d are any matrices making*

$$A_{ce} = \begin{bmatrix} \Phi(0_{0\sim n-1}) & B_c E \\ 0 & \Phi(S_{0\sim k-1}) \end{bmatrix} + \begin{bmatrix} L_x \\ L_d \end{bmatrix} \begin{bmatrix} C_m & 0 \end{bmatrix} \tag{4.63}$$

   stable;

2. *the parameter F_1 is Hurwitz, and also satisfies, together with the parameter Z_1, the condition $\det V(F_1, Z_1) \neq 0$; and*

3. *the parameter Z_2 in the expressions of Π and Γ given by (4.62) satisfies*

$$C_r \begin{bmatrix} Z_2 \\ Z_2 \Phi(S_{0\sim k-1}) \\ \vdots \\ Z_2 \Phi^{k-1}(S_{0\sim k-1}) \end{bmatrix} + E_r = 0. \tag{4.64}$$

Then the following closed-loop system is internally asymptotically stable:

$$\begin{bmatrix} \dot{x}^{(0\sim n-1)} \\ \dot{\hat{x}}^{(0\sim n-1)} \\ \dot{\hat{d}}^{(0\sim k-1)} \end{bmatrix} = \begin{bmatrix} \Phi(0_{0\sim n-1}) & -B_c K_{0\sim n-1} & A_{c2} \\ -L_x C_m & A_{c1} & B_c E + A_{c2} \\ -L_d C_m & L_d C_m & \Phi(S_{0\sim k-1}) \end{bmatrix} \begin{bmatrix} x^{(0\sim n-1)} \\ \hat{x}^{(0\sim n-1)} \\ \hat{d}^{(0\sim k-1)} \end{bmatrix}, \tag{4.65}$$

where

$$A_{c1} = \Phi(0_{0\sim n-1}) + L_x C_m - B_c K_{0\sim n-1},$$
$$A_{c2} = B_c(\Gamma + K_{0\sim n-1}\Pi).$$

Furthermore, for any initial values $x^{(0\sim n-1)}(0)$, $\zeta(0)$ *and* $d^{(0\sim k-1)}(0)$, *the following asymptotic output regulation requirement is always met:*

$$\lim_{t\to\infty} y_r(t) = 0. \tag{4.66}$$

Proof. If we take the following control law

$$u = -B^{-1}(y_m, \zeta, t)(f(y_m, \zeta, t) - \upsilon), \tag{4.67}$$

then the corresponding closed-loop system is given by

$$\begin{cases} x^{(n)} = E d^{(0\sim k-1)} + \upsilon \\ y_m = C_m x^{(0\sim n-1)} \\ y_r = C_r x^{(0\sim n-1)} + E_r d^{(0\sim k-1)}. \end{cases} \tag{4.68}$$

The above equation (4.68) can be also equivalently rewritten in the following state-space form:

$$\begin{cases} \dot{x}^{(0\sim n-1)} = \Phi(0_{0\sim n-1}) x^{(0\sim n-1)} + B_c \upsilon + B_c E d^{(0\sim k-1)} \\ y_m = C_m x^{(0\sim n-1)} \\ y_r = C_r x^{(0\sim n-1)} + E_r d^{(0\sim k-1)}. \end{cases} \tag{4.69}$$

It is noted that the above (4.69) is now a constant linear system. Applying Lemma 4.4 to the above system (4.69), we can design the following controller υ to meet the asymptotic output regulation requirement (4.46):

$$\begin{cases} \begin{bmatrix} \dot{\hat{x}}^{(0\sim n-1)} \\ \dot{\hat{d}}^{(0\sim k-1)} \end{bmatrix} = \begin{bmatrix} \Phi(0_{0\sim n-1}) & B_c E \\ 0 & \Phi(S_{0\sim k-1}) \end{bmatrix} \begin{bmatrix} \hat{x}^{(0\sim n-1)} \\ \hat{d}^{(0\sim k-1)} \end{bmatrix} + \begin{bmatrix} B_c \\ 0 \end{bmatrix} \upsilon \\ \qquad\qquad + \begin{bmatrix} L_x \\ L_d \end{bmatrix} \left(C_m \hat{x}^{(0\sim n-1)} - y_m\right) \\ \upsilon = \begin{bmatrix} -K_{0\sim n-1} & \Gamma + K_{0\sim n-1}\Pi \end{bmatrix} \begin{bmatrix} \hat{x}^{(0\sim n-1)} \\ \hat{d}^{(0\sim k-1)} \end{bmatrix}, \end{cases} \tag{4.70}$$

where the matrices L_x and L_d satisfy the first condition in the theorem, $K_{0\sim n-1}$ makes the matrix $\Phi(K_{0\sim k-1})$ stable, while Π and Γ are two matrices satisfying the equations

$$\Phi(0_{0\sim n-1})\Pi + B_c\Gamma = \Pi\Phi(S_{0\sim k-1}) - B_c E, \tag{4.71}$$

and

$$C_r \Pi + E_r = 0. \tag{4.72}$$

Combining the above controller (4.70) with (4.67), gives controller (4.60). Further, it is easy to see that the following three facts hold:

1. according to Theorem 4.1, the matrix $K_{0\sim n-1}$ which makes $\Phi(K_{0\sim k-1})$ stable is given by (4.61) with the parameter F_1 Hurwitz and the parameter matrices F_1 and Z_1 satisfying the condition $\det V(F_1, Z_1) \neq 0$;

2. applying Lemma 4.5, with F and R substituted by $\Phi(S_{0\sim k-1})$ and $-E$, respectively, it is easily proven that the matrices Π and Γ given by (4.62) satisfy the above nonhomogeneous GSE (4.71); and

3. with the matrices Π and Γ given by (4.62), the matrix equation (4.72) is equivalent to condition (4.64).

With the above three facts, we immediately have the conclusion of the theorem. ■

Remark 4.4 The above theorem has provided a parametric approach for the problem of disturbance decoupling in FAS (4.45). The design DOFs are explicitly given by the parameters $F_1 \in \mathbb{C}^{nr\times nr}$, $Z_1 \in \mathbb{R}^{r\times nr}$ and $Z_2 \in \mathbb{R}^{r\times kp}$, which can be further properly selected to meet additional performance requirements of the closed-loop system. It should be also noted that explicit parametric solutions of L_x and L_d satisfying the first condition in the above Theorem 4.3 can be also readily established using state feedback ESA results for linear systems (Duan (1992b, 1993b, 1998, 2004, 2005)). We point out that there exist usually also certain DOFs in the solution of L_x and L_d satisfying the first condition in the theorem, and the case of dynamic state feedback will be further addressed in the next subsection.

4.3.3 Case of Dynamical State Feedback

In the case of $C_m = I_{nr}$, we have $y_m = x^{(0\sim n-1)}$, and controller (4.60) turns into a dynamical state feedback controller. In this subsection, we aim to further give a parametric solution to the design of L_x and L_d satisfying the first condition in Theorem 4.3 in the case of dynamical state feedback.

Firstly, let us give a necessary and sufficient condition for the observability (detectability) of the matrix pair (Φ_E, C_I) defined by

$$\Phi_E = \begin{bmatrix} \Phi(0_{0\sim n-1}) & B_c E \\ 0 & \Phi(S_{0\sim k-1}) \end{bmatrix},\ C_I = \begin{bmatrix} I_{nr} & 0 \end{bmatrix}. \tag{4.73}$$

Theorem 4.4 *The matrix pair (Φ_E, C_I) defined by (4.73) is observable (detectable) if and only if $(\Phi(S_{0\sim k-1}), E)$ is observable (detectable).*

Proof. In view of the expression of B_c given in (4.22), we have

$$\begin{aligned}
&\operatorname{rank}\begin{bmatrix} sI_{nr+kp}-\Phi_E \\ C_I \end{bmatrix} \\
=\ &\operatorname{rank}\begin{bmatrix} sI_{nr}-\Phi(0_{0\sim n-1}) & -B_cE \\ 0 & sI_{kp}-\Phi(S_{0\sim k-1}) \\ I_{nr} & 0 \end{bmatrix} \\
=\ &\operatorname{rank}\begin{bmatrix} 0 & B_cE \\ 0 & sI_{kp}-\Phi(S_{0\sim k-1}) \\ I_{nr} & 0 \end{bmatrix} \\
=\ &\operatorname{rank}\begin{bmatrix} 0 & 0 \\ 0 & E \\ 0 & sI_{kp}-\Phi(S_{0\sim k-1}) \\ I_{nr} & 0 \end{bmatrix}, \forall s\in\mathbb{C}.
\end{aligned} \tag{4.74}$$

This implies

$$\begin{aligned}
&\operatorname{rank}\begin{bmatrix} sI_{nr+kp}-\Phi_E \\ C_I \end{bmatrix} \\
=\ &nr+\operatorname{rank}\begin{bmatrix} sI_{kp}-\Phi(S_{0\sim k-1}) \\ E \end{bmatrix},\ \forall s\in\mathbb{C}.
\end{aligned} \tag{4.75}$$

Therefore, the conclusion holds according to the well-known PBH (Popov-Belevitch-Hautus) criterion. ■

If the matrix pair $(\Phi(S_{0\sim k-1}), E)$ is observable, then it follows from Duan (2015) that there exists a pair of polynomial matrices $N(s)\in\mathbb{R}^{kp\times nr}[s]$ and $D(s)\in\mathbb{R}^{nr\times nr}[s]$ satisfying the following generalized right coprime factorization (RCF):

$$\left[sI_{kp}-\Phi^{\mathrm{T}}(S_{0\sim k-1})\right]N(s)=E^{\mathrm{T}}D(s). \tag{4.76}$$

If we denote $D(s)=\left[d_{ij}(s)\right]$ and

$$\omega=\max\left\{\deg\left(d_{ij}(s)\right),\ i,j=1,2,\ldots,nr\right\}, \tag{4.77}$$

then $N(s)$ and $D(s)$ can be written in the form of

$$\begin{cases} N(s)=\displaystyle\sum_{i=0}^{\omega}N_is^i,\ N_i\in\mathbb{R}^{kp\times nr} \\ D(s)=\displaystyle\sum_{i=0}^{\omega}D_is^i,\ D_i\in\mathbb{R}^{nr\times nr}. \end{cases} \tag{4.78}$$

With the above preparations, the following result can be stated.

Theorem 4.5 *Suppose that Φ_E and C_I are given by (4.73) and $(\Phi(S_{0\sim k-1}), E)$ is observable. Let $N(s)$ and $D(s)$ be a pair of right coprime polynomial matrices in the*

form of (4.78) and satisfy RCF (4.76), and $F_0 \in R^{(nr+kp)\times(nr+kp)}$ *be a given Hurwitz matrix. Then all the gain matrices* L_x *and* L_d *and the nonsingular matrix* T *satisfying*

$$\left(\Phi_E + \begin{bmatrix} L_x \\ L_d \end{bmatrix} C_I\right)^{\mathrm{T}} = T F_0 T^{-1}, \tag{4.79}$$

can be given by

$$\begin{cases} \begin{bmatrix} L_x \\ L_d \end{bmatrix} = T^{-\mathrm{T}} G^{\mathrm{T}} \\ T = \sum_{i=0}^{\omega} \begin{bmatrix} D_i \\ N_i \end{bmatrix} Z_0 F_0^i \\ G = \sum_{i=0}^{\omega+1} \left[D_{i-1} - \Phi^{\mathrm{T}}\left(0_{0\sim n-1}\right) D_i\right] Z_0 F_0^i, \end{cases} \tag{4.80}$$

where

$$D_{\omega+1} = D_{-1} = 0, \tag{4.81}$$

and $Z_0 \in \mathbb{R}^{nr\times(nr+kp)}$ *is a parameter matrix satisfying*

$$\det\left(\sum_{i=0}^{\omega} \begin{bmatrix} D_i \\ N_i \end{bmatrix} Z_0 F_0^i\right) \neq 0. \tag{4.82}$$

Proof. If we take

$$\hat{N}(s) = \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} \tag{4.83}$$

and

$$\hat{D}(s) = \left[s I_{nr} - \Phi^{\mathrm{T}}\left(0_{0\sim n-1}\right)\right] D(s), \tag{4.84}$$

then it can be easily verified using relation (4.76) that

$$\left(s I_{nr+kp} - \Phi_E^{\mathrm{T}}\right) \hat{N}(s) = C_I^{\mathrm{T}} \hat{D}(s). \tag{4.85}$$

Further note that

$$\begin{bmatrix} \hat{N}(s) \\ \hat{D}(s) \end{bmatrix} = \begin{bmatrix} D(s) \\ N(s) \\ \left(s I_{nr} - \Phi^{\mathrm{T}}\left(0_{0\sim n-1}\right)\right) D(s) \end{bmatrix}, \tag{4.86}$$

$\hat{N}(s)$ and $\hat{D}(s)$ are clearly right coprime since $D(s)$ and $N(s)$ are right coprime. Thus the above (4.85) is a generalized RCF of the matrix pair $\left(\Phi_E^{\mathrm{T}}, C_I^{\mathrm{T}}\right)$.

In view of (4.81), $\hat{N}(s)$ and $\hat{D}(s)$ can be obviously rewritten into the form of

$$\hat{N}(s) = \sum_{i=0}^{\omega} \begin{bmatrix} D_i \\ N_i \end{bmatrix} s^i, \tag{4.87}$$

and

$$\begin{aligned}\hat{D}(s) &= \left[sI_{nr} - \Phi^{\mathrm{T}}(0_{0\sim n-1})\right]\sum_{i=0}^{\omega} D_i s^i \\ &= \sum_{i=0}^{\omega+1}\left[D_{i-1} - \Phi^{\mathrm{T}}(0_{0\sim n-1})\, D_i\right]s^i, \end{aligned} \tag{4.88}$$

respectively. Thus it follows from the Theorems 2.1 and 4.3 in Duan (2015) that all the matrices L_x and L_d satisfying (4.79) can be given by (4.80), where $Z_0 \in \mathbb{R}^{nr\times(nr+kp)}$ is a parameter matrix satisfying constraint (4.82). Then the proof is completed. ■

Remark 4.5 Note that, in the case of $C_m = I$, we have

$$\begin{aligned} A_{ce} &= \begin{bmatrix} \Phi(0_{0\sim n-1}) & B_c E \\ 0 & \Phi(S_{0\sim k-1}) \end{bmatrix} + \begin{bmatrix} L_x \\ L_d \end{bmatrix}\begin{bmatrix} C_m & 0 \end{bmatrix} \\ &= \Phi_E + \begin{bmatrix} L_x \\ L_d \end{bmatrix} C_I. \end{aligned} \tag{4.89}$$

In this case, the above theorem gives a parametric approach to design the gains L_x and L_d satisfying the first condition in Theorem 4.3. Combining Theorems 4.3 and 4.5 gives a complete parametric approach for asymptotic disturbance decoupling via dynamical state feedback. The complete explicit DOFs are composed of the parameters F_0 and F_1, and Z_i, $i = 0, 1, 2$. These DOFs can be further properly utilized to achieve additional system performance requirements (Duan (1992a, 1993a); Duan et al. (2000); Duan and Zhao (2020)).

Remark 4.6 This whole book only provides a demonstration of the global FAS approach, while control of sub-FASs is actually a more important investigation topic of the FAS approach. Certain ideas about the control of sub-FASs have been addressed in some of the author's papers, e.g., Duan (2021g, 2023b,c,e), while a more comprehensive introduction to the sub-FAS approach is given in the second volume of the book (Duan (2025g)). Eventually, disturbance attenuation and decoupling in sub-FASs is certainly one of the important research topics.

4.4 An Illustrative Example

Let us consider a coarse-fine tracking system shown in Fig. 4.1, the design objective is to attenuate (or decouple) the effect of the disturbance d, and let the final angle θ_2 track a constant reference signal (which can be set to zero without loss of generality). The function of the coarse system is to acquisite the object and provide a "good" input for the fine system.

4.4.1 The FAS Model

According to Fig. 4.1, the original model of the coarse tracking subsystem can be set up as

$$\begin{cases} c_1(d+\mu\dot{d}+u_1)=\dot{\tau}_1+a_1\tau_1 \\ \ddot{\theta}_1+D_1(\theta_1,\dot{\theta}_1)\dot{\theta}_1+K_1(\theta_1)=\tau_1. \end{cases} \tag{4.90}$$

Taking differentials to the second equation in (4.90), gives

$$\begin{aligned} \dot{\tau}_1 &= \frac{\mathrm{d}}{\mathrm{d}t}\ddot{\theta}_1+\frac{\mathrm{d}}{\mathrm{d}t}\left(D_1\left(\theta_1,\dot{\theta}_1\right)\dot{\theta}_1\right)+\frac{\mathrm{d}}{\mathrm{d}t}K_1(\theta_1) \\ &= \dddot{\theta}_1+\frac{\mathrm{d}}{\mathrm{d}t}\left(D_1\left(\theta_1,\dot{\theta}_1\right)\right)\dot{\theta}_1+D_1\left(\theta_1,\dot{\theta}_1\right)\ddot{\theta}_1+\frac{\mathrm{d}}{\mathrm{d}t}K_1(\theta_1) \\ &= \dddot{\theta}_1+\frac{\partial D_1}{\partial\theta_1}\dot{\theta}_1^2+\frac{\partial D_1}{\partial\dot{\theta}_1}\ddot{\theta}_1\dot{\theta}_1+D_1\left(\theta_1,\dot{\theta}_1\right)\ddot{\theta}_1+\frac{\mathrm{d}K_1}{\mathrm{d}\theta_1}\dot{\theta}_1. \end{aligned} \tag{4.91}$$

Substituting the above equation (4.91), together with the second equation in (4.90), into the first one in (4.90), yields

$$\begin{aligned} \dddot{\theta}_1 &= -\frac{\partial D_1}{\partial\theta_1}\dot{\theta}_1^2-\frac{\partial D_1}{\partial\dot{\theta}_1}\ddot{\theta}_1\dot{\theta}_1-D_1(\theta_1,\dot{\theta}_1)\ddot{\theta}-\frac{\mathrm{d}K_1}{\mathrm{d}\theta_1}\dot{\theta}_1 \\ &\quad +c_1(d+\mu\dot{d}+u_1)-a_1\tau_1 \\ &= -\frac{\partial D_1}{\partial\theta_1}\dot{\theta}_1^2-\frac{\partial D_1}{\partial\dot{\theta}_1}\ddot{\theta}_1\dot{\theta}_1-D_1(\theta_1,\dot{\theta}_1)\ddot{\theta}_1-\frac{\mathrm{d}K_1}{\mathrm{d}\theta_1}\dot{\theta}_1 \\ &\quad +c_1(d+\mu\dot{d}+u_1)-a_1(\ddot{\theta}_1+D_1(\theta_1,\dot{\theta}_1)\dot{\theta}_1+K_1(\theta_1)), \end{aligned}$$

which can be further arranged into the following FAS model for the coarse subsystem:

$$\dddot{\theta}_1=f_1\left(\theta_1^{(0\sim2)}\right)+c_1u_1+c_1d+c_1\mu\dot{d}, \tag{4.92}$$

where

$$\begin{aligned} f_1\left(\theta_1^{(0\sim2)}\right) &= -\frac{\partial D_1}{\partial\theta_1}\dot{\theta}_1^2-\frac{\partial D_1}{\partial\dot{\theta}_1}\ddot{\theta}_1\dot{\theta}_1-D_1\left(\theta_1,\dot{\theta}_1\right)\ddot{\theta}_1-\frac{\mathrm{d}K_1}{\mathrm{d}\theta_1}\dot{\theta}_1 \\ &\quad -a_1\ddot{\theta}_1-a_1D_1\left(\theta_1,\dot{\theta}_1\right)\dot{\theta}_1-a_1K_1(\theta_1). \end{aligned} \tag{4.93}$$

In parallel, the original model for the fine tracking system is given by

$$\begin{cases} c_2u_2-c_2\theta_1=\dot{\tau}_2+a_2\tau_2 \\ M_2(\theta_2)\ddot{\theta}_2+D_2\left(\theta_2,\dot{\theta}_2\right)\dot{\theta}_2+K_2(\theta_2)=\tau_2. \end{cases} \tag{4.94}$$

Through a similar process, the FAS model for the fine subsystem can be also obtained, as,

$$\ddot{\theta}_2=f_2\left(\theta_2^{(0\sim2)}\right)-c_2M_2^{-1}(\theta_2)\theta_1+c_2M_2^{-1}(\theta_2)u_2, \tag{4.95}$$

where

$$f_2\left(\theta_2^{(0\sim2)}\right)=-M_2^{-1}(\theta_2)\,\tilde{f}_2\left(\theta_2^{(0\sim2)}\right), \tag{4.96}$$

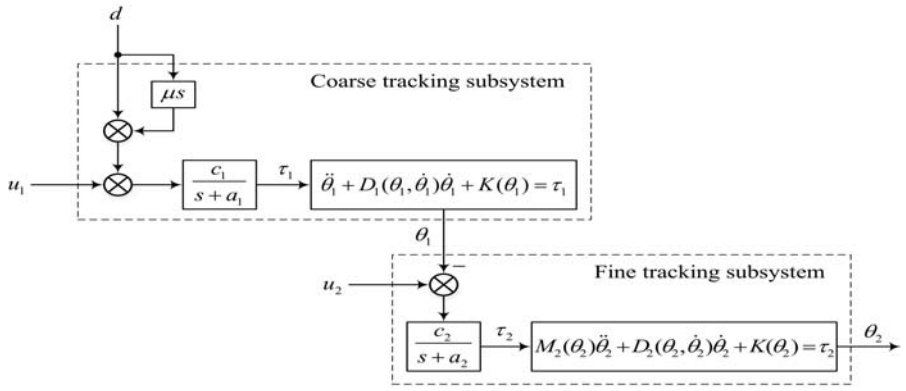


Figure 4.1: The coarse-fine tracking system

with

$$\begin{aligned}\tilde{f}_2\left(\theta_2^{(0\sim 2)}\right) &= \frac{\partial M_2}{\partial \theta_2}\dot{\theta}_2\ddot{\theta}_2+\frac{\partial D_2}{\partial \theta_2}\dot{\theta}_2^2+\frac{\partial D_2}{\partial \dot{\theta}_2}\ddot{\theta}_2\dot{\theta}_2\\ &\quad +D_2\left(\theta_2,\dot{\theta}_2\right)\ddot{\theta}_2+\frac{\mathrm{d}K_2}{\mathrm{d}\theta_2}\dot{\theta}_2+a_2M_2\left(\theta_2\right)\ddot{\theta}_2\\ &\quad +a_2D_2\left(\theta_2,\dot{\theta}_2\right)\dot{\theta}_2+a_2K_2\left(\theta_2\right).\end{aligned} \tag{4.97}$$

Defining

$$\theta=\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix},\ u=\begin{bmatrix}u_1\\ u_2\end{bmatrix}, \tag{4.98}$$

and combining (4.92) and (4.95), give the following entire FAS model for the whole system:

$$\ddot{\theta}=f\left(\theta_{1\sim 2}^{(0\sim 2)}\right)+B\left(\theta_2\right)u+E_0d+E_1\dot{d}, \tag{4.99}$$

where

$$f\left(\theta_{1\sim 2}^{(0\sim 2)}\right) = \begin{bmatrix} f_1\left(\theta_1^{(0\sim 2)}\right)\\ f_2\left(\theta_2^{(0\sim 2)}\right)-c_2M_2^{-1}\left(\theta_2\right)\theta_1\end{bmatrix}, \tag{4.100}$$

$$B\left(\theta_2\right) = \begin{bmatrix} c_1 & 0\\ 0 & c_2M_2^{-1}\left(\theta_2\right)\end{bmatrix}, \tag{4.101}$$

and

$$E_0=\begin{bmatrix}c_1\\ 0\end{bmatrix},\ E_1=\begin{bmatrix}c_1\mu\\ 0\end{bmatrix}. \tag{4.102}$$

In the rest of this section, we assume

$$c_1=c_2=a_1=a_2=1,\ \mu=0.5,$$

and

$$\begin{cases} D_1\left(\theta_1, \dot{\theta}_1\right) = 20\dot{\theta}_1 \sin\theta_1 \\ K_1\left(\theta_1\right) = 15\cos\theta_1 \\ M_2\left(\theta_2\right) = 2 + \cos\theta_2 \\ D_2\left(\theta_2, \dot{\theta}_2\right) = \dot{\theta}_2 \sin\theta_2 \\ K_2\left(\theta_2\right) = 2\cos\theta_2. \end{cases} \tag{4.103}$$

4.4.2 Disturbance Attenuation

In this subsection, it is required that the effect of the disturbance d on the angle θ_2 is as small as possible, thus the output equation is assumed to be

$$y = \theta_2. \tag{4.104}$$

Therefore, the system to be considered is

$$\begin{cases} \ddot{\theta} = f\left(\theta^{(0\sim 2)}\right) + B\left(\theta_2\right)u + E_0 d' \\ y = \theta_2, \end{cases} \tag{4.105}$$

where E_0 is given by (4.102), and

$$d' = d + \mu\dot{d}. \tag{4.106}$$

The control law for system (4.105) is designed as

$$u = -B^{-1}\left(\theta_2\right)\left[A_{0\sim 2}\theta^{(0\sim 2)} + f\left(\theta^{(0\sim 2)}\right) - \upsilon\right], \tag{4.107}$$

and the corresponding closed-loop system can be written as

$$\begin{cases} \dddot{\theta} + A_2\ddot{\theta} + A_1\dot{\theta} + A_0\theta = \upsilon + E_0 d' \\ y = \theta_2, \end{cases} \tag{4.108}$$

which can be also written in the following state-space form:

$$\begin{cases} \dot{\theta}^{(0\sim 2)} = \Phi\left(A_{0\sim 2}\right)\theta^{(0\sim 2)} + B_c\upsilon + B_c E_0 d' \\ y = C\theta^{(0\sim 2)}, \end{cases} \tag{4.109}$$

where

$$C = \begin{bmatrix} 0 & 1 & 0_{1\times 4} \end{bmatrix}, \quad B_c = \begin{bmatrix} 0_{4\times 2} \\ I_2 \end{bmatrix}. \tag{4.110}$$

Parameterization of $A_{0\sim 2}$

Choose

$$F = \operatorname{diag}\left(-a_1, -a_2, -a_3, -a_4, \begin{bmatrix} -\alpha & \beta \\ -\beta & -\alpha \end{bmatrix}\right), \tag{4.111}$$

where $\alpha, \beta > 0$ and $a_i > 0,\ i = 1, 2, 3, 4$. Then it follows from Theorem 4.1 that all the matrix $A_{0\sim 2}$ and the nonsingular matrix $V \in \mathbb{R}^{6\times 6}$ satisfying

$$\Phi\left(A_{0\sim 2}\right) = VFV^{-1} \tag{4.112}$$

are given by

$$A_{0\sim 2} = -ZF^3V^{-1}(Z,F), \tag{4.113}$$

$$V(Z,F) = \begin{bmatrix} Z \\ ZF \\ ZF^2 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \cdots & v_6 \end{bmatrix}, \tag{4.114}$$

where

$$v_i = \begin{bmatrix} z_{1i} \\ z_{2i} \\ -a_i z_{1i} \\ -a_i z_{2i} \\ a_i^2 z_{1i} \\ a_i^2 z_{2i} \end{bmatrix}, \ i = 1,2,3,4, \tag{4.115}$$

$$v_5 = \begin{bmatrix} z_{15} \\ z_{25} \\ -\beta z_{16} - \alpha z_{15} \\ -\beta z_{26} - \alpha z_{25} \\ -z_{15}^2\beta^2 + 2z_{16}\alpha\beta + z_{15}\alpha^2 \\ -z_{25}^2\beta^2 + 2z_{26}\alpha\beta + z_{25}\alpha^2 \end{bmatrix}, \tag{4.116}$$

$$v_6 = \begin{bmatrix} z_{16} \\ z_{26} \\ \beta z_{15} - \alpha z_{16} \\ \beta z_{25} - \alpha z_{26} \\ -z_{16}\beta^2 - 2z_{15}\beta + z_{16}\alpha^2 \\ -z_{26}\beta^2 - 2z_{25}\alpha\beta + z_{26}\alpha^2 \end{bmatrix}, \tag{4.117}$$

and

$$Z = \begin{bmatrix} z_{11} & z_{12} & z_{13} & z_{14} & z_{15} & z_{16} \\ z_{21} & z_{22} & z_{23} & z_{24} & z_{25} & z_{26} \end{bmatrix} \tag{4.118}$$

is a parameter matrix satisfying

$$\det V(Z,F) \neq 0. \tag{4.119}$$

Parameter optimization

It follows from Theorem 4.2 that the H_2 norm of the transfer function

$$G_{d'y}(s) = C(sI_6 - \Phi(A_{0\sim 2}))^{-1} B_c E_0 \tag{4.120}$$

is given by

$$\tilde{J}(Z,F) = \left\| G_{d'y}(s) \right\|_2^2 = CVP_1^* V^{\mathrm{T}} C^{\mathrm{T}}, \tag{4.121}$$

where

$$P_1^* = -\sum_{i=0}^{5} F_i V^{-1} B_c E_0 E_0^{\mathrm{T}} B_c^{\mathrm{T}} V^{-\mathrm{T}} P_{10}^{-1} \left(F^{\mathrm{T}}\right)^i, \tag{4.122}$$

with

$$P_{10} = \sum_{i=0}^{6} f_i \left(F^{\mathrm{T}}\right)^i, \tag{4.123}$$

and the variables f_i, $i = 1 \sim 6$ and F_i, $i = 1 \sim 5$ are determined by

$$\det(sI_6 + F) = \sum_{i=0}^{6} f_i s^i, \tag{4.124}$$

$$\mathrm{adj}(sI_6 + F) = \sum_{i=0}^{5} F_i s^i. \tag{4.125}$$

In order to give simultaneously a smaller control input, we also include the norm of the coefficient matrix $A_{0\sim 2}$ in the optimization index, and this leads to the following index:

$$J(Z, F) = \alpha_1 \tilde{J}(Z, F) + \alpha_2 \|A_{0\sim 2}\|, \tag{4.126}$$

where $\alpha_1 = 10^9$ and $\alpha_2 = 2 \times 10^{-6}$ are the selected weight factors. Solving the following optimization problem

$$\begin{array}{ll} \min & J(Z, F) \\ \text{s.t.} & \det V(Z, F) \neq 0 \\ & a_i^{\min} \leq a_i \leq a_i^{\max},\ i = 1, 2, 3, 4 \\ & \alpha_{\min} \leq \alpha \leq \alpha_{\max} \\ & \beta_{\min} \leq \beta \leq \beta_{\max}, \end{array} \tag{4.127}$$

with

$$\begin{cases} a_1^{\min} = 1,\ a_1^{\max} = 1.5 \\ a_2^{\min} = 10,\ a_2^{\max} = 15 \\ a_3^{\min} = 25,\ a_3^{\max} = 35 \\ a_4^{\min} = 75,\ a_4^{\max} = 115 \\ \alpha_{\min} = 120,\ \alpha_{\max} = 180 \\ \beta_{\min} = 550,\ \beta_{\max} = 820, \end{cases} \tag{4.128}$$

we obtain a sub-optimal solution as

$$\begin{cases} Z = \begin{bmatrix} Z_1 & Z_2 \end{bmatrix} \\ Z_1 = \begin{bmatrix} -51.4807 & 139.1198 & 225.5578 \\ 25.0825 & 61.6422 & 148.3187 \end{bmatrix} \\ Z_2 = \begin{bmatrix} -128.3527 & -132.9885 & 2.4666 \\ 107.6435 & 12.7195 & -43.1168 \end{bmatrix}, \end{cases} \tag{4.129}$$

and

$$\begin{cases} F = \mathrm{diag}\left(-a_1, -a_2, -a_3, -a_4, \begin{bmatrix} -\alpha & \beta \\ -\beta & -\alpha \end{bmatrix}\right) \\ a_1 = 1.15987,\ a_2 = 10.69849 \\ a_3 = 24.98112,\ a_4 = 96.37437 \\ \alpha = 132.58205,\ \beta = 550.26164. \end{cases} \tag{4.130}$$

Corresponding to these optimized values, the set of closed-loop poles are

$$\begin{array}{l} s_1 = -1.15987,\ s_2 = -10.69849, \\ s_3 = -24.98112,\ s_4 = -96.37437, \\ s_{5,6} = -132.58205 \pm 550.26164\,\mathrm{i}, \end{array} \tag{4.131}$$

and

$$\left\| G_{d'y}(s) \right\|_2 = 3.1215 \times 10^{-5}. \tag{4.132}$$

Comparative simulation results

Let the initial values be taken as

$$\theta_1(0) = 1,\ \theta_2(0) = -0.5, \tag{4.133}$$

$$\dot{\theta}_1(0) = \ddot{\theta}_1(0) = \dot{\theta}_2(0) = \ddot{\theta}_2(0) = 0. \tag{4.134}$$

In order to clearly see the effect of disturbance attenuation, we have taken a large disturbance d as

$$d = 10^4 \times (1 + 0.5 \sin 2t). \tag{4.135}$$

Then the simulation of the system is carried out, and the results are shown in Figs. 4.2 and 4.3.

In view of

$$\Phi(A_{0\sim 2}) = \Phi(0_{0\sim 2}) - B_c A_{0\sim 2}, \tag{4.136}$$

pole assignment methods can be also applied to design the coefficient matrix $A_{0\sim 2}$ in control law (4.107). For comparison, we also obtained another solution of $A_{0\sim 2}$ using the `place` function. To make a fair comparison, the closed-loop poles are taken as in (4.131), and the gain matrix $A_{0\sim 2}$ is obtained as

$$\left\{ \begin{array}{l} A_{0\sim 2} = \left[\begin{array}{ccc} A_0 & A_1 & A_2 \end{array} \right] \\ A_0 = \left[\begin{array}{cc} 7.5039 \times 10^4 & 3.4648 \times 10^5 \\ -1.0785 \times 10^5 & -3.7044 \times 10^5 \end{array} \right] \\ A_1 = \left[\begin{array}{cc} 1.2860 \times 10^4 & 3.7057 \times 10^4 \\ -3.6006 \times 10^4 & -2.7607 \times 10^4 \end{array} \right] \\ A_2 = \left[\begin{array}{cc} -1.6119 \times 10^2 & 7.4253 \times 10^2 \\ -4.1287 \times 10^2 & -2.3719 \times 10^2 \end{array} \right]. \end{array} \right. \tag{4.137}$$

Simulation corresponding to the above solution is also carried out, and the results are also shown in Figs. 4.2 and 4.3 for comparison. It is clearly seen that the proposed approach is much more effective in the sense of disturbance attenuation and control effort consumption.

4.4.3 Asymptotic Disturbance Decoupling

In this subsection, the measurement output and the regulated output are assumed to be

$$y_m = \theta^{(0\sim 2)}, \tag{4.138}$$

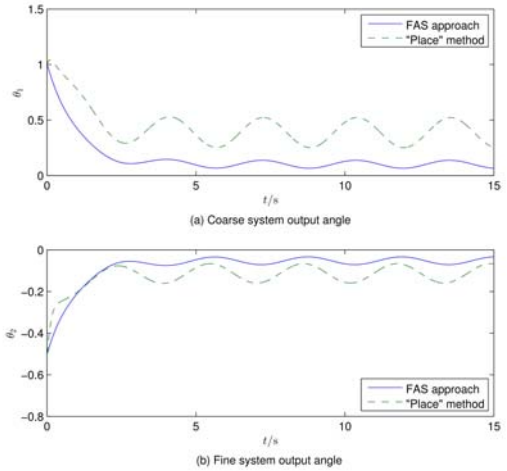


Figure 4.2: The regulated angles

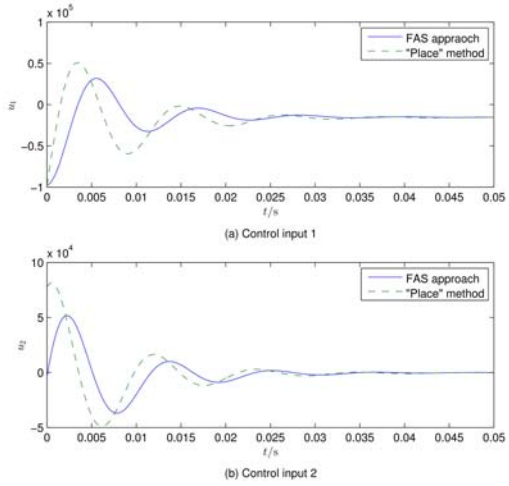


Figure 4.3: The control inputs

and

$$y_r = \theta_2 + d + 0.1\dot{d}, \tag{4.139}$$

respectively. These correspond to

$$C_m = I_6,\ C_r = \begin{bmatrix} 0 & 1 & 0_{1\times 4} \end{bmatrix}. \tag{4.140}$$

Taking the above two output equations into consideration, we obtain the following FAS:

$$\begin{cases} \ddot{\theta} = f\left(\theta^{(0\sim 2)}\right) + B(\theta_2)u + E_{0\sim 1}d^{(0\sim 1)} \\ y_r = C_r\theta^{(0\sim 2)} + E_r d^{(0\sim 1)}(t), \end{cases} \tag{4.141}$$

where

$$E_{0\sim 1} = \begin{bmatrix} E_0 & E_1 \end{bmatrix},\ E_r = \begin{bmatrix} 1 & 0.1 \end{bmatrix}. \tag{4.142}$$

The disturbance d is assumed to be generated by the following exogenous system

$$\ddot{d} = -d = -S_{0\sim 1}d^{(0\sim 1)}, \tag{4.143}$$

with

$$S_{0\sim 1} = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{4.144}$$

Design of Π and Γ

According to Theorem 4.3, a parametric expression of the matrices Π and Γ in control law (4.60) can be given by

$$\begin{cases} \Pi = \begin{bmatrix} Z \\ Z\Phi(S_{0\sim 1}) \\ Z[\Phi(S_{0\sim 1})]^2 \end{bmatrix} \\ \Gamma = -E_{0\sim 1} + Z\Phi^3(S_{0\sim 1}), \end{cases} \tag{4.145}$$

with the real parameter matrix

$$Z = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \tag{4.146}$$

satisfying

$$C_r \begin{bmatrix} Z \\ Z\Phi(S_{0\sim 1}) \\ Z[\Phi(S_{0\sim 1})]^2 \end{bmatrix} = 0. \tag{4.147}$$

Solving the above constraint (4.147), gives

$$z_{21} = z_{22} = 0, \tag{4.148}$$

thus Z can be taken as

$$Z = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}. \tag{4.149}$$

Substituting (4.149) into (4.145), gives

$$\Pi = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ -2 & 1 \\ 0 & 0 \\ -1 & -2 \\ 0 & 0 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 1 & -1.5 \\ 0 & 0 \end{bmatrix}. \tag{4.150}$$

Other parameters

A general parametric solution of the matrix $K_{0\sim 2}$ is given by (4.61). For simplicity, here only a specific solution is given, as

$$K_{0\sim 2} = \begin{bmatrix} K^1_{0\sim 2} & K^2_{0\sim 2} \end{bmatrix}, \tag{4.151}$$

where

$$K^1_{0\sim 2} = \begin{bmatrix} 30.4907 & -17.2697 & 33.0225 \\ -15.6643 & 32.4859 & -10.0397 \end{bmatrix}, \tag{4.152}$$

$$K^2_{0\sim 2} = \begin{bmatrix} -10.9311 & 10.4463 & -1.5547 \\ 33.9725 & -1.4422 & 10.5537 \end{bmatrix}. \tag{4.153}$$

The corresponding set of eigenvalues is

$$\mathrm{eig}\left(\Phi\left(K_{0\sim 2}\right)\right) = \{-1, -2, \cdots, -6\}. \tag{4.154}$$

The parametric solutions of the gain matrices L_x and L_d can be obtained using Theorem 4.5. Again, for simplicity only a specific pair of solutions is given, as follows:

$$L_x = \begin{bmatrix} -12 & 0 & -1 & 0 & 0 & 0 \\ 0 & -13 & 0 & -1 & 0 & 0 \\ 0 & 0 & -14 & 0 & -1 & 0 \\ 0 & 0 & 0 & -11 & 0 & -1 \\ 0 & 0 & 0 & 0 & -25 & 0 \\ 0 & 0 & 0 & 0 & 0 & -9 \end{bmatrix}, \tag{4.155}$$

$$L_d = \begin{bmatrix} 0 & 0 & 0 & 0 & 50 & 0 \\ 0 & 0 & 0 & 0 & -510 & 0 \end{bmatrix}. \tag{4.156}$$

In such a case, the set of eigenvalues of A_{ce} is

$$\mathrm{eig}\left(A_{ce}\right) = \{-7, -8, \cdots, -14\}. \tag{4.157}$$

Simulation results

According to Theorem 4.3, the control law is finally designed as

$$\begin{cases} \begin{bmatrix} \dot{\hat{\theta}}^{(0\sim 2)} \\ \dot{\hat{d}}^{(0\sim 1)} \end{bmatrix} = \begin{bmatrix} \Phi(0_{0\sim 2}) & B_c E_{0\sim 1} \\ 0 & \Phi(S_{0\sim 1}) \end{bmatrix} \begin{bmatrix} \hat{\theta}^{(0\sim 2)} \\ \hat{d}^{(0\sim 1)} \end{bmatrix} + \begin{bmatrix} B_c \\ 0 \end{bmatrix} \upsilon \\ \qquad\qquad + \begin{bmatrix} L_x \\ L_d \end{bmatrix} \left(\hat{\theta}^{(0\sim 2)} - \theta^{(0\sim 2)} \right) \\ \upsilon = \begin{bmatrix} -K_{0\sim 2} & \Gamma + K_{0\sim 2}\Pi \end{bmatrix} \begin{bmatrix} \hat{\theta}^{(0\sim 2)} \\ \hat{d}^{(0\sim 1)} \end{bmatrix} \\ u = -B^{-1}(\theta_2) \left(f\left(\theta^{(0\sim 2)}\right) - \upsilon \right), \end{cases} \tag{4.158}$$

where Π and Γ are given by (4.150), $K_{0\sim 2}$ is given by (4.151)-(4.153), and L_x and L_d are given by (4.155) and (4.156), respectively.

When the initial values of the angles are taken as

$$\theta_1(0) = 1,\ \theta_2(0) = -0.5, \tag{4.159}$$

with other system initial values being zeros, simulation of the system is carried out and the results are shown in Figs. 4.4 and 4.5. It is clearly seen that $\theta_2(t)$ approaches to zero with a quite accurate precision in spite of the big sine disturbance. The magnitudes of the control inputs are relatively large because they need to take the burden to compensate the strong nonlinearities in the system.

4.5 Notes and References

Firstly, let us comment on some of the results on disturbance attenuation and decoupling.

4.5.1 Disturbance Attenuation

There are many excellent results on disturbance attenuation in the history. In Huang (1995), a robust control law is proposed for a type of uncertain nonlinear systems, which can realize local asymptotic tracking and disturbance rejection regardless of certain plant uncertainties. A similar problem is also considered for fully actuated passive mechanical systems in Jayawardhana and Weiss (2008).

H_∞ and H_2 techniques are two of the mainstream methods to solve the problem of disturbance attenuation. In the celebrated works, Zhou et al. (1994); Doyle et al. (1994), the mixed H_2 and H_∞ performance control for linear systems is transformed into a problem of solving a type of Riccati equations. Another type of solution is the LMI approach. In fact, the H_∞ and H_2 control problems are treated by many scholars through solving a group of LMIs, and numerous systematic results are obtained (see Duan and Yu (2013) and the references therein). Regarding nonlinear systems, H_∞ control problem is considered for the case of measurement output feedback in Isidori and Astolfi (1992); Ball et al. (1993). Wang et al. (2009, 2012) develop a robust H_∞

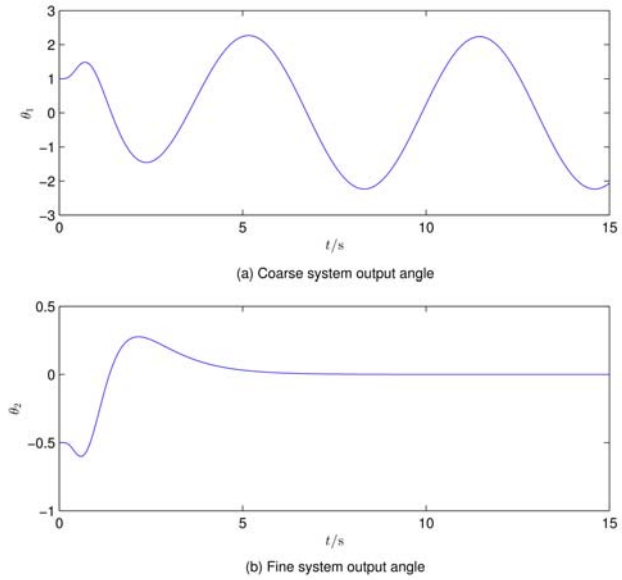


Figure 4.4: The regulated angles

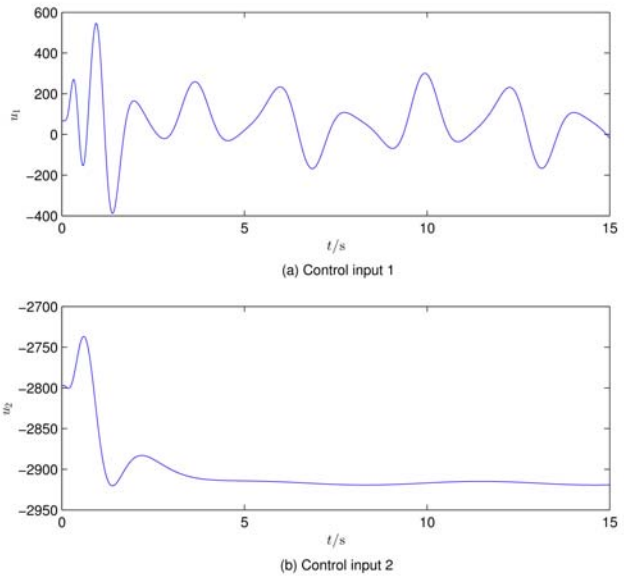


Figure 4.5: The control inputs

controller and a quantized H_∞ controller, respectively, for a class of nonlinear discrete time-delay stochastic systems with missing measurements.

Another type of approaches for disturbance attenuation in linear systems is the so-called parametric approaches. These include the case of model reference tracking (Huang et al. (2006)), and dynamic compensator (Duan et al. (2000)).

Active disturbance rejection control is another effective way to solve the problem of disturbance attenuation, and many valuable results have been obtained in the past decades (see the review paper Huang and Xue (2014) and the references therein).

4.5.2 Disturbance Decoupling

Regarding disturbance decoupling, there are also many reported results. For linear systems, a series of systematic methods are presented in the book Saberi et al. (2012). For nonlinear systems, the problem of disturbance decoupling may be originally proposed and investigated in Huijberts et al. (1992) by making an analogy with the dynamic input-output decoupling problem, and a local solution to this problem is obtained in the case that the system under consideration is invertible. The case of output feedback is further considered for continuous- and discrete-time nonlinear systems in Andiarti and Moog (1996) and Kaldmae et al. (2013), respectively. Furthermore, parametric approaches have also been proposed for disturbance decoupling. For instance, Duan et al. (1997) proposed a parametric design method for the problem of disturbance decoupling and simultaneously applied it to fault diagnosis.

It is well-known that, for many nonlinear systems, global stabilization can seldom be achieved with the commonly used state-space models of the dynamical systems, while disturbance attenuation and decoupling are generally dependent on the stabilization of the system. Therefore, many disturbance attenuation and decoupling results for nonlinear control systems, such as Huang (1995), Isidori and Astolfi (1992) and Ran et al. (2020), can only be obtained in a local sense.

4.5.3 Further Comments

Secondly, let us make some further comments.

In a wide sense, the problem of automatic control is, to a large extent, to realize disturbance rejection. Disturbance rejection has remained a long-standing difficult problem in control systems design. Particularly, for nonlinear systems the problem appears even harder, since it heavily depends on the stabilization of the systems, while it is well-known that, for many nonlinear systems, global stabilization can seldom be achieved with the commonly used state-space model representation of the dynamical systems.

It has been shown in former chapters that many dynamical control systems can be represented by FAS models. It is demonstrated in this chapter that, with the proposed FAS approach, the problems of disturbance attenuation and decoupling in a nonlinear system can be conveniently converted into corresponding ones in a linear system. Based on this great feature, parametric approaches for disturbance attenuation and decoupling in nonlinear systems are easily proposed. The content of this chapter is closely related to Duan (2021f).

For disturbance attenuation, it is shown that a parametric state feedback controller exists, with which the H_2 norm of the disturbance-to-output transfer function of the closed-loop system can also be established and hence minimized. For asymptotic disturbance decoupling, it is shown that a parametric solution can be well established by using complete parametric solutions to a type of homogeneous and nonhomogeneous GSEs.

The proposed results and methodology can be further generalized in several directions. Particularly, disturbance compensation based on disturbance estimators is also worth investigating, disturbance rejection problems for discrete-time systems and time-delay systems can be similarly treated. Like the treatment in Duan (2021c,d), method of high-order backstepping can also be proposed for high-order SFSs with disturbances. Furthermore, disturbance rejection in sub-FASs certainly remains to be a very important future research topic.

Disturbance attenuation and decoupling are important issues in the field of control theory, and have attracted much attention from control scientists and engineers. As is claimed in Gao (2014) that "the problem of automatic control is, in essence, that of disturbance rejection". In many applications, we encounter systems with completely unknown disturbances. In such applications, attenuation of the effect of the disturbances on the output of the system is desired.

To conclude this chapter, let us point out that the FAS approach also possesses a natural anti-disturbance feature which is illustrated in Subsection 6.6.3.

Chapter 5

Robust and Adaptive Control

This chapter treats three types of FASs with uncertainties:

- *Robust control*. For a single-order FAS model with nonlinear uncertainties, a direct approach for the design of robust stabilizing controllers and robust tracking controllers is proposed based on the Lyapunov stability theory. The proposed approach possesses several merits, particularly, the state vectors converge globally into an ellipsoid with a radius which can be made as small as desired. Furthermore, in the case of a single FAS model, the closed-loop system becomes a constant linear one when uncertainties do not exist.
- *Adaptive control*. For a FAS model with parametric uncertainties, direct design approaches of adaptive stabilizing controllers and adaptive tracking controllers are proposed based on the Lyapunov stability theory. Again, the proposed approach possesses several merits, particularly, the state vectors converge globally to the origin, and in the case of a single FAS model, the closed-loop system becomes a constant linear one when the unknown parameters are not present.
- *Robust adaptive control*. For a type of FAS models with both nonlinear uncertainties and time-varying unknown parameters, a direct approach for the designs of robust adaptive stabilizing controllers and robust adaptive tracking controllers is proposed based on the Lyapunov stability theory. The established controller is composed of three parts. The basic part cancels the known nonlinearities in the system and simultaneously assigns the linear dominant term in the closed-loop system, the robustness part overcomes the effects of the nonlinear uncertainties in the system, and the adaptation part adjusts online the controller to suit the effect of the unknown time-varying parameter vector. The proposed controller guarantees that the tracking error of the state with respect to a given signal and the estimation error of the parameter vector finally converge globally into a bounded ellipsoid. Particularly, in the case that the unknown parameter vector is constant, an adaptive scheme that enables global asymptotical tracking is presented.

An example demonstrates the effect and the application procedure of the proposed robust and/or adaptive control approach.

5.1 Robust Control

5.1.1 Robust Stabilization

Uncertain FAS model

In Duan (2020a,b, 2021a,b), it has been shown that many systems without uncertainties can be converted into FAS models in the following form:

$$x^{(n)} = f(x^{(0\sim n-1)}) + L(x^{(0\sim n-1)})u, \tag{5.1}$$

where $x, u \in \mathbb{R}^r$ are the state vector and the control input vector, respectively, $f(x^{(0\sim n-1)}) \in \mathbb{R}^r$ is a continuous vector function, and $L(x^{(0\sim n-1)}) \in \mathbb{R}^{r\times r}$ is a continuous matrix function satisfying the following fully actuated assumption:

Assumption 5.1 $\det L(X) \neq 0$ *or* ∞, $\forall X \in \mathbb{R}^{nr}$.

Correspondingly, for some nonlinear systems subject to uncertainties, often their FAS models can be also derived, but in the following form:

$$x^{(n)} = f(x^{(0\sim n-1)}) + \Delta f(x^{(0\sim n-1)}) + L(x^{(0\sim n-1)})u, \tag{5.2}$$

where $\Delta f(x^{(0\sim n-1)}) \in \mathbb{R}^r$ is the nonlinear uncertainty of the system. The derivation of the above uncertain FAS model may be of two folds: one is through physical modelling, see, e.g., the example 2 in Duan (2021b), when the disturbances d_i, $i = 1, 2, 3$, are taken as uncertainties; the other is through model conversion.

In this section, let us consider the control of systems represented by the single-order FAS model (5.2) under the following condition:

Assumption 5.2 *There exists a non-negative continuous scalar function* $\rho(x^{(0\sim n-1)})$ *such that the nonlinear uncertainty* $\Delta f(x^{(0\sim n-1)}) \in \mathbb{R}^r$ *satisfies*

$$\left\| \Delta f(x^{(0\sim n-1)}) \right\| \leqslant \rho(x^{(0\sim n-1)}).$$

We point out that, in this section, the following particular notation is also used:

$$I_n^{\circ} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \iddots & 0 \\ 1 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{n\times n}.$$

Preliminaries

In order to give a control law for the above high-order system (5.2) with dynamic uncertainties, let us give two preliminary results.

Lemma 5.1 *Let* $A \in \mathbb{R}^{n\times n}$ *satisfy*

$$\mathrm{Re}\,\lambda_i(A) < -\frac{\gamma}{2},\ i = 1, 2, \cdots, n, \tag{5.3}$$

where $\gamma > 0$*, then there exists a positive definite matrix* $P \in \mathbb{R}^{n\times n}$ *satisfying*

$$A^{\mathrm{T}}P + PA < -\gamma P. \tag{5.4}$$

Proof. It follows from (5.3) that the matrix $A+\frac{\gamma}{2}I$ is stable, thus there exists a positive definite matrix P satisfying

$$\left(A+\frac{\gamma}{2}I\right)^{\mathrm{T}}P+P\left(A+\frac{\gamma}{2}I\right)<0.$$

Expanding the above equation gives equation (5.4). ■

Lemma 5.2 *For any $\mu>0$, there exist a set of matrices $A_i\in\mathbb{R}^{r\times r},\ i=0,1,\cdots,n-1$ satisfying*

$$\mathrm{Re}\,\lambda_i\left(\Phi\left(A_{0\sim n-1}\right)\right)<-\frac{\mu}{2},\ i=1,2,\cdots,nr. \tag{5.5}$$

Proof. In view of

$$\det\left(sI_{nr}-\Phi\left(A_{0\sim n-1}\right)\right)=\det\left(s^nI_r+\sum_{i=0}^{n-1}A_is^i\right),$$

the eigenvalues of $\Phi\left(A_{0\sim n-1}\right)$ can be arbitrarily assigned by properly choosing $A_i,\ i=0,1,\cdots,n-1$, thus the conclusion holds. ■

It is known from Lemma 5.1 that, when the condition in (5.5) holds for some $\mu>0$, there exists a positive definite matrix $P(A_{0\sim n-1})$ satisfying

$$\Phi^{\mathrm{T}}(A_{0\sim n-1})P(A_{0\sim n-1})+P(A_{0\sim n-1})\Phi(A_{0\sim n-1})<-\mu P(A_{0\sim n-1}). \tag{5.6}$$

Partitioning $P(A_{0\sim n-1})$ as

$$P(A_{0\sim n-1})=\left[\begin{array}{cccc}P_1 & P_2 & \cdots & P_n\end{array}\right],\ P_i\in\mathbb{R}^{nr\times r}, \tag{5.7}$$

we can further introduce the following notation which will be frequently used in this chapter:

$$P_L(A_{0\sim n-1})=P(A_{0\sim n-1})\left[\begin{array}{c}0\\ I_r\end{array}\right]=P_n. \tag{5.8}$$

To conclude this subsection, let us state the following simple result.

Lemma 5.3 *Let a,b be two real numbers, and $b>0$. Then the following relation holds:*

$$a-\frac{a^2}{4b}\leqslant b. \tag{5.9}$$

Stabilizing controller

About the robust stabilization for the high-order system (5.2) with nonlinear uncertainties, we have the following theorem.

Theorem 5.1 *Suppose that system (5.2) satisfies Assumptions 5.1 and 5.2. Let μ and ε be two arbitrarily given positive numbers, and $A_i\in\mathbb{R}^{r\times r},\ i=0,1,\ldots,n-1$, be a set of matrices satisfying (5.5). Then, the following control law*

$$\left\{\begin{array}{l}u=-L^{-1}(x^{(0\sim n-1)})\left(A_{0\sim n-1}x^{(0\sim n-1)}+u^*\right)\\ u^*=f(x^{(0\sim n-1)})+\frac{1}{4\varepsilon}\rho^2(x^{(0\sim n-1)})P_L^{\mathrm{T}}(A_{0\sim n-1})x^{(0\sim n-1)},\end{array}\right. \tag{5.10}$$

for the uncertain system (5.2) guarantees that the state $x^{(0\sim n-1)}$ converges into the following ellipsoid centered at the origin:

$$\Theta_{\mu,\varepsilon}(0)=\left\{x^{(0\sim n-1)}\left|\left(x^{(0\sim n-1)}\right)^{\mathrm{T}}P(A_{0\sim n-1})x^{(0\sim n-1)}\leq\frac{\varepsilon}{\mu}\right.\right\}.$$

For a proof of the above result, please refer to Appendix B.

In the case of $n=1$, system (5.2) becomes

$$\dot{x}=f(x)+\Delta f(x)+L(x)u, \tag{5.11}$$

and the robust control law turns into

$$u=L^{-1}(x)\left(Ax-f(x)-\frac{1}{4\varepsilon}\rho^2(x)P_L^{\mathrm{T}}(A)x\right). \tag{5.12}$$

In such a case, the state x of the uncertain control system eventually converges into the following ellipsoid:

$$\Theta_{\mu,\varepsilon}(0)=\left\{x\left|x^{\mathrm{T}}P(A)x\leq\frac{\varepsilon}{\mu}\right.\right\}.$$

5.1.2 Robust Tracking

Let $x^*(t)\in\mathbb{R}^r$ be a reference signal to be tracked by the state $x(t)$, and define

$$z=x-x^*,$$

then we have

$$z^{(i)}=x^{(i)}-\left(x^*\right)^{(i)},\ i=0,1,\ldots,n, \tag{5.13}$$

and $f\left(x^{(0\sim n-1)}\right)$, $\Delta f\left(x^{(0\sim n-1)}\right)$ and $L\left(x^{(0\sim n-1)}\right)$ can all be transformed into functions with respect to $z^{(0\sim n-1)}$ and t, that is,

$$\begin{cases} f\left(x^{(0\sim n-1)}\right)=f\left(z^{(0\sim n-1)},t\right)\\ \Delta f\left(x^{(0\sim n-1)}\right)=\Delta f\left(z^{(0\sim n-1)},t\right)\\ L\left(x^{(0\sim n-1)}\right)=L\left(z^{(0\sim n-1)},t\right). \end{cases} \tag{5.14}$$

Thus system (5.2) can be turned into

$$z^{(n)}=f\left(z^{(0\sim n-1)},t\right)+\Delta f\left(z^{(0\sim n-1)},t\right)+L\left(z^{(0\sim n-1)},t\right)u-\left(x^*\right)^{(n)}. \tag{5.15}$$

Applying Theorem 5.1 to the above system, gives the following robust stabilizing control law:

$$\begin{cases} u=-L^{-1}\left(z^{(0\sim n-1)},t\right)\left(A^{0\sim n-1}z^{(0\sim n-1)}+u^*\right)\\ u^*=\frac{1}{4\varepsilon}\rho^2(z^{(0\sim n-1)},t)P_L^{\mathrm{T}}(A^{0\sim n-1})z^{(0\sim n-1)}+f\left(z^{(0\sim n-1)},t\right)-(x^*)^{(n)}. \end{cases}$$

Then, using the relations in (5.13)-(5.14) again reversely, we can obtain the robust tracking control law for the original system (5.2).

Theorem 5.2 *Suppose that system (5.2) satisfies Assumptions 5.1 and 5.2. Let μ and ε be two arbitrarily given positive numbers, and $A_i \in \mathbb{R}^{r\times r}$, $i = 0, 1, \ldots, n-1$, be a set of matrices satisfying (5.5). Then, the following control law*

$$\begin{cases} u = -L^{-1}(x^{(0\sim n-1)})\left(A^{0\sim n-1}x^{(0\sim n-1)} + u^*\right) \\ u^* = \frac{1}{4\varepsilon}\rho^2(x^{(0\sim n-1)})P_L^{\mathrm{T}}(A^{0\sim n-1})x^{(0\sim n-1)} + f(x^{(0\sim n-1)}) - u_0(x^*), \end{cases}$$

with

$$\begin{aligned} u_0(x^*) &= (x^*)^{(n)} + A^{0\sim n-1}(x^*)^{(0\sim n-1)} \\ &\quad + \frac{1}{4\varepsilon}\rho^2(x^{(0\sim n-1)})P_L^{\mathrm{T}}(A^{0\sim n-1})(x^*)^{(0\sim n-1)}, \end{aligned}$$

guarantees that the state $x^{(0\sim n-1)}$ converges globally into the following ellipsoidal belt-shaped region centered at $(x^*)^{(0\sim n-1)}$:

$$\Theta_{\mu,\varepsilon}\left((x^*)^{(0\sim n-1)}\right) = \left\{ x^{(0\sim n-1)} \,\middle|\, \left((x-x^*)^{(0\sim n-1)}\right)^{\mathrm{T}} P\,(x-x^*)^{(0\sim n-1)} \le \frac{\varepsilon}{\mu} \right\}.$$

Remark 5.1 In applications, the radii of the ellipsoids $\Theta_{\mu,\varepsilon}(0)$ and $\Theta_{\mu,\varepsilon}\left((x^*)^{(0\sim n-1)}\right)$ can be comprehensively adjusted through two ways to make it as small as desired: one is to increase the stability threshold μ of the linear part of the system, the other is to reduce the value of ε, or equivalently, to increase the magnitude of the control.

Remark 5.2 The proposed method can be extended to some sub-FASs (Duan (2020a), Duan (2021a)). One of the natural ideas of doing this is to select properly a set of initial values and a reference signal $x^*(t)$ such that the solution of the system can pass safely around the "dangerous" area which possibly makes the matrix $L\left(x^{(0\sim n-1)}\right)$ off-rank by following the "lead" of the reference signal $x^*(t)$, so as to preserve the boundedness and realizability of the controller.

Remark 5.3 When the nonlinear uncertainty is removed, we can choose $\rho\left(x^{(0\sim n-1)}\right) = 0$. In this case the closed-loop system is in the following constant linear form:

$$x^{(n)} + A_{0\sim n-1}x^{(0\sim n-1)} = 0,$$

which can be equivalently written in the following state-space form:

$$\dot{x}^{(0\sim n-1)} = \Phi(A_{0\sim n-1})x^{(0\sim n-1)}.$$

This tells us that, different from many other nonlinear control approaches, the proposed FAS approach produces a constant linear closed-loop system when the uncertainties are not present (see also, Duan (2020a,b), and Duan (2021a)). This huge advantage eventually allows most of the theories and techniques for linear control systems design to be applicable.

Remark 5.4 It is seen from the above result that in order to meet condition (5.5), it suffices only to select a matrix F satisfying

$$\mathrm{Re}\,\lambda_i(F) < -\frac{\mu}{2},\ i = 1, 2, \ldots, nr. \tag{5.16}$$

Clearly, there are quite some degrees of freedom (DOFs) in the selection of F, while on the other hand we also have the parameter matrix Z, which provides $r^2 n$ DOFs. All these DOFs can be further utilized to achieve additional performance of the system (see, e.g., Duan (1992a, 1993a)).

Finally, let us point out that, for solution of the matrix $A_{0\sim n-1}$, Proposition 1.1 can be readily used.

5.1.3 Illustrative Example

Example 5.1

Applications of the proposed robust FAS approach to some practical systems will be investigated elsewhere. In this section, we only demonstrate the application procedures of the proposed approaches via an illustrative example obtained from the following first-order strict feedback system (SFS) (Kokotovic and Arcak (2001), p.648):

$$\begin{cases} \dot{x}_1 = x_1^2 + x_2 \\ \dot{x}_2 = u, \end{cases} \tag{5.17}$$

where all variables are scalar ones. By changing the first-order derivatives into second-order ones, and also adding uncertainties, we obtain the following second-order uncertain SFS:

$$\begin{cases} \ddot{x}_1 = x_1^2 + \Delta f_1(x_1) + x_2 \\ \ddot{x}_2 = \Delta f_2(x_1, \dot{x}_2) + u. \end{cases} \tag{5.18}$$

Obviously, we have

$$G_1 = G_2 = 1,\ f_1 = x_1^2,\ f_2 = 0.$$

The uncertain terms are supposed to satisfy

$$|\Delta f_1(x_1)| \le \sigma_1 x_1^2,\ |\Delta f_2(x_1, \dot{x}_2)| \le \sigma_2 |x_1 \dot{x}_2|, \tag{5.19}$$

and

$$\left|\frac{\partial \Delta f_1(x_1)}{\partial x_1}\right| \le \gamma_1 |\dot{x}_1|,\ \left|\frac{\partial^2 \Delta f_1(x_1)}{\partial x_1^2}\right| \le \gamma_2 |\ddot{x}_1|, \tag{5.20}$$

with σ_i and γ_i, $i = 1, 2$, being some non-negative scalars.

By the Corollary 3.3 in Duan (2021b), we have $L = 1$, and

$$\begin{aligned} h(x_1, x_2) &= [1\ G_1] \begin{bmatrix} \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(x_1^2 + \Delta f_1(x_1)\right) \\ \Delta f_2(x_1, \dot{x}_2) \end{bmatrix} \\ &= [1\ 1] \begin{bmatrix} 2\dot{x}_1^2 + 2x_1\ddot{x}_1 + \Delta \ddot{f}_1(x_1) \\ \Delta f_2(x_1, \dot{x}_2) \end{bmatrix} \\ &= 2\dot{x}_1^2 + 2x_1\ddot{x}_1 + \Delta \ddot{f}_1(x_1) + \Delta f_2(x_1, \dot{x}_2). \end{aligned} \tag{5.21}$$

Thus an equivalent FAS model for system (5.18) can be derived, with $z = x_1$, as follows:

$$z^{(4)} = f\left(z^{(0\sim 2)}\right) + \Delta f\left(z^{(0\sim 3)}\right) + u, \tag{5.22}$$

where

$$\begin{aligned} f\left(z^{(0\sim 2)}\right) &= 2\dot{z}^2 + 2z\ddot{z}, & (5.23)\\ \Delta f\left(z^{(0\sim 3)}\right) &= \Delta \ddot{f}_1(z) + \Delta f_2(z, \dot{x}_2) \\ &= \Delta \ddot{f}_1(z) + \Delta f_2\left(z, \left(\dddot{z} - 2z\dot{z} + \Delta \dot{f}_1(z)\right)\right). & (5.24) \end{aligned}$$

Proposition 5.1 *The uncertain term in system (5.22) satisfies*

$$\left|\Delta f\left(z^{(0\sim 3)}\right)\right| \le \rho\left(z^{(0\sim 3)}\right),$$

with

$$\rho\left(z^{(0\sim 3)}\right) = \left(\gamma_1 |\dot{z}| + \gamma_2 \dot{z}^2\right)|\ddot{z}| + \sigma_2 |z|\left(|\dddot{z} - 2z\dot{z}| + \gamma_1 |\dot{z}|^2\right).$$

Proof. Since

$$\begin{aligned} \Delta \ddot{f}_1(z) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \Delta f_1(z)}{\partial z}\dot{z}\right) \\ &= \frac{\partial^2 \Delta f_1(z)}{\partial z^2}\dot{z}^2 + \frac{\partial \Delta f_1(z)}{\partial z}\ddot{z}, \end{aligned}$$

we have, using (5.20),

$$\begin{aligned} \left|\Delta \ddot{f}_1(z)\right| &\le \left|\frac{\partial^2 \Delta f_1(z)}{\partial z^2}\right|\dot{z}^2 + \left|\frac{\partial \Delta f_1(z)}{\partial z}\right||\ddot{z}| \\ &\le \left(\gamma_1 |\dot{z}| + \gamma_2 \dot{z}^2\right)|\ddot{z}|. \end{aligned}$$

Further using (5.19), gives

$$\begin{aligned} &\left|\Delta f_2\left(z, \left(\dddot{z} - 2z\dot{z} + \Delta \dot{f}_1(z)\right)\right)\right| \\ \le\ & \sigma_2 |z|\left|\dddot{z} - 2z\dot{z} + \frac{\partial \Delta f_1(z)}{\partial z}\dot{z}\right| \\ \le\ & \sigma_2 |z|\left(|\dddot{z} - 2z\dot{z}| + \left|\frac{\partial \Delta f_1(z)}{\partial z}\right||\dot{z}|\right) \\ \le\ & \sigma_2 |z|\left(|\dddot{z} - 2z\dot{z}| + \gamma_1 |\dot{z}|^2\right). \end{aligned}$$

Finally, using the above two relations and definition (5.24), yields

$$\begin{aligned} \left|\Delta f\left(z^{(0\sim 3)}\right)\right| &\le \left|\Delta \ddot{f}_1(z)\right| + \left|\Delta f_2\left(z, \left(\dddot{z} - 2z\dot{z} + \Delta \dot{f}_1(z)\right)\right)\right| \\ &\le \left(\gamma_1 |\dot{z}| + \gamma_2 \dot{z}^2\right)|\ddot{z}| + \sigma_2 |z|\left(|\dddot{z} - 2z\dot{z}| + \gamma_1 |\dot{z}|^2\right). \end{aligned}$$

Thus the conclusion holds. ■

Using the function $\rho\left(z^{(0\sim 3)}\right)$ defined in the above proposition, we can design for system (5.18), or FAS (5.22)-(5.24), the robust stabilizing control law

$$\begin{cases} u = -\left(A_{0\sim 3} z^{(0\sim 3)} + f(z^{(0\sim 2)}) + u^*\right) \\ u^* = \frac{1}{4\varepsilon}\rho^2(z^{(0\sim 3)}) P_L^{\mathrm{T}}(A_{0\sim 3}) z^{(0\sim 3)}, \end{cases} \tag{5.25}$$

where $A_{0\sim 3}$ is chosen to satisfy

$$\mathrm{Re}\,\lambda_i\left(\Phi\left(A_{0\sim 3}\right)\right) < -\frac{\mu}{2},\ i = 1, 2, 3, 4, \tag{5.26}$$

μ and ε are two arbitrarily given positive numbers. The controller guarantees that the vector $z^{(0\sim 3)}$ converges into the following ellipsoid centered at the origin:

$$\Theta_{\mu,\varepsilon}(0) = \left\{ z^{(0\sim 3)} \,\middle|\, \left(z^{(0\sim 3)}\right)^{\mathrm{T}} P z^{(0\sim 3)} \le \frac{\varepsilon}{\mu} \right\},$$

where $P = P(A_{0\sim 3})$ and $P_L^{\mathrm{T}}(A_{0\sim 3})$ are determined by (5.6) and (5.8), respectively.

The above design process demonstrates that the design based on a single FAS model is even simpler than the proposed high-order method of backstepping, but it should be noted that the two approaches need different prior knowledge of the nonlinear uncertainties.

5.2 Adaptive Control

5.2.1 Adaptive Stabilization

Consider the following FAS with an unknown parameter vector:

$$x^{(n)} = H^{\mathrm{T}}(x^{(0\sim n-1)})\theta + q(x^{(0\sim n-1)}) + L(x^{(0\sim n-1)})u, \tag{5.27}$$

where $x,\ u \in \mathbb{R}^r$ are the state vector and the control input vector, respectively, $\theta \in \mathbb{R}^m$ is an unknown constant vector, $q(x^{(0\sim n-1)}) \in \mathbb{R}^r$ is a sufficiently smooth vector function, $H(x^{(0\sim n-1)}) \in \mathbb{R}^{m\times r}$ and $L(x^{(0\sim n-1)}) \in \mathbb{R}^{r\times r}$ are two sufficiently smooth matrix functions, and $L(x^{(0\sim n-1)})$ satisfies the full-actuation Assumption 5.1.

The above FAS (5.27) represents a quite general model for "completely controllable" systems, and can either be obtained through physical modelling, or through conversion from other types of models (Duan (2020a,b, 2021a,b)).

In order to establish the control law of FAS (5.27) with unknown parameters, we need the following preparations.

When the matrix $A_{0\sim n-1}$ is so chosen that the matrix $\Phi(A_{0\sim n-1})$ is stable, it follows from the well-known Lyapunov Theorem that there exists a positive definite matrix $P(A_{0\sim n-1})$ satisfying

$$\Phi^{\mathrm{T}}(A_{0\sim n-1})P(A_{0\sim n-1}) + P(A_{0\sim n-1})\Phi(A_{0\sim n-1}) \le -I_{nr}. \tag{5.28}$$

Partition $P(A_{0\sim n-1})$ as

$$P(A_{0\sim n-1}) = \begin{bmatrix} P_1 & P_2 & \cdots & P_n \end{bmatrix},\ P_i \in \mathbb{R}^{nr\times r}. \tag{5.29}$$

Again, for convenience, we introduce the following notation which is frequently used in this chapter:

$$P_L(A_{0\sim n-1}) \triangleq P(A_{0\sim n-1}) \begin{bmatrix} 0 \\ I_r \end{bmatrix} = P_n. \tag{5.30}$$

Regarding the control of FAS (5.27) with unknown parameters, we have the following theorem.

Theorem 5.3 *Suppose that Assumption 5.1 holds. Let $A_i \in \mathbb{R}^{r\times r}$, $i = 0, 1, \ldots, n-1$, be a set of matrices such that the matrix $\Phi(A_{0\sim n-1})$ is stable, and $P_L(A_{0\sim n-1})$ be given by (5.28)-(5.30). Then, for any unknown constant vector $\theta \in \mathbb{R}^m$, there exists an adaptive control law*

$$\begin{cases} \dot{\hat{\theta}} = H(x^{(0\sim n-1)})P_L^{\mathrm{T}}(A_{0\sim n-1})x^{(0\sim n-1)} \\ u = -L^{-1}(x^{(0\sim n-1)})\left(A_{0\sim n-1}x^{(0\sim n-1)} + u^*\right) \\ u^* = H^{\mathrm{T}}(x^{(0\sim n-1)})\hat{\theta} + q(x^{(0\sim n-1)}), \end{cases} \tag{5.31}$$

for system (5.27) such that the states of the closed-loop system satisfy

1. $x^{(0\sim n-1)} \to 0,\ t \to \infty$;
2. $\left\|\theta - \hat{\theta}\right\|$ *is bounded.*

For a proof of the above result, please refer to Appendix B.

5.2.2 Adaptive Tracking

Let $x^*(t) \in \mathbb{R}^r$ be a reference signal to be tracked by $x(t)$, and let

$$z = x - x^*.$$

Then

$$z^{(i)} = x^{(i)} - \left(x^*\right)^{(i)},\ i = 0, 1, \ldots, n, \tag{5.32}$$

and $H\left(x^{(0\sim n-1)}\right)$, $q\left(x^{(0\sim n-1)}\right)$ and $L\left(x^{(0\sim n-1)}\right)$ can be transformed into functions with respect to $z^{(0\sim n-1)}$ and t, that is,

$$\begin{cases} H\left(x^{(0\sim n-1)}\right) = H\left(z^{(0\sim n-1)}, t\right) \\ q\left(x^{(0\sim n-1)}\right) = q\left(z^{(0\sim n-1)}, t\right) \\ L\left(x^{(0\sim n-1)}\right) = L\left(z^{(0\sim n-1)}, t\right). \end{cases} \tag{5.33}$$

Thus system (5.27) turns into

$$z^{(n)} = H^{\mathrm{T}}\left(z^{(0\sim n-1)}, t\right)\theta + q\left(z^{(0\sim n-1)}, t\right) + L\left(z^{(0\sim n-1)}, t\right)u - \left(x^*\right)^{(n)}. \tag{5.34}$$

Applying Theorem 5.3 to the above system, we can obtain the following adaptive stabilizing controller

$$\begin{cases} \dot{\hat{\theta}} = H(z^{(0\sim n-1)},t)P_L^{\mathrm{T}}(A_{0\sim n-1})z^{(0\sim n-1)} \\ u = -L^{-1}\left(z^{(0\sim n-1)},t\right)\left(A_{0\sim n-1}z^{(0\sim n-1)}+u^*\right) \\ u^* = H^{\mathrm{T}}(z^{(0\sim n-1)},t)\hat{\theta}+q(z^{(0\sim n-1)},t)-(x^*)^{(n)}. \end{cases}$$

Using the relations in (5.32)-(5.33) again, gives the adaptive tracking control law for the original system (5.27), as follows.

Theorem 5.4 *Suppose that system (5.27) satisfies Assumption 5.1. Let $A_i \in \mathbb{R}^{r\times r}$, $i = 0,1,\ldots,n-1$, be a set of matrices such that $\Phi(A_{0\sim n-1})$ is stable, and $P_L(A_{0\sim n-1})$ be given by (5.28)-(5.30). Then, the following control law*

$$\begin{cases} \dot{\hat{\theta}} = H(x^{(0\sim n-1)})P_L^{\mathrm{T}}(A_{0\sim n-1})\left(x^{(0\sim n-1)}-(x^*)^{(0\sim n-1)}\right) \\ u = -L^{-1}\left(x^{(0\sim n-1)}\right)\left(A_{0\sim n-1}x^{(0\sim n-1)}+u^*\right) \\ u^* = H^{\mathrm{T}}(x^{(0\sim n-1)})\hat{\theta}+q(x^{(0\sim n-1)})+u_0(x^*) \\ u_0(x^*) = -(x^*)^{(n)}-A_{0\sim n-1}(x^*)^{(0\sim n-1)}, \end{cases}$$

when applied to system (5.27), guarantees the following:

1. $x^{(0\sim n-1)}-(x^*)^{(0\sim n-1)}\to 0,\ t\to\infty$;
2. $\left\|\theta-\hat{\theta}\right\|$ *is bounded.*

Remark 5.5 The proposed method can be extended to some sub-FASs (Duan (2021a)). One of the natural ideas of doing this is to select properly a set of initial values and a reference signal $x^*(t)$ such that the solution of the system can pass safely around the dangerous areas which possibly make the matrix $L\left(x^{(0\sim n-1)}\right)$ off-rank by following the "leading" signal $x^*(t)$, so as to preserve the boundedness and realizability of the controller.

5.2.3 Solution of $P(A_{0\sim n-1})$

Regarding the solution of the matrix $A_{0\sim n-1}$ which makes $\Phi(A_{0\sim n-1})$ stable, we can apply Proposition 1.1.

To meet the requirement, it suffices to choose a diagonal matrix F with negative diagonal elements. In the case that complex eigenvalues are expected, we may have such a block

$$\begin{bmatrix} -a & -b \\ b & -a \end{bmatrix}, \tag{5.35}$$

with a and b being two positive scalars, included as a diagonal block in F.

Further, to solve the matrix $P(A_{0\sim n-1})$ satisfying the Lyapunov matrix inequality (5.28), it suffice to consider the solution to the following Lyapunov matrix equation:

$$\Phi^{\mathrm{T}}(A_{0\sim n-1})P(A_{0\sim n-1})+P(A_{0\sim n-1})\Phi(A_{0\sim n-1})=-Q, \tag{5.36}$$

where $Q \in \mathbb{R}^{n\times n}$ is an arbitrary positive definite matrix. If we may introduce the following notations related to a square matrix $\Phi \in \mathbb{R}^{n\times n}$:

$$\begin{aligned}\det(sI+\Phi) &\triangleq \sum_{i=0}^{n} c_i^{\Phi} s^i,\\ \operatorname{adj}\left(sI+\Phi^{\mathrm{T}}\right) &\triangleq \sum_{i=0}^{n-1} C_i^{\Phi} s^i,\end{aligned}$$

then, it follows from Lemma 4.3 (see, also, Duan (2015), p.356), the unique solution $P(A_{0\sim n-1})$ to the Lyapunov equation (5.36) is given by

$$P = -\sum_{i=0}^{n-1} C_i^{\Phi} Q P_0^{-1} \Phi^i, \tag{5.37}$$

with

$$P_0 = \sum_{i=0}^{n} c_i^{\Phi} \Phi^i. \tag{5.38}$$

Clearly, there are quite some DOFs in the selection of F, while on the other hand we also have the parameter matrix Z, which provides $r^2 n$ DOFs. All these DOFs can be further utilized to achieve additional performance of the system.

5.2.4 Illustrative Example

Applications of the proposed adaptive FAS approach to some practical systems will be investigated elsewhere. In this section, we only consider a simple illustrative example (Kokotovic and Arcak (2001), p.649), which was originally presented by Kokotovic and Kanellakopoulos (1990):

$$\begin{cases} \dot{x}_1 = \theta x_1^2 + x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = u, \end{cases} \tag{5.39}$$

where all variables are scalar ones, θ is an unknown parameter with a true value of 0.8. According to Kokotovic and Arcak (2001) (p.649), the real difficulties were encountered in the "bench-mark problem". Global stabilization and convergence were finally achieved with the first, overparametrized version of adaptive backstepping by Kanellakopoulos et al. (1991), which also employed the nonlinear damping of Feuer and Morse (1978). Jiang and Praly (1991) reduced the overparametrization on one-half, and the tuning function method of Krstic et al. (1992) completely removed it.

The FAS model and the controller

To derive the FAS model of the system, let us repeat the process in Duan (2021b). Let $z = x_1$. Then, using (5.39), we can obtain

$$\begin{aligned}\dot{z} &= \dot{x}_1 = \theta x_1^2 + x_2 = \theta z^2 + x_2,\\ \ddot{z} &= 2\theta z\dot{z} + \dot{x}_2 = 2\theta z\dot{z} + x_3,\\ \dddot{z} &= 2\theta z\ddot{z} + 2\theta\dot{z}^2 + \dot{x}_3 = 2\theta z\ddot{z} + 2\theta\dot{z}^2 + u.\end{aligned}$$

Therefore, system (5.39) is equivalent to the following third-order FAS:

$$\dddot{z} = 2\left(\dot{z}^2 + z\ddot{z}\right)\theta + u, \tag{5.40}$$

while the original states of system (5.39) are given by the transformation

$$\begin{cases} x_1 = z \\ x_2 = \dot{z} - \theta z^2 \\ x_3 = \ddot{z} - 2\theta z\dot{z}. \end{cases} \tag{5.41}$$

Corresponding to the form of (5.27), we have

$$\begin{cases} H\left(z^{(0\sim 2)}\right) = 2\left(\dot{z}^2 + z\ddot{z}\right) \\ q\left(z^{(0\sim 2)}\right) = 0 \\ L\left(z^{(0\sim 2)}\right) = 1, \end{cases}$$

thus by Theorem 5.3 the adaptive stabilizing controller for system (5.40) is

$$\begin{cases} \dot{\hat{\theta}} = 2\left(\dot{z}^2 + z\ddot{z}\right) P_L^{\mathrm{T}}\left(A_{0\sim 2}\right) z^{(0\sim 2)} \\ u = -A_{0\sim 2} z^{(0\sim 2)} - 2\left(\dot{z}^2 + z\ddot{z}\right)\hat{\theta}. \end{cases} \tag{5.42}$$

Solution of $A_{0\sim 2}$ and $P_L\left(A_{0\sim 2}\right)$

Choose

$$F = \begin{bmatrix} -a & -b & 0 \\ b & -a & 0 \\ 0 & 0 & -c \end{bmatrix}, \tag{5.43}$$

where a, b and c are three positive scalars. Obviously, the eigenvalues of the matrix F are $-a \pm jb$ and $-c$. Further choosing

$$Z = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix},$$

by Proposition 1.1, we have

$$V = \begin{bmatrix} Z \\ ZF \\ ZF^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -a+b & -a-b & -c \\ a^2-2ab-b^2 & a^2+2ab-b^2 & c^2 \end{bmatrix}, \tag{5.44}$$

and

$$A_{0\sim 2} = -ZF^3V^{-1} = -\frac{1}{\Delta}\begin{bmatrix} a_0 & a_1 & a_2 \end{bmatrix}, \tag{5.45}$$

where

$$\begin{cases} a_0 = -c\left(a^2+b^2\right)\left(a^2-2ac+b^2+c^2\right) \\ a_1 = \left(3a^2-2ac-b^2\right)c^2 - \left(a^2+b^2\right)^2 \\ a_2 = -(2a+c)\left(a^2-2ac+b^2+c^2\right), \end{cases} \tag{5.46}$$

and

$$\Delta = a^2 - 4ac + 2b^2 + 2c^2. \tag{5.47}$$

Therefore, with the above obtained matrix $A_{0\sim 2}$, the following matrix

$$\Phi(A_{0\sim 2}) = VFV^{-1}$$

also has eigenvalues $-a \pm jb$ and $-c$.

Particularly choosing

$$a = b = c = 1,$$

we have

$$V = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ -2 & 2 & 1 \end{bmatrix}, \tag{5.48}$$

and

$$A_{0\sim 2} = \begin{bmatrix} 2 & 4 & 3 \end{bmatrix}. \tag{5.49}$$

Thus we can obtain

$$\begin{aligned} \det(sI + \Phi) &= s^3 - 3s^2 + 4s - 2, \\ \mathrm{adj}(sI + \Phi^{\mathrm{T}}) &= \begin{bmatrix} s^2 - 3s + 4 & -2 & 2s \\ -s + 3 & s^2 - 3s & 4s - 2 \\ 1 & -s & s^2 \end{bmatrix}. \end{aligned}$$

These produce

$$c_0^{\Phi} = -2,\ c_1^{\Phi} = 4,\ c_2^{\Phi} = -3,\ c_3^{\Phi} = 1,$$

and

$$C_0^{\Phi} = \begin{bmatrix} 4 & -2 & 0 \\ 3 & 0 & -2 \\ 1 & 0 & 0 \end{bmatrix},\ C_1^{\Phi} = \begin{bmatrix} -3 & 0 & 2 \\ -1 & -3 & 4 \\ 0 & -1 & 0 \end{bmatrix},\ C_2^{\Phi} = I_3.$$

Thus we further obtain

$$P_0 = \begin{bmatrix} -4 & 0 & -6 \\ 12 & 20 & 18 \\ -36 & -60 & -34 \end{bmatrix},$$

and

$$P\,(A_{0\sim 2}) = \frac{1}{40}\begin{bmatrix} 78 & 56 & 10 \\ 56 & 99 & 19 \\ 10 & 19 & 13 \end{bmatrix},$$

hence

$$P_L^{\mathrm{T}}\,(A_{0\sim 2}) = \frac{1}{40}\begin{bmatrix} 10 & 19 & 13 \end{bmatrix}.$$

Simulation results

Based on the above obtained $A_{0\sim 2}$ and $P_L^{\mathrm{T}}(A_{0\sim 2})$, the adaptive controller of the system can be finally obtained as

$$\begin{cases} \dot{\hat{\theta}} = \frac{1}{20}\left(\dot{z}^2 + z\ddot{z}\right)(10z + 19\dot{z} + 13\ddot{z}) \\ u = -(2z + 4\dot{z} + 3\ddot{z}) - 2\left(\dot{z}^2 + z\ddot{z}\right)\hat{\theta}. \end{cases}$$

In the simulation, two sets of initial values are considered:
Case A. $\hat{\theta}(0) = -1.0$, and

$$z(0) = 3.0, \dot{z}(0) = 1.0, \ddot{z}(0) = 2.0,$$

which corresponds to

$$x_1(0) = 3.0, x_2(0) = -6.2, x_3(0) = -2.8.$$

Case B. $\hat{\theta}(0) = 1.5$, and

$$z(0) = 3.5, \dot{z}(0) = -0.5, \ddot{z}(0) = -1.0,$$

which corresponds to

$$x_1(0) = 3.5, x_2(0) = -10.3, x_3(0) = 1.8.$$

The simulation results are shown in Figs. 5.1 and 5.2.

It can be observed from the simulation results that

1. the forms of the responses of this nonlinear system are quite different from that of a linear system; and
2. the responses of the system depend more heavily on the initial values.

5.3 Robust Adaptive Control

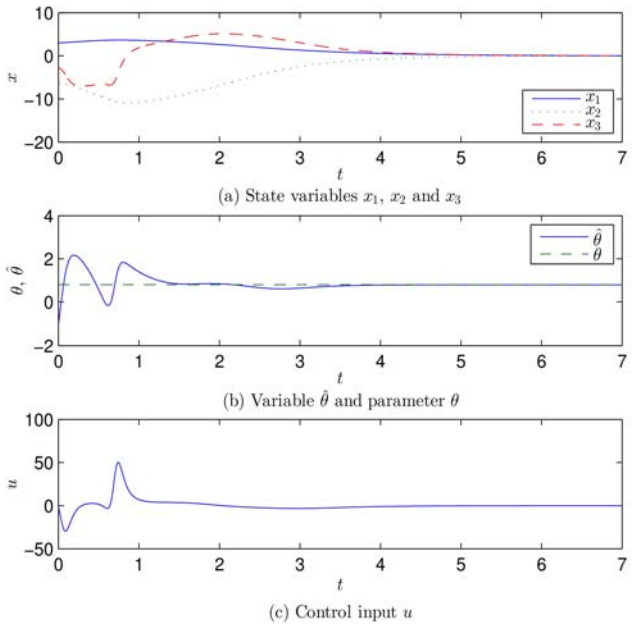
Robust control and adaptive control are both very important issues in the field of systems and control. In many engineering problems, we often encounter systems with both nonlinear uncertainties and parametric uncertainties, and this gives rise to the problem of robust adaptive control (Tao (2014)). It is worth pointing out that robust adaptive control is not simply the superposition of robust control and adaptive control, while is a more challenging topic worth extensive investigation.

5.3.1 Robust Adaptive Stabilization

FAS model and preliminaries

This section considers the control of the following uncertain FAS:

$$x^{(n)} = f(x^{(0\sim n-1)}) + \Delta f(x^{(0\sim n-1)}) + H^{\mathrm{T}}(x^{(0\sim n-1)})\theta + L(x^{(0\sim n-1)})u, \qquad (5.50)$$



(a) State variables x_1, x_2 and x_3

(b) Variable $\hat{\theta}$ and parameter θ

(c) Control input u

Figure 5.1: Simulation results for Case A

where $x \in \mathbb{R}^r$ and $u \in \mathbb{R}^r$ are the state vector and the control input vector, respectively, $f(x^{(0\sim n-1)}) \in \mathbb{R}^r$ is a continuous vector function, $H(x^{(0\sim n-1)}) \in \mathbb{R}^{m\times r}$ and $L(x^{(0\sim n-1)}) \in \mathbb{R}^{r\times r}$ are two continuous matrix functions, and $L(x^{(0\sim n-1)})$ satisfies the fully actuated Assumption 5.1. Furthermore, $\Delta f\left(x^{(0\sim n-1)}\right) \in \mathbb{R}^r$ is an uncertain nonlinearity of the system satisfying the Assumption 5.2. Furthermore, $\theta = \theta(t) \in \mathbb{R}^m$ is an unknown time-varying parameter vector with a pre-estimate $\theta_0 = \theta_0(t) \in \mathbb{R}^m$, and satisfies

Assumption 5.3 *For two non-negative real numbers δ_0 and δ_1, there hold*

$$\|\theta(t) - \theta_0(t)\| \leq \delta_0, \ \left\|\dot{\theta}(t) - \dot{\theta}_0(t)\right\| \leq \delta_1, \ \forall t \geq 0.$$

Remark 5.6 The above FAS model (5.50) might be easily mistaken to represent a very small portion of systems due to the full-actuated Assumption 5.1. While as a matter of fact, many systems which are in state-space forms can be converted into FASs (see Duan (2020a,b, 2021a,b)). Furthermore, many practical systems can also be modeled as FASs. In fact, most physical systems are governed by certain physical laws, such as the Newton's Law, the Lagrangian Equation, the Theorem of Linear and Angular Momentum, Kirchhoff's Laws of Current and Voltage. When such physical laws are used in modelling, a series of second-order subsystems are originally obtained. From this stage we can further get a state-space model by variable extension, and on the other side, we can get a FAS model by variable elimination. It should be also noted

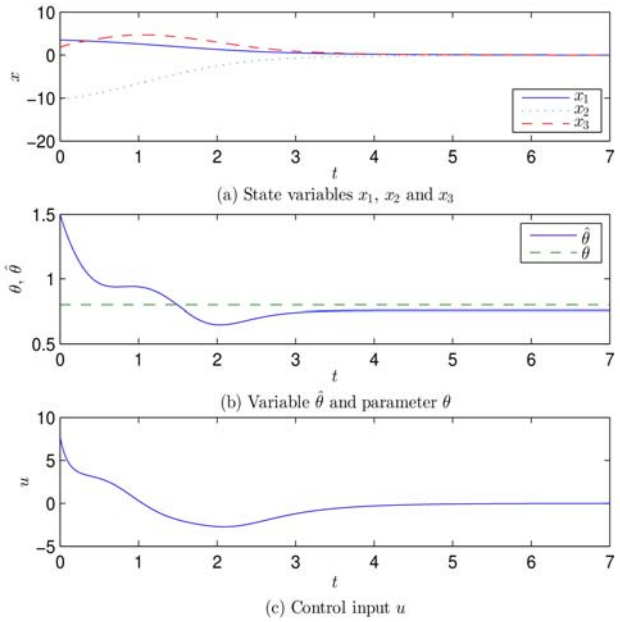


Figure 5.2: Simulation results for Case B

that many physically FASs exist, which are already in the form of second-order FASs at the modelling stage.

Remark 5.7 Assumption 5.2, as well as Assumption 5.4 which is imposed for the case of constant unknown parameter vectors in Section 5.3.3, are both very common ones. Further studies may consider the case where the nonlinearities possess some structural properties. Theoretically, Assumption 5.3 is also quite general and allows the parameter $\theta(t)$ to go to infinity. However, it is clear that the above Assumption 5.3 holds for some δ_0 and δ_1 when both $\theta(t)$ and its pre-estimation $\theta_0(t)$ are uniformly bounded with respect to $t \in [0, \infty)$.

Technical lemmas

For control of the above FAS (5.50), we need some preliminary results.

Firstly, let $\hat{\theta}$ be the estimate of θ, and define

$$\begin{cases} \tilde{\theta} = \theta - \hat{\theta} \\ \tilde{\theta}_{e0} = \hat{\theta} - \theta_0 \\ \tilde{\theta}_{r0} = \theta - \theta_0, \end{cases} \tag{5.51}$$

then, we have the following lemma.

Lemma 5.4 *Suppose that Assumption 5.3 is met, then, for the variables defined in (5.51), the following relations hold:*

1. $\tilde{\theta} = \tilde{\theta}_{r0} - \tilde{\theta}_{e0}$;
2. $\tilde{\theta}_{r0}^{\mathrm{T}}\tilde{\theta} \leq \frac{1}{2}\left(\delta_0^2 + \left\|\tilde{\theta}\right\|^2\right)$;
3. $\dot{\tilde{\theta}}_{r0}^{\mathrm{T}}\tilde{\theta} \leq \frac{1}{2}\left(\delta_1^2 + \left\|\tilde{\theta}\right\|^2\right)$.

Proof. The first conclusion can be easily verified. In view of Assumption 5.3, we have

$$\begin{aligned}
\tilde{\theta}_{r0}^{\mathrm{T}}\tilde{\theta} &\leq \left\|\tilde{\theta}_{r0}\right\|\left\|\tilde{\theta}\right\| \\
&\leq \frac{1}{2}\left(\left\|\tilde{\theta}_{r0}\right\|^2 + \left\|\tilde{\theta}\right\|^2\right) \\
&\leq \frac{1}{2}\left(\delta_0^2 + \left\|\tilde{\theta}\right\|^2\right), \\
\dot{\tilde{\theta}}_{r0}^{\mathrm{T}}\tilde{\theta} &\leq \left\|\dot{\tilde{\theta}}_{r0}\right\|\left\|\tilde{\theta}\right\| \\
&\leq \frac{1}{2}\left(\left\|\dot{\tilde{\theta}}_{r0}\right\|^2 + \left\|\tilde{\theta}\right\|^2\right) \\
&\leq \frac{1}{2}\left(\delta_1^2 + \left\|\tilde{\theta}\right\|^2\right).
\end{aligned}$$

Thus the second and the third conclusions also hold. ■

Secondly, if for any $\mu > 0$, there exist a set of matrices $A_i \in \mathbb{R}^{r\times r}$, $i = 0, 1, \ldots, n-1$, satisfying the following condition

$$\mathrm{Re}\,\lambda_i\left(\Phi\left(A_{0\sim n-1}\right)\right) < -\frac{\mu}{2},\ i = 1, 2, \ldots, nr. \tag{5.52}$$

Then, it follows from Lemmas 5.1 and 5.2 that, for any $\mu > 0$, there exists a positive definite matrix $P\left(A_{0\sim n-1}\right)$ satisfying

$$\Phi^{\mathrm{T}}(A_{0\sim n-1})P(A_{0\sim n-1}) + P(A_{0\sim n-1})\Phi(A_{0\sim n-1}) \leq -\mu P(A_{0\sim n-1}). \tag{5.53}$$

As done in the previous two sections, we can then define the matrix $P_L(A_{0\sim n-1})$, which will be often used in the following sections.

Remark 5.8 In the following discussion, we often omit some independent variables in the case of not causing confusion, e.g., $P(A_{0\sim n-1})$ and $P_L(A_{0\sim n-1})$ are denoted by P and P_L, respectively; $H(x^{(0\sim n-1)})$ and $L(x^{(0\sim n-1)})$ are denoted by H and L, respectively.

Stabilizing controller

For FAS (5.50) with both uncertain nonlinearities and unknown parameters, the following robust adaptive stabilizing control law is designed:

$$u = -L^{-1}(x^{(0\sim n-1)})\,(u_0 + u_r + u_a)\,, \tag{5.54}$$

where

$$u_0 = A_{0\sim n-1}x^{(0\sim n-1)} + f(x^{(0\sim n-1)}) \tag{5.55}$$

is the basic part, which aims to cancel the known nonlinear term $f(x^{(0\sim n-1)})$ in the system and assign the linear term $A_{0\sim n-1}x^{(0\sim n-1)}$ to the closed-loop system; and

$$u_r = \frac{1}{4\varepsilon}\rho^2(x^{(0\sim n-1)})P_L^{\mathrm{T}}(A_{0\sim n-1})x^{(0\sim n-1)}, \tag{5.56}$$

with ε being an arbitrarily selected positive real number, is the robustness part whose function is to overcome the in uence of the uncertain nonlinearity $\Delta f(x^{(0\sim n-1)})$; while

$$\begin{cases} \dot{\hat{\theta}} = H(x^{(0\sim n-1)})P_L^{\mathrm{T}}(A_{0\sim n-1})x^{(0\sim n-1)} - (\mu+1)\left(\hat{\theta}-\theta_0\right) + \dot{\theta}_0 \\ u_a = H^{\mathrm{T}}(x^{(0\sim n-1)})\hat{\theta} \end{cases} \tag{5.57}$$

is the adaptation part, which aims to estimate the parameters online and adjust the control input accordingly based on the estimated results.

A common aspect among the three parts of the controllers is that they are all the static or dynamical feedback of the state $x^{(0\sim n-1)}$. Their expressions are really determined by the Lyapunov stability theory, as is shown in the proof of the following result, which reveals the capability of the above described controller for FAS (5.50).

Theorem 5.5 *Suppose that FAS (5.50), together with the pre-estimation θ_0 of the unknown parameter vector θ, satisfies Assumptions 5.1, 5.2 and 5.3. Let μ and ε be two arbitrarily selected non-negative real numbers, and $A_i \in \mathbb{R}^{r\times r}$, $i = 0, 1, \ldots, n-1$, be a set of matrices satisfying (5.52). Then, control law (5.54)-(5.57) for system (5.50) guarantees that the state $x^{(0\sim n-1)}$ of the closed-loop system and the parameter estimation error $\tilde{\theta}$ converge to the following ellipsoid:*

$$\Theta_{(\mu,\varepsilon,\delta)}(0) = \left\{ \begin{bmatrix} x^{(0\sim n-1)} \\ \tilde{\theta} \end{bmatrix} \middle| \left\| x^{(0\sim n-1)} \right\|_P^2 + \left\| \tilde{\theta} \right\|^2 \le 2\frac{\varepsilon+\delta}{\mu} \right\},$$

where

$$\delta = \frac{1}{2}\left(\delta_1^2 + (\mu+1)\,\delta_0^2\right). \tag{5.58}$$

For a proof of the above result, please refer to Appendix B.

In the case of $n = 1$, system (5.50) turns into

$$\dot{x} = f(x) + \Delta f(x) + H^{\mathrm{T}}(x)\theta + L(x)u, \tag{5.59}$$

and the corresponding robust adaptive control law is turned into

$$\begin{cases} \dot{\hat{\theta}} = H(x)P_L^{\mathrm{T}}(A)x - (\mu+1)\left(\hat{\theta}-\theta_0\right) + \dot{\theta}_0 \\ u = L^{-1}(x)\left(Ax - f(x) - \frac{1}{4\varepsilon}\rho^2(x)P_L^{\mathrm{T}}(A)x - H^{\mathrm{T}}(x)\hat{\theta}\right). \end{cases} \tag{5.60}$$

In such a case the state x and the parameter estimation error $\tilde{\theta}$ eventually converge into the following ellipsoid:

$$\Theta_{(\mu,\varepsilon,\delta)}(0) = \left\{ \begin{bmatrix} x \\ \tilde{\theta} \end{bmatrix} \middle| x^{\mathrm{T}}P(A)x + \tilde{\theta}^{\mathrm{T}}\tilde{\theta} \le 2\frac{\varepsilon+\delta}{\mu} \right\}.$$

5.3.2 Robust Adaptive Tracking

Let $x^*(t) \in \mathbb{R}^r$ be a sufficiently smooth reference signal to be tracked by $x(t)$, and put

$$\tilde{x} = x - x^*,$$

then we have

$$\tilde{x}^{(i)} = x^{(i)} - \left(x^*\right)^{(i)},\ i = 0, 1, \ldots, n, \tag{5.61}$$

and the functions in FAS (5.50) can all be transformed into functions with respect to $\tilde{x}^{(0\sim n-1)}$ and t, that is,

$$\begin{cases} f\left(x^{(0\sim n-1)}\right) = f\left(\tilde{x}^{(0\sim n-1)}, t\right) \\ \Delta f\left(x^{(0\sim n-1)}\right) = \Delta f\left(\tilde{x}^{(0\sim n-1)}, t\right) \\ \rho\left(x^{(0\sim n-1)}\right) = \rho\left(\tilde{x}^{(0\sim n-1)}, t\right) \\ H\left(x^{(0\sim n-1)}\right) = H\left(\tilde{x}^{(0\sim n-1)}, t\right) \\ L\left(x^{(0\sim n-1)}\right) = L\left(\tilde{x}^{(0\sim n-1)}, t\right). \end{cases} \tag{5.62}$$

Then FAS (5.50) turns into

$$\begin{aligned} \tilde{x}^{(n)} &= f\left(\tilde{x}^{(0\sim n-1)}, t\right) + \Delta f\left(\tilde{x}^{(0\sim n-1)}, t\right) \\ &\quad + H^{\mathrm{T}}\left(\tilde{x}^{(0\sim n-1)}, t\right)\theta + L\left(\tilde{x}^{(0\sim n-1)}, t\right)u - \left(x^*\right)^{(n)}. \end{aligned} \tag{5.63}$$

Applying Theorem 5.5 to the above system, gives the following robust adaptive control law:

$$u = -L^{-1}(\tilde{x}^{(0\sim n-1)}, t)\left(u_0 + u_r + u_a\right), \tag{5.64}$$

where

$$u_0 = A_{0\sim n-1}\tilde{x}^{(0\sim n-1)} + f(\tilde{x}^{(0\sim n-1)}, t) - \left(x^*\right)^{(n)}, \tag{5.65}$$

$$u_r = \frac{1}{4\varepsilon}\rho^2(\tilde{x}^{(0\sim n-1)}, t)P_L^{\mathrm{T}}(A_{0\sim n-1})\tilde{x}^{(0\sim n-1)}, \tag{5.66}$$

$$\begin{cases} \dot{\hat{\theta}} = H(\tilde{x}^{(0\sim n-1)}, t)P_L^{\mathrm{T}}(A_{0\sim n-1})\tilde{x}^{(0\sim n-1)} \\ \qquad -(\mu+1)\left(\hat{\theta}-\theta_0\right) + \dot{\theta}_0 \\ u_a = H^{\mathrm{T}}(\tilde{x}^{(0\sim n-1)}, t)\hat{\theta}. \end{cases} \tag{5.67}$$

Substituting the relations in (5.61)-(5.62) back to (5.64)-(5.67), yields the following robust adaptive tracking control law for the original FAS (5.50):

$$u = -L^{-1}(x^{(0\sim n-1)})\left(\hat{u}_0 + \hat{u}_r + \hat{u}_a\right), \tag{5.68}$$

where

$$\begin{cases} \hat{u}_0 = A_{0\sim n-1}(x-x^*)^{(0\sim n-1)} + f(x^{(0\sim n-1)}) - (x^*)^{(n)} \\ \hat{u}_r = \frac{1}{4\varepsilon}\rho^2(x^{(0\sim n-1)})P_L^{\mathrm{T}}(A_{0\sim n-1})(x-x^*)^{(0\sim n-1)} \\ \hat{u}_a = H^{\mathrm{T}}(x^{(0\sim n-1)})\hat{\theta}, \end{cases} \tag{5.69}$$

with $\hat{\theta}$ being given by the following regulation law:

$$\begin{aligned} \dot{\hat{\theta}} = {} & H(x^{(0\sim n-1)})P_L^{\mathrm{T}}(A_{0\sim n-1})(x-x^*)^{(0\sim n-1)} \\ & -(\mu+1)\left(\hat{\theta}-\theta_0\right)+\dot{\theta}_0. \end{aligned} \tag{5.70}$$

Theorem 5.6 *Suppose that FAS (5.50), together with a pre-estimation θ_0 of the unknown parameter vector θ, satisfies Assumptions 5.1∼5.3. Let μ and ε be two arbitrarily selected positive real numbers, and $A_i \in \mathbb{R}^{r\times r}$, $i = 0, 1, \ldots, n-1$, be a set of matrices satisfying (5.52). Then, control law (5.68)-(5.70) for the uncertain system (5.50) guarantees that the tracking error $\tilde{x} = x - x^*$ and the parameter estimation error $\tilde{\theta}$ converge to the following ellipsoid:*

$$\begin{aligned} & \Theta_{(\mu,\varepsilon,\delta)}\left(\begin{bmatrix} \tilde{x}^{(0\sim n-1)} \\ \tilde{\theta} \end{bmatrix}\right) \\ = {} & \left\{\begin{bmatrix} \tilde{x}^{(0\sim n-1)} \\ \tilde{\theta} \end{bmatrix} \middle| \left\|\tilde{x}^{(0\sim n-1)}\right\|_P^2 + \left\|\tilde{\theta}\right\|^2 \leqslant 2\frac{\varepsilon+\delta}{\mu}\right\}. \end{aligned}$$

To end this section, we further present some related remarks.

Remark 5.9 Obviously, the radii of the ellipsoids $\Theta_{(\mu,\varepsilon,\delta)}(0)$ and $\Theta_{(\mu,\varepsilon,\delta)}\left((\tilde{x}^*)^{(0\sim n-1)}\right)$ can be comprehensively adjusted through three ways: one is to increase the stability threshold μ of the linear part; the second is to reduce the value of ε, or equivalently, to increase the magnitude of the control; the third is to reduce the values of δ_0 and δ_1 by improving the pre-estimation accuracy of the parameter vector. The first two aspects can be adjusted at will, but the third aspect cannot. Since the proposed method depends on the pre-estimation θ_0 of the parameter θ, it is very difficult to obtain δ_0 and δ_1 with very small values in the absence of prior information. This leads to the fact that the radii of the ellipsoids $\Theta_{(\mu,\varepsilon,\delta)}(0)$ and $\Theta_{(\mu,\varepsilon,\delta)}\left((\tilde{x}^*)^{(0\sim n-1)}\right)$ cannot be arbitrarily small.

Remark 5.10 When the unknown time-varying parameter θ does not exist, we have $\delta = 0$, and the adaptive parts of the control laws u_a and $\hat{u}_a$ vanish. In such a case, the results in this section reduce to that in Duan (2021a). In this situation, the radius of the final ellipsoid can be made as small as desired through increasing the stability threshold μ and increasing the magnitude of control (equally, reducing the value of ε), that is, the desired accuracy of signal tracking can be guaranteed. However, the situation becomes complicated when there exist unknown time-varying parameters in the system.

5.3.3 Further Discussions

Case of θ being constant

When the unknown parameter θ in FAS (5.50) is constant, its pre-estimation is naturally taken to be constant. Then, we have $\dot{\theta} = \dot{\theta}_0 = 0$, $\delta_1 = 0$, and

$$\delta_0 = \|\theta - \theta_0\|, \tag{5.71}$$

$$\delta = \frac{1}{2}(\mu+1)\delta_0^2. \tag{5.72}$$

In such a case, the robust adaptive control law for the system is given by (5.68)-(5.69) and the following parameter regulation law:

$$\dot{\hat{\theta}} = H(x^{(0\sim n-1)})P_L^{\mathrm{T}}(A_{0\sim n-1})\left(x - x^*\right)^{(0\sim n-1)} - (\mu+1)\left(\hat{\theta} - \theta_0\right), \tag{5.73}$$

and the radius of the eventually relevant ellipsoid is

$$r_{(\varepsilon,\mu,\delta_0)} = \frac{2\varepsilon + (\mu+1)\delta_0^2}{\mu}. \tag{5.74}$$

Considering that the parameter adaptive law (5.73) and the radius of the ellipsoid into which the state finally converges are both related to the pre-estimation θ_0 of the parameter θ, it is very important to find a θ_0 with higher precision (a smaller δ_0). Without a good prior information, the parameter vector θ_0 has to be chosen by guess, and the value of δ_0 may be eventually very large, and this consequently gives an attracting ellipsoid with a big radius.

In this section, let us replace Assumption 5.2 with

Assumption 5.4 *There exists a non-negative continuous scalar function $\rho(x^{(0\sim n-1)})$, such that*

$$\left\|\Delta f(x^{(0\sim n-1)})\right\| \le \rho(x^{(0\sim n-1)})\left\|P_L^{\mathrm{T}}x^{(0\sim n-1)}\right\|.$$

Under Assumption 5.1 and the above Assumption 5.4, we will present a modified adaptation scheme for the case of constant θ, which has the following two advantages:

1. the adaptation scheme does not depend on a pre-estimate of the parameter θ;
2. the robust adaptive controller guarantees that the state $x^{(0\sim n-1)}$ converges to zero.

In the case that the parameter vector θ is constant, the modification of our robust adaptive controller consists of the following two aspects:

- replacing the robustness part in the robust adaptive controller (5.54)-(5.56) by

$$u_r = \frac{1}{\varepsilon}\rho(x^{(0\sim n-1)})P_L^{\mathrm{T}}(A_{0\sim n-1})x^{(0\sim n-1)}, \tag{5.75}$$

  where ε is an arbitrarily selected positive real number not greater than 1, and

- replacing the adaptation part in the robust adaptive controller (5.54)-(5.56) by the following:

$$\begin{cases} \dot{\hat{\theta}} = H(x^{(0\sim n-1)})P_L^{\mathrm{T}}(A_{0\sim n-1})x^{(0\sim n-1)} \\ u_a = H^{\mathrm{T}}(x^{(0\sim n-1)})\hat{\theta}. \end{cases} \tag{5.76}$$

As a result, we have the following theorem.

Theorem 5.7 *Suppose that FAS (5.50) satisfies Assumptions 5.1 and 5.4, and the parameter vector θ is constant. Let μ and ε be two selected positive real numbers, with $\varepsilon \leq 1$, and $A_i \in \mathbb{R}^{r\times r}$, $i = 0, 1, \ldots, n-1$, be a set of matrices satisfying (5.52). Then, the control law given by (5.54)-(5.55) and (5.75)-(5.76), for system (5.50), guarantees*

- $\lim_{t\to\infty} \left\| x^{(0\sim n-1)} \right\| = 0$;
- $\left\| \tilde{\theta} \right\|$ is bounded.

For a proof of the above result, please refer to Appendix B.

In the case that the state $x(t)$ is required to track a given signal $x^*(t)$, the robust adaptive tracking control law for the system turns into

$$u = -L^{-1}(x^{(0\sim n-1)})\left(\widehat{u}_0 + \widehat{u}_r + \widehat{u}_a\right), \tag{5.77}$$

where

$$\begin{cases} \widehat{u}_0 = A_{0\sim n-1}(x-x^*)^{(0\sim n-1)} + f(x^{(0\sim n-1)}) - (x^*)^{(n)} \\ \widehat{u}_r = \frac{1}{\varepsilon}\rho(x^{(0\sim n-1)})P_L^{\mathrm{T}}(A_{0\sim n-1})(x-x^*)^{(0\sim n-1)} \\ \widehat{u}_a = H^{\mathrm{T}}(x^{(0\sim n-1)})\hat{\theta}, \end{cases} \tag{5.78}$$

and $\hat{\theta}$ is given by the adaptation law

$$\dot{\hat{\theta}} = H(x^{(0\sim n-1)})P_L^{\mathrm{T}}(A_{0\sim n-1})\left(x-x^*\right)^{(0\sim n-1)}. \tag{5.79}$$

When this robust adaptive control law is applied to FAS (5.50), $||\tilde{\theta}||$ is guaranteed to be bounded, and

$$\lim_{t\to\infty} \left\| \left(x-x^*\right)^{(0\sim n-1)} \right\| = 0. \tag{5.80}$$

Therefore, asymptotic tracking of the system state vector to the reference signal can be realized in such a case. In the particular case of $\Delta f(x^{(0\sim n-1)}) = 0$, this result reduces to the one given in Duan (2021d).

Further remarks

Remark 5.11 The work in this section once again confirms the linear-dominant design idea described in Duan (2020a), that is, to overcome the in uence of uncertainties and unknown parameters by firstly cancelling the known nonlinearities in the system and

then appropriately designing the linear part in the closed-loop. The linear dominant part $A_i \in \mathbb{R}^{r\times r}$, $i = 0, 1, \ldots, n-1$, or $\Phi(A_{0\sim n-1})$, of the robust adaptive stabilizing control law proposed in this chapter can still be solved by the parametric design method in Duan (2020a). Concretely, we can directly use Proposition 1.1.

Remark 5.12 The adaptive control of nonlinear systems with time-varying unknown parameters has always been a difficult problem in the field of systems and control. The result in Section 5.3.1 shows that although there exist uncertain nonlinear terms and time-varying unknown parameters in the system, we can still obtain a control law which guarantees the stability (boundedness) of the closed-loop system. However, due to the time-varying nature of the unknown parameters, asymptotical stability is generally not achievable.

Remark 5.13 Parallelly, it can be also seen that the solution to the tracking problem considered in Section 5.3.2 is not in an asymptotic sense, and the established control law may not ensure a high tracking accuracy. What the proposed control law can achieve is to ensure that the actual operating state of the system will not deviate too far away from the desired one. Such kind of tracking requirements have numerous practical backgrounds, e.g., mid-course guidance of long-distance missiles, monitoring and tracking of space targets, etc. A common characteristic in these applications is to keep the system running with lower energy while maintaining an endurable and controllable performance before the time to take further "action".

Remark 5.14 In the situation where the unknown parameter is time-varying, generally speaking, it is very difficult to realize asymptotic tracking or arbitrary-precision tracking. To increase the tracking accuracy, it is recommended to obtain as much as possible *a priori* information about unknown parameters based on theoretical and technical analysis, numerical experiments as well as man's experience in order to give smaller δ_0 and δ_1, so as to improve the tracking accuracy. After all, it is not necessary to argue whether the parameter estimation is realized or not, but only to regard the proposed controller as a dynamic compensator. The conclusion is that the stabilization of the system in the sense of global boundedness is achieved under this dynamic compensator.

Remark 5.15 Disturbance attenuation or compensation has always been an important issue in control systems design. The unknown parameter θ in the FAS model (5.50) in this section can also be regarded as an unknown disturbance vector. The disturbance term $H^{\mathrm{T}}\theta$ represents a structural perturbation of the system. From this perspective, the work of this section can be also regarded as the robust control of systems with structural disturbances. In particular, in disturbance compensation theory, constant disturbance is an important case and is widely accepted. When the parameter vector θ is constant, the work in this section can also be regarded as robust control with disturbance compensation using a disturbance estimator.

Remark 5.16 The proposed method can be extended to some sub-FASs. One of the natural ideas of doing this is to select properly a set of initial values and a reference

signal $x^*(t)$ such that the solution of the system can safely pass around the "dangerous area" which possibly makes the matrix $L\left(x^{(0\sim n-1)}\right)$ off-rank by following the "leading" signal $x^*(t)$, so as to preserve the boundedness and realizability of the controller. This idea will be demonstrated by several examples in the Part VII (Duan (2021g)) of the series.

5.3.4 Illustrative Example

Recall the following system treated in Subsection 5.2.4 which was considered by Kokotovic and Kanellakopoulos (1990) and Kokotovic and Arcak (2001), p.649:

$$\begin{cases} \dot{x}_1 = \theta x_1^2 + x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = u, \end{cases} \tag{5.81}$$

where all variables are scalar ones, θ is an unknown parameter. According to Kokotovic and Arcak (2001) (p.649), real difficulties were encountered in this "bench-mark problem". Global stabilization and convergence were finally achieved with the first, over-parametrized version of adaptive backstepping by Kanellakopoulos et al. (1991), which also employed the nonlinear damping of Feuer and Morse (1978). Jiang and Praly (1991) reduced the overparametrization to one-half, and the tuning function method of Krstic et al. (1992) completely removed it.

Subsection 5.2.4 converts this system into the following third-order FAS:

$$\dddot{x} = 2\left(\dot{x}^2 + x\ddot{x}\right)\theta + u, \tag{5.82}$$

and solved the adaptive control of the system using the FAS approach. To derive an example system with both an unknown parameter and a nonlinear uncertainty, let us further add a nonlinear uncertain term $\Delta f(x, \dot{x}, \ddot{x})$ to the above model (5.82), and obtain the following FAS:

$$\dddot{x} = 2\left(\dot{x}^2 + x\ddot{x}\right)\theta + \Delta f(x, \dot{x}, \ddot{x}) + u, \tag{5.83}$$

where the uncertainty $\Delta f(x, \dot{x}, \ddot{x})$ satisfies

$$|\Delta f(x, \dot{x}, \ddot{x})| \leq \rho(x, \dot{x}, \ddot{x}),$$

with $\rho(x, \dot{x}, \ddot{x})$ being some non-negative scalar function.

Different from Duan (2021d), in this example we allow the parameter θ in system (5.83) to be time-varying. Applying Theorem 5.5, we obtain the robust adaptive controller for system (5.83) as follows:

$$u = -\left(A_{0\sim 2}x^{(0\sim 2)} + u_r + u_a\right), \tag{5.84}$$

where

$$u_r = \frac{1}{4\varepsilon}\rho^2(x^{(0\sim 2)})P_L^{\mathrm{T}}(A_{0\sim 2})x^{(0\sim 2)}, \tag{5.85}$$

and

$$\begin{cases} u_a = 2\left(\dot{x}^2 + x\ddot{x}\right)\hat{\theta} \\ \dot{\hat{\theta}} = 2\left(\dot{x}^2 + x\ddot{x}\right) P_L^{\mathrm{T}}(A_{0\sim 2}) x^{(0\sim 2)} - (\mu+1)\left(\hat{\theta} - \theta_0\right) + \dot{\theta}_0. \end{cases} \tag{5.86}$$

Solution of $A_{0\sim 2}$ and $P_L\,(A_{0\sim 2})$

According to Duan (2021d), in order that the matrix $\Phi(A_{0\sim 2})$ has eigenvalues $-a \pm \mathrm{j}b$ and $-c$, we need to choose $A_{0\sim 2}$ as

$$A_{0\sim 2} = \begin{bmatrix} c\left(a^2+b^2\right) & a^2+2ca+b^2 & 2a+c \end{bmatrix}. \tag{5.87}$$

Particularly choosing

$$a = 5,\ b = 2,\ c = 10,$$

yields

$$A_{0\sim 2} = \begin{bmatrix} 290 & 129 & 20 \end{bmatrix}. \tag{5.88}$$

Consider the Lyapunov equation

$$\left(\Phi(A_{0\sim 2}) + 4I_3\right)^{\mathrm{T}} P + P\left(\Phi(A_{0\sim 2}) + 4I_3\right) = -10^{-3} I_3, \tag{5.89}$$

which clearly gives

$$\Phi^{\mathrm{T}}(A_{0\sim 2})P + P\Phi(A_{0\sim 2}) \le -\mu P,$$

with $\mu = 8$. By solving (5.89), we obtain

$$P\,(A_{0\sim 2}) = 10^{-4}\begin{bmatrix} 22005.24 & 5322.264 & 303.5377 \\ 5322.264 & 1495.840 & 87.67925 \\ 303.5377 & 87.67925 & 5.792453 \end{bmatrix},$$

hence

$$P_L^{\mathrm{T}}\,(A_{0\sim 2}) = 10^{-4}\begin{bmatrix} 303.5377 & 87.67925 & 5.792453 \end{bmatrix}.$$

Simulation results

The uncertainty $\Delta f\,(x, \dot{x}, \ddot{x})$ and its bound function $\rho\,(x, \dot{x}, \ddot{x})$ are chosen, respectively, as

$$\Delta f\,(x, \dot{x}, \ddot{x}) = \frac{1}{100}\left(3x^2 + 2x\dot{x} + 98\right),$$

and

$$\rho\,(x, \dot{x}, \ddot{x}) = \frac{1}{100}\left(4x^2 + \dot{x}^2 + 100\right).$$

Further take $\varepsilon = 10^{-5}$, then we obtain the robust part of the controller as

$$u_r = \frac{5}{2}\left(4x^2 + \dot{x}^2 + 100\right)^2 P_L^{\mathrm{T}}(A_{0\sim 2}) x^{(0\sim 2)}. \tag{5.90}$$

Let the parameter θ be given as

$$\theta(t) = 2 + 0.2\sin(10t),$$

and a pre-estimate of $\theta(t)$ be taken as

$$\theta_0(t) = 1,\ t \geq 0.$$

Thus the adaptive part of the controller becomes

$$\begin{cases} u_a = 2\left(\dot{x}^2 + x\ddot{x}\right)\hat{\theta} \\ \dot{\hat{\theta}} = 2\left(\dot{x}^2 + x\ddot{x}\right)P_L^{\mathrm{T}}(A_{0\sim 2})x^{(0\sim 2)} - 9\left(\hat{\theta} - 1\right). \end{cases} \tag{5.91}$$

In the simulation, two sets of initial values are considered:
Case A. $\hat{\theta}(0) = -1.0$, and

$$x(0) = 3.0, \dot{x}(0) = 1.0, \ddot{x}(0) = 2.0.$$

Case B. $\hat{\theta}(0) = 1.5$, and

$$x(0) = 3.5, \dot{x}(0) = -0.5, \ddot{x}(0) = -1.0.$$

The simulation results are shown in Figs. 5.3 and 5.4. It can be observed that the simulation results are consistent with the theoretical indications. Furthermore, it can be noted that the magnitude of the control u is relatively large, this is because the control u has to take the burden to cancel the nonlinearity in the system.

Remark 5.17 The adaptive control of nonlinear systems with time-varying unknown parameters has always been a difficult problem in the field of systems and control. In this section the problem of robust adaptive control of a FAS model is further investigated. It is shown that for a FAS with nonlinear uncertainties and time-varying unknown parameters, a robust adaptive control law still exists under very mild conditions, which guarantees the global stability (boundedness) of the closed-loop system.

Remark 5.18 When the unknown parameter vector is constant and the nonlinear uncertainty has a quasi-linear bound, it is further shown that a modified adaptation scheme exists, which does not depend on a pre-estimate of the parameter θ, and guarantees that the state $x^{(0\sim n-1)}$ globally converges to zero while keeping the estimation of the parameter vector bounded. The proposed approach is very simple and convenient. In the special case that both the nonlinear uncertainties and the unknown parameters do not exist, the closed-loop system reduces to a constant linear system with arbitrarily assignable eigenstructure.

Remark 5.19 The work of this section can be extended to the case of high-order sub-FASs with nonlinear and parametric uncertainties. It can also be extended to different types of systems, such as discrete-time systems and stochastic systems. Furthermore, based on the proposed results, the high-order method of backstepping can also be proposed as treated in Duan (2021c,d).

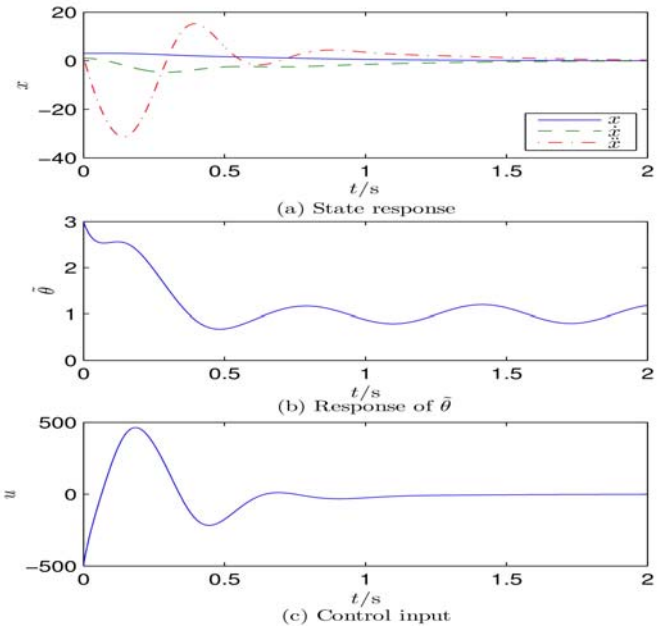


Figure 5.3: Simulation results for Case A

5.4 Notes and References

5.4.1 Relevant References

Robust control

Most practical systems are subject to uncertainties. Robust control aims to preserve the stability and performance of systems under the existence of uncertainties, and has been widely recognized as one of the most important problems in the field of control systems, and eventually has attracted tremendous attention in the last several decades. Robust control of linear systems with uncertainties has been relatively well-solved (Dorato (1987); Park et al. (2006); Bhattacharyya (2017); Bhattacharyya and Keel (1995); Zhou et al. (1996); Zhou and Doyle (1998); Green and Limebeer (2012)). While for robust control of nonlinear systems by now many problems are still left open (Blongdel et al. (1995)).

Nonlinear systems widely exist, and appear in various forms. On one hand, many practical applications give rise to various types of practical nonlinear systems; on the other hand, there are also numerous "man-made" nonlinear systems of different forms (Isidori (1995); Qu (1998); Khalil (2002a); Lin (2007)). Among the various types of nonlinear systems, probably the SFSs are the largest type of nonlinear systems which have been fully investigated (see Duan (2021b) and the references therein). The specific triangular structure allows a systematic recursive design procedure of stabilizing con-

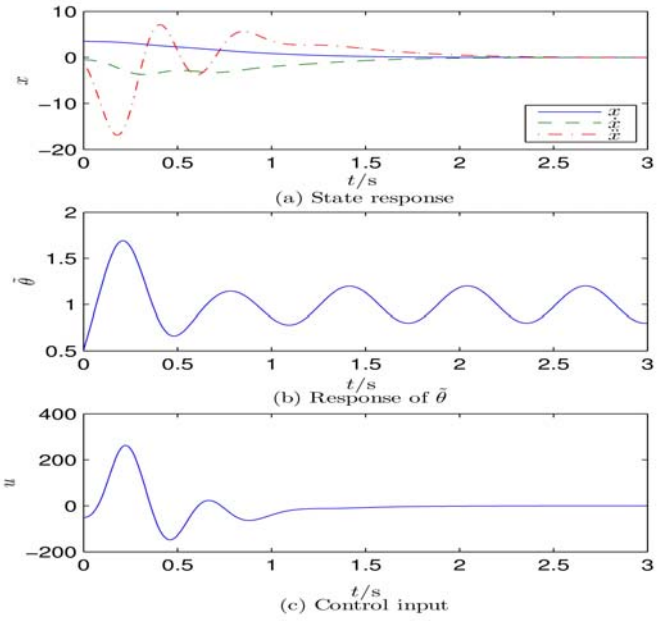


Figure 5.4: Simulation results for Case B

trollers, which is well-known as the method of backstepping. Robust control of SFSs using the method of backstepping has been tackled by a great number of researchers (e.g., Kokotovic and Arcak (2001); Itou et al. (1999); Krstic et al. (1996); Jiang and Nijmeijer (1997)).

Like most results in control systems theory, works in robust control of nonlinear systems are mainly restricted to the first-order state-space system framework.

The first-order state-space approaches have remained in a dominant position for over a half-century (Kalman (1959, 1960a); Kalman et al. (1969)). Most results in the field of control systems analysis and design are derived and presented in the first-order system framework. A state-space model focuses on the state of a system, and is more suitable and convenient for problems of state solution (response analysis) and observation (estimation). It can be used to deal with the control problem of dynamical systems, but is not the best choice. It was the preconceived occurrence which made it so popular, while it does have certain limitations in dealing with the control of dynamical systems (Duan (2020a,b, 2021a)).

FASs are originally a type of physical systems, and are hence considered to be a minor portion of the whole set of control systems since underactuated systems widely exist (Fantoni and Lozano (2002); Spong et al. (2008)). Therefore, FASs have not attracted enough attention although their control can be solved extremely simply with the help of the full-actuation feature. However, the concept of full-actuation can be generalized, as done in Duan (2020a,b, 2021a), to give a type of FAS models which

can describe many practical and theoretical systems. This stimulates us to propose the so-called FAS approaches. As a matter of fact, the key step of FAS approaches is to derive a FAS model for the considered nonlinear system, which is usually composed of a single FAS model or a set of FAS models. The control of the system can then be well solved by using the full-actuation feature of the model (Duan (2020a,b, 2021a)).

This chapter proposes an approach for robust control of nonlinear systems, which involves the following aspects.

Firstly, similar to the case of FAS models without uncertainties (Duan (2020b, 2021a)), an uncertain FAS model is also composed of either one single FAS model or a set of FAS models.

Secondly, for a single uncertain nonlinear FAS model, it is shown that a very effective FAS approach exists for designing a stabilizing or signal tracking controller, which enables that the state and their relevant orders of derivatives all converge globally into an ellipsoid with an arbitrarily small radius. Moreover, in the case that the uncertainties do not exist, the designed control law results in a constant linear closed-loop system with an arbitrarily assignable eigenstructure.

Adaptive control

Adaptive control is a control methodology capable of controlling systems with parametric uncertainties to ensure desired performance. Adaptation is often realized through an adaptive law which automatically adjusts the parameters of a feedback controller for a dynamical system with uncertain parameters, and guarantees desired stability behavior. Due to the dramatic practical demands and wide theoretical challenges, adaptive control has attracted tremendous attention in the literature (see, e.g., the survey papers Kokotovic (1991) and Tao (2014)).

Two aspects of early remarkable contributions in the field of adaptive control are self-tuning control and model reference adaptive control (MRAC). The fundamental theory, design techniques and applications of self-tuning control systems are widely reported in the literature (Aström (1980); Aström and Wittenmark (1995); Goodwin and Sin (1984); Mosca (1995); Nassiri-Toussi and Ren (1997)). A main feature of an MRAC system is that the closed-loop control system is matched to a reference model system. Early works in MRAC include Monopoli and Hsing (1975); Landau (1979); Goodwin et al. (1980); Goodwin and Long (1980); Chan and Goodwin (1982).

For nonlinear adaptive control, significant progress is also made. Firstly, adaptive control of uncertain nonlinear systems in certain parametrizable forms such as feedback linearizable (and parametrizable) systems, parametric-strict-feedback and output feedback (canonical-form) systems, has achieved remarkable successes (see, e.g., Kanellakopoulos et al. (1991); Isidori (1995); Krstic et al. (1995); Sastry and Isidori (1989); Marino and Tomei (1995)). In Seto et al. (1994); Seto and Baillieul (1994), design methods for several classes of nonlinear control systems with parametric uncertainties are presented, and it is shown how these methods may be applied without the need for linearizing coordinate transformations. Secondly, nonlinear adaptive control designs based on backstepping control techniques for nonlinear systems with a cascade structure have gained much attention (see, e.g., Krstic et al. (1995); Zhou and Wen (2008)). Thirdly, adaptive control designs and analysis for continuous-time and

discrete-time multivariable systems have achieved great advances (see, Fradkov et al. (1999); Tao (2003); Astolfi et al. (2008); Ding (2013)). Finally, of those so-called universal approximators, neural networks and fuzzy functions have been widely employed for approximation-based adaptive control of some classes of complex uncertain nonlinear dynamical systems (e.g., Brown and Harris (1994); Farrell and Polycarpou (2006); Ge et al. (2001); Lewis et al. (2002); Zhang and Liu (2006)). These have also significantly in uenced different research areas of adaptive control, especially, adaptive control of nonlinear systems.

Robust adaptive control

Backstepping designs have remained a dominant effective control methodology for nonlinear system control for decades. A systematic and fundamental introduction to adaptive backstepping designs for nonlinear SFSs is presented in the classical book Krstic et al. (1995). An excellent overview on the developments and advancements in adaptive backstepping as well as general adaptive control schemes can be found in the survey paper Tao (2014).

For SFSs with both nonlinear uncertainties and unknown parameters, many researchers developed robust adaptive control schemes by combining the ideas of robust control and adaptive control with the method of backstepping. These schemes adjust online an estimation of the parameter vector and meanwhile ensure the robust stability with respect to the system nonlinear uncertainties. Works of this aspect include systems with bounded uncertainties (Freeman et al. (1996)), single-input and single-output semi-SFSs (Yao and Tomizuka (1997)), SFSs with nonlinear uncertainties satisfying a class of triangular conditions (Polycarpou and Ioannou (1996)), and a type of SFSs with simultaneously unmodeled dynamics and nonlinear uncertainties (Jiang and Praly (1998)).

It is well-known that sliding mode control has a natural robustness feature and has attracted much attention. By combining the sliding mode methods and the modified methods of backstepping, robust adaptive control algorithms for uncertain SFSs are proposed (Hou and Duan (2011); Rios-Bolivar et al. (1996); Bartolini et al. (1996)).

For systems with very complicated nonlinear and parametric uncertainties, neural networks and fuzzy analysis are also introduced to design robust adaptive control laws. For instance, a robust adaptive control algorithm for uncertain semi-SFSs is presented based on neural networks in Gong and Yao (2001), and a general robust adaptive controller is established in Farrell and Polycarpou (2006) by combining neural networks and fuzzy analysis with the adaptive approximation approach. Furthermore, variable structure control methods utilizing neural networks for robust adaptive control of SFSs are also proposed (Li et al. (2009)).

5.4.2 Further Comments on Results

It has been shown that FAS models are a type of general system representations, and have great advantages in dealing with the control of dynamical systems. In this chapter, the problems of robust control, adaptive control and robust adaptive control of FASs have all been effectively treated. Closely related references are Duan (2021c,d,e).

The adaptive control of nonlinear systems with time-varying unknown parameters has always been a difficult problem in the field of systems and control. It is shown that for a FAS system with nonlinear uncertainties and time-varying unknown parameters, a robust adaptive control law still exists under very mild conditions, which guarantees the global stability (boundedness) of the closed-loop system.

When the unknown parameter vector is constant and the nonlinear uncertainty has a quasi-linear bound, it is further shown that a modified adaptation scheme exists, which does not depend on a pre-estimate of the parameter θ, and guarantees that the state $x^{(0\sim n-1)}$ globally converges to zero while keeping the estimation of the parameter vector bounded.

The proposed approach is very simple and convenient. In the special case that both the nonlinear uncertainties and the unknown parameters do not exist, the closed-loop system reduces to a constant linear system with arbitrarily assignable eigenstructure.

The work of this chapter can be extended to the case of high-order sub-FASs with nonlinear and parametric uncertainties. It can also be extended to different types of systems, such as, discrete-time systems and stochastic systems. Furthermore, based on the proposed results, high-order method of backstepping can also be proposed as treated in Duan (2021c) and Duan (2021d).

Part III

Generalized Strict Feedback Systems

Chapter 6

Strict Feedback Systems with Uniform Dimensions

An advantage of a high-order FAS is that there exists a controller such that a constant linear closed-loop system with an arbitrarily assignable eigenstructure can be obtained. In this chapter, a generalized form of the conventional first-order strict feedback systems (SFSs) is firstly proposed, and a recursive solution is proposed to convert equivalently the generalized SFS into a FAS model. Then the second- and high-order SFSs are defined and their equivalent FAS models are also derived. It is further shown that, under certain common conditions, the recursive solutions for converting the generalized SFSs into FAS models can be rearranged into direct analytical explicit solutions. Such a high-order system approach is more direct and simpler than the first-order system approach since it avoids the process of converting firstly these second- and high-order SFSs into first-order ones for control, and can finally produce a constant linear closed-loop system. Particularly, it is more effective than the well-known method of backstepping since, for the generalized complicated SFSs with more subsystems, the method of backstepping may simply be not applicable due to more serious "differential explosion" problem. Two examples are worked out to demonstrate the effect of the approach.

6.1 Strict Feedback Systems and Backstepping

6.1.1 A Brief Overview

Stabilization of complex dynamical systems has been a long-standing hard problem in control systems theory (Krstic and Deng (1998); Karafyllis and Jiang (2011); Chen and Huang (2015); Terrell (2009); Lin and Antsaklis (2009); Jiang and Astolfi (2020)). The well-known Lyapunov second method gives a general guideline for stabilization of a general dynamical system, but it is usually hard to find a Lyapunov function in control applications of complicated systems (Kalman and Bertram (1960); LaSalle (1960); Daafouz et al. (2002); Pukdeboon (2011)). As a matter of fact, most researchers in

Table 6.1: Searching results in Web of Science

Searched items	by title	by topic
strict-feedback systems	953	3025
strict feedback systems	971	5418
backstepping method	343	15774
back-stepping method	35	804
backstepping design	914	15485
back-stepping design	54	1048
total	3270	41554

— October 2024

the nonlinear control field have paid attention only to certain special types of systems. Maybe the first type of systems which are perfectly solvable is the feedback linearizable ones, since in the feedback linearization designs the closed-loop systems can be made linear and stable. Perhaps the largest type of systems which are considered is the type of SFSs, which stimulates the well-known backstepping designs (see Table 6.1). Besides, time-varying nonlinear systems and time-delay systems have also attracted much attention (Gao and Chen (2007); Lakshmanan and Senthilkumar (2011); Mazenc and Bliman (2006); Zhang et al. (2019); Chen et al. (2022); Guo and Han (2018); Hadj Taieb (2022)). Relatively speaking, nonlinear systems with state and/or control constraints have not gained as much attention, although they have also attracted a substantial number of researchers (Gilbert and Tan (1991); Mhaskar et al. (2006); Min et al. (2020)). Prior to 2021, almost all of the results on stabilization of complex nonlinear systems are obtained within the state-space framework.

6.1.2 SFSs and FASs

State-space approaches for control system analysis and design have remained absolutely dominant for over half a century, yet it is found that state-space models may not be the best choice for dealing with the control of the system, although they are probably the best choice for state solutions (response analysis) and observations (estimations) (see, Duan (2020a,b,c, 2021a)). In contrast, a type of high-order FAS models behave more powerfully in the control of many nonlinear systems (Duan (2020a, 2021a)). This chapter is concerned with the derivation of the FAS models of three types of generalized SFSs.

Firstly, more general nonlinear systems can be equivalently converted into the form of a FAS. In fact, it has been shown that the general nonlinear systems in the following affine form:

$$\dot{x} = f(x) + g(x)u,$$

with $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^r$ being respectively the state and control vectors, can be converted into the form of a FAS under some conditions (Duan (2020a,b,c)). Very recently, Duan (2021a) investigated a more general system in the form of

$$x^{(m)} = f(x, \dot{x}, \cdots, x^{(m-1)}) + G(x, \dot{x}, \cdots, x^{(m-1)})u,$$

where $m \geq 1$ is some integer, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^r$ are respectively the state and control vectors. It is shown that the system, when n is a multiple of r, can also be converted, under certain conditions, into a similar strict-feedback form of order m (Duan (2021a)), which reduces to the normal FAS form in the case of $m = 1$. Such facts indicate, from the theoretical aspect, that FASs are a type of very common nonlinear systems.

Secondly, SFSs describe a variety of practical nonlinear systems. Such a fact has been verified by a great deal of applications, including, but certainly not limited to, circuit system (Khalil (2002a)), pendulum systems (Khalil (2002a), Jiang and Nijmeijer (1997); Yang et al. (2004)), robotic systems (Khalil (2002a); Marino and Tomei (1995); Dawson et al. (1994); Feuer and Giacomini (2000); Yang et al. (2004)), missile and satellite control systems (Farrell et al. (2005); Kim et al. (2004); Sun et al. (2017)), and ship fin roll stabilization systems (Yang et al. (2004)).

Thirdly, SFSs probably form the largest class of nonlinear systems for which stabilizing controllers can be systematically designed. A very typical recursive design method for this class of systems is the well-known method of backstepping, which was initiated by a group of researchers, e.g., Tsinias (1989, 1991); Byrnes and Isidori (1989); Sontag and Sussmann (1988); Kokotovic and Sussmann (1989); Saberi et al. (1990). Since 1990, works on control of SFSs using improved method of backstepping have dramatically increased (Kokotovic and Arcak (2001); Farrell et al. (2009); Tee and Ge (2011); Huang et al. (2005); Hong and Jiang (2006)).

6.2 First-order SFSs

Let us consider the following time-varying SFS with uniform dimensions and a parameter vector $\zeta \in \mathbb{R}^m$:

$$\begin{cases} \dot{x}_1 = f_1(x_1, \zeta, t) + G_1(x_1, \zeta, t)x_2 \\ \dot{x}_2 = f_2(x_{1\sim 2}, \zeta, t) + G_2(x_{1\sim 2}, \zeta, t)x_3 \\ \quad\quad \vdots \\ \dot{x}_{n-1} = f_{n-1}(x_{1\sim n-1}, \zeta, t) + G_{n-1}(x_{1\sim n-1}, \zeta, t)x_n \\ \dot{x}_n = f_n(x_{1\sim n}, \zeta, t) + G_n(x_{1\sim n}, \zeta, t)u, \end{cases} \tag{6.1}$$

where $x_i \in \mathbb{R}^r$, $i = 1, 2, \ldots, n$, are the state vectors, $u \in \mathbb{R}^r$ is the control input, $f_i(x_{1\sim i}, \zeta, t) \in \mathbb{R}^r$ and $G_i(x_{1\sim i}, \zeta, t) \in \mathbb{R}^{r\times r}$, $i = 1, 2, \ldots, n$, are sufficiently smooth vector functions and matrix functions, respectively.

As a normal requirement to SFSs, we assume the following:

Assumption 6.1 *For arbitrary $x_i \in \mathbb{R}^r$, $i = 1, 2, \ldots, n$, and $\zeta \in \mathbb{R}^m$, there holds*

$$\det G_i(x_{1\sim i}, \zeta, t) \neq 0, \ \forall t \geq 0. \tag{6.2}$$

In certain high-dimensional cases, the above Assumption 6.1 may be difficult to verify. It is recommended that symbolic computation techniques may be considered to help in such cases.

More concretely, the f and B terms in FASs (2.39) or (2.56) are finally obtained by iterative formulas using $f_i(x_{1\sim i})$ and $G_i(x_{1\sim i})$, $i = 1, 2, \ldots, n$. Moreover, a discrete-time version of similar results can be found in Duan (2022f), and generalization of the results to SFSs with time-varying delays can be found in Duan (2023b,c).

Remark 6.1 It has been shown in Duan (2021b) that SFS (6.1) can be equivalently converted into a single-order global FAS in the form of (2.39). Here we present this result. Meanwhile, let us point out that, when the dimension of $x_i \in \mathbb{R}^r$, $i = 1, 2, \ldots, n$, are not equal, but increasing, it has been shown in Xu and Duan (2024) (see also Xu (2023) for the discrete-time case) that SFS (6.1) can be equivalently converted into a multi-order global FAS in the form of (2.56). Such a result will be presented in the next chapter.

6.2.1 Recursive Solution

For this generalized form (6.1), some existing control methods, such as the backstepping methods, may become extremely difficult or even impossible to apply when r and n are large due to the "differential explosion" problem. However, if we can transform it into a FAS model, its control problems can then be solved easily and conveniently (Duan (2020a)).

Theorem 6.1 *Let Assumption 6.1 be satisfied, then, under the following transformation*

$$\begin{cases} x_1 = z \\ x_2 = G_1^{-1}(z, \zeta, t)[\dot{z} - f_1(z, \zeta, t)] \\ x_{k+1} = L_k^{-1}(z^{(0\sim k-1)}, \zeta, t)[z^{(k)} - h_k(z^{(0\sim k-1)}, \zeta, t)], \\ \qquad\qquad k = 2, 3, \ldots, n-1, \end{cases} \tag{6.3}$$

the generalized SFS (6.1) can be transformed into the following FAS:

$$z^{(n)} = h_n(z^{(0\sim n-1)}, \zeta, t) + L_n(z^{(0\sim n-1)}, \zeta, t)u, \tag{6.4}$$

where the matrix function $L_k(z^{(0\sim k-1)}, \zeta, t)$ and the vector function $h_k(z^{(0\sim k-1)}, \zeta, t)$ are recursively given by

$$L_k(z^{(0\sim k-1)}, \zeta, t) = L_{k-1}(z^{(0\sim k-2)}, \zeta, t)G_k(x_{1\sim k}, \zeta, t), \tag{6.5}$$

and

$$\begin{aligned} h_k(z^{(0\sim k-1)}, \zeta, t) \;=\; & \dot{h}_{k-1}(z^{(0\sim k-2)}, \zeta, t) + \dot{L}_{k-1}(z^{(0\sim k-2)}, \zeta, t)x_k \\ & + L_{k-1}(z^{(0\sim k-2)}, \zeta, t)\, f_k(x_{1\sim k}, \zeta, t), \\ & \qquad k = 1, 2, \ldots, n, \end{aligned} \tag{6.6}$$

with the initial values

$$L_0(\zeta, t) = I_r,\ h_0(\zeta, t) = 0, \tag{6.7}$$

where in (6.5)-(6.6) the x_i, $i = 2, 3, \ldots, n$, and their derivatives are all given or determined by transformation (6.3).

For a proof of the theorem, please refer to Appendix B. Please note that the above iteration (6.5)-(6.7) gives

$$h_1(x_1,\zeta,t)=f_1(x_1,\zeta,t),\ L_1(x_1,\zeta,t)=G_1(x_1,\zeta,t).$$

Remark 6.2 Assumption 6.1 is a global one, it requires (6.2) to hold for $x_i\in\mathbb{R}^r$, $i=1,2,\ldots,n$. Relaxation of this assumption to a local one can be considered. If this assumption is valid only on a certain set $\Omega\subset\mathbb{R}^{nr}$, then the derived high-order system (6.4) is a FAS model on a certain set Ω' determined from Ω by transformation (6.3).

Remark 6.3 Once the FAS model (6.4) is obtained, the controller of the system can then be simply designed as (Duan (2020a), Duan (2021a))

$$\begin{cases} u=-L_n^{-1}\left(A_{0\sim n-1}z^{(0\sim n-1)}+u^*\right)\\ u^*=h_n(z^{(0\sim n-1)},\zeta,t)-v, \end{cases} \tag{6.8}$$

which gives the following constant linear closed-loop system:

$$z^{(n)}+A_{0\sim n-1}z^{(0\sim n-1)}=v,$$

where v is some external signal, while $A_{0\sim n-1}$ is an arbitrary matrix which makes $\Phi(A_{0\sim n-1})$ stable. A complete parametric approach for solving the matrix $A_{0\sim n-1}$ is given in Duan (2020a) (see also the Proposition 2 in Duan (2021a)).

An illustration of transformation (6.3) is given in Fig. 6.1. Through the proof of the above theorem, which is given in Appendix B, we can observe that transformation (6.3) is in fact the inverse of the following transformation $\left(x_{1\sim n}\longmapsto z^{(0\sim n-1)}\right)$:

$$z^{(0\sim n-1)}=\begin{bmatrix} x_1\\ h_1(x_1,\zeta,t)+L_1(x_1,\zeta,t)x_2\\ h_2(x_{1\sim 2},\zeta,t)+L_2(x_{1\sim 2},\zeta,t)x_3\\ \vdots\\ h_{n-1}(x_{1\sim n-1},\zeta,t)+L_{n-1}(x_{1\sim n-1},\zeta,t)x_n \end{bmatrix}, \tag{6.9}$$

which can be compactly written as

$$z^{(0\sim n-1)}=f_{\rm e}(x_{1\sim n-1},\zeta,t)+B_{\rm e}(x_{1\sim n-1},\zeta,t)x_{1\sim n},$$

where

$$\begin{aligned} f_{\rm e}(x_{1\sim n-1},\zeta,t) &= \begin{bmatrix} 0\\ h_1(x_1,\zeta,t)\\ h_2(x_{1\sim 2},\zeta,t)\\ \vdots\\ h_{n-1}(x_{1\sim n-1},\zeta,t) \end{bmatrix},\\ B_{\rm e}(x_{1\sim n-1},\zeta,t) &= \operatorname{diag}\left(I_r,L_1(x_1,\zeta,t),L_2(x_{1\sim 2},\zeta,t),\cdots,L_{n-1}(x_{1\sim n-1},\zeta,t)\right). \end{aligned}$$

Clearly, transformation (6.3), or (6.9), is globally one-to-one. Furthermore, it is clearly seen from (6.9) that $x_{1\sim n}=0$ implies $z^{(0\sim n-1)}=0$. Thus transformation (6.3), or (6.9), also keep the origin unchanged.

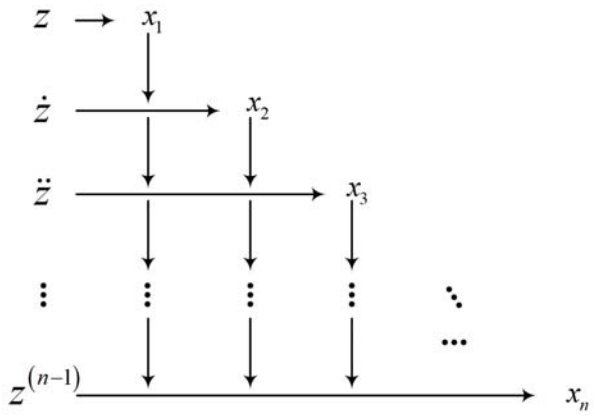


Figure 6.1: Illustration of transformation (6.3)

6.2.2 Explicit Solution

Let us consider again the generalized SFS (6.1), but with the following condition imposed:

Condition 6.1 *The coefficient matrices* $G_i(x_{1\sim i},\zeta,t)=G_i\in\mathbb{R}^{r\times r}$, $i=1,2,\ldots,n-1$, *are restricted to be constant nonsingular ones, and*

$$\det G_n(x_{1\sim n},\zeta,t)\neq 0,\ \forall t\geq 0 \tag{6.10}$$

holds for arbitrary $x_k\in\mathbb{R}^r$, $k=1,2,\ldots,n$, *and* $\zeta\in\mathbb{R}^m$.

Using the above condition and Theorem 6.1, we can derive the following result, which provides a direct analytical explicit solution to the problem of converting SFS (6.1) into a FAS model.

Theorem 6.2 *Let Condition 6.1 be satisfied, then, under the following transformation*

$$\begin{cases} x_1=z \\ x_2=G_1^{-1}[\dot{z}-f_1(z,\zeta,t)] \\ x_{k+1}=L_k^{-1}[z^{(k)}-h_k(z^{(0\sim k-1)},\zeta,t)], \\ \qquad\qquad k=2,3,\ldots,n-1, \end{cases} \tag{6.11}$$

the generalized SFS (6.1) can be equivalently transformed into the following FAS:

$$z^{(n)}=h_n(z^{(0\sim n-1)},\zeta,t)+L_n(z^{(0\sim n-1)},\zeta,t)u, \tag{6.12}$$

with

$$\begin{cases} L_k=G_1G_2\cdots G_k,\ k=1,2,\ldots,n-1 \\ L_n(z^{(0\sim n-1)},\zeta,t)=L_{n-1}G_n(x_{1\sim n},\zeta,t), \end{cases} \tag{6.13}$$

and

$$h_n(z^{(0\sim n-1)},\zeta,t)=L_{0\sim n-1}\begin{bmatrix} f_1^{(n-1)}(z,\zeta,t)\\ f_2^{(n-2)}(z,x_2,\zeta,t)\\ \vdots\\ \dot{f}_{n-1}(x_{1\sim n-1},\zeta,t)\\ f_n(x_{1\sim n},\zeta,t)\end{bmatrix}, \tag{6.14}$$

where $L_0=I$, and in (6.13)-(6.14) the x_k, $k=2,3,\ldots,n$, and their derivatives are all given or determined by transformation (6.11).

Proof. According to (6.5), we have

$$\begin{aligned} L_0 &= I,\\ L_1 &= L_0G_1=G_1,\\ L_2 &= L_1G_2=G_1G_2,\\ &\vdots\\ L_k &= L_{k-1}G_k=G_1G_2\cdots G_k. \end{aligned}$$

Thus the matrix $L_k(z^{(0\sim k-1)},\zeta,t)$ can be given recursively by

$$\begin{aligned} L_k(z^{(0\sim k-1)},\zeta,t) &= L_{k-1}(z^{(0\sim k-2)},\zeta,t)G_k(x_{1\sim k},\zeta,t), \\ &\qquad k=1,2,\ldots,n, \end{aligned} \tag{6.15}$$

with the initial value $L_0=I$. This is obviously equivalent to (6.13).

Simultaneously, using (6.6), and taking $\dot{L}_k=0, k=1,2,\ldots,n-1$, into consideration, we can write the recursive formula for $h_k(z^{(0\sim k-1)},\zeta,t)$ as

$$\begin{aligned} h_k(z^{(0\sim k-1)},\zeta,t) &= \dot{h}_{k-1}(z^{(0\sim k-2)},\zeta,t)+L_{k-1}\,f_k(x_{1\sim k},\zeta,t), \\ &\qquad k=1,2,\ldots,n, \end{aligned} \tag{6.16}$$

with the initial value $h_0(\zeta,t)=0$. Thus, we further have, from the above formula,

$$h_1(z,\zeta,t)=f_1(z,\zeta,t), \tag{6.17}$$

$$\begin{aligned} h_2(z^{(0\sim1)},\zeta,t) &= \dot{h}_1(z,\zeta,t)+L_1\,f_2(x_{1\sim2},\zeta,t)\\ &= [I \quad L_1]\begin{bmatrix}\dot{f}_1(z,\zeta,t)\\ f_2(x_{1\sim2},\zeta,t)\end{bmatrix}, \end{aligned}$$

$$\begin{aligned} & h_3(z^{(0\sim2)},\zeta,t)\\ =\ & \dot{h}_2(z^{(0\sim1)},\zeta,t)+L_2\,f_3(x_{1\sim3},\zeta,t)\\ =\ & [I \quad L_1]\begin{bmatrix}\ddot{f}_1(z,\zeta,t)\\ \dot{f}_2(x_{1\sim2},\zeta,t)\end{bmatrix}+L_2\,f_3(x_{1\sim3},\zeta,t)\\ =\ & L_{0\sim2}\begin{bmatrix}\ddot{f}_1(z,\zeta,t)\\ \dot{f}_2(x_{1\sim2},\zeta,t)\\ f_3(x_{1\sim3},\zeta,t)\end{bmatrix}. \end{aligned}$$

Continuing this process, we can finally get formula (6.14). Noting that (6.11) can be obtained easily, we then complete the proof. ■

6.3 Second-order SFSs

Lagrangian Equation and Theorem of Momentum (moment) are often used in modeling physical systems. As a result in such applications, the models of the obtained subsystems are of second-order. For this reason, we introduce the following second-order SFS:

$$\begin{cases} \ddot{x}_1 = f_1(x_1^{(0\sim 1)},\zeta,t) + G_1(x_1^{(0\sim 1)},\zeta,t)x_2 \\ \ddot{x}_2 = f_2(x_{1\sim 2}^{(0\sim 1)},\zeta,t) + G_2(x_{1\sim 2}^{(0\sim 1)},\zeta,t)x_3 \\ \qquad\vdots \\ \ddot{x}_{n-1} = f_{n-1}(x_{1\sim n-1}^{(0\sim 1)},\zeta,t) + G_{n-1}(x_{1\sim n-1}^{(0\sim 1)},\zeta,t)x_n \\ \ddot{x}_n = f_n(x_{1\sim n}^{(0\sim 1)},\zeta,t) + G_n(x_{1\sim n}^{(0\sim 1)},\zeta,t)u, \end{cases} \tag{6.18}$$

where $x_i \in \mathbb{R}^r$, $i = 1,2,\ldots,n$, are the state vectors, $u \in \mathbb{R}^r$ is the control input vector, $\zeta \in \mathbb{R}^m$ is a parameter vector; $f_i(x_{1\sim i}^{(0\sim 1)},\zeta,t) \in \mathbb{R}^r$, $i = 1,2,\ldots,n$, are a set of sufficiently smooth vector functions, $G_i(x_{1\sim i}^{(0\sim 1)},\zeta,t) \in \mathbb{R}^{r\times r}$, $i = 1,2,\ldots,n$, are a set of sufficiently smooth matrix functions, and satisfy the following assumption:

Assumption 6.2 *For arbitrary $x_i \in \mathbb{R}^r$, $i = 1,2,\ldots,n$, and $\zeta \in \mathbb{R}^m$, there holds*

$$\det G_i(x_{1\sim i}^{(0\sim 1)},\zeta,t) \neq 0, \forall t \geq 0.$$

6.3.1 Recursive Solution

Parallel to the first-order case, we have the following conclusion.

Theorem 6.3 *Let Assumption 6.2 be met, then, under the following transformation*

$$\begin{cases} x_1 = z \\ x_2 = G_1^{-1}(z^{(0\sim 1)},\zeta,t)[\ddot{z} - f_1(z^{(0\sim 1)},\zeta,t)] \\ x_{k+1} = L_k^{-1}(z^{(0\sim 2k-1)},\zeta,t)[z^{(2k)} - h_k(z^{(0\sim 2k-1)},\zeta,t)], \\ \qquad\qquad k = 2,3,\ldots,n-1, \end{cases} \tag{6.19}$$

SFS (6.18) can be equivalently converted into the following FAS:

$$z^{(2n)} = h_n(z^{(0\sim 2n-1)},\zeta,t) + L_n(z^{(0\sim 2n-1)},\zeta,t)u, \tag{6.20}$$

where the functions $L_k(z^{(0\sim 2k-1)},\zeta,t)$ and $h_k(z^{(0\sim 2k-1)},\zeta,t)$ are given recursively by

$$L_k(z^{(0\sim 2k-1)},\zeta,t) = L_{k-1}(z^{(0\sim 2k-3)},\zeta,t)G_k(x_{1\sim k}^{(0\sim 1)},\zeta,t), \tag{6.21}$$

$$\begin{aligned} h_k(z^{(0\sim 2k-1)},\zeta,t) = {} & \ddot{h}_{k-1}(z^{(0\sim 2k-3)},\zeta,t) + \ddot{L}_{k-1}(z^{(0\sim 2k-3)},\zeta,t)x_k \\ & +2\dot{L}_{k-1}(z^{(0\sim 2k-3)},\zeta,t)\dot{x}_k \\ & +L_{k-1}(z^{(0\sim 2k-3)},\zeta,t)\, f_k(x_{1\sim k}^{(0\sim 1)},\zeta,t), \\ & \qquad k = 1,2,\ldots,n, \end{aligned} \tag{6.22}$$

with the initial values

$$L_0(\zeta,t) = I_r,\ h_0(\zeta,t) = 0, \tag{6.23}$$

where in (6.21)-(6.22) the $x_i, i = 2, 3, \ldots, n$, and their derivatives are given or determined by transformation (6.19).

For a proof of the theorem, please refer to Appendix B.

It is noted that facts similar to those mentioned in the Remarks 6.2 and 6.3 also hold true in this second-order case.

SFS (6.18) clearly has the following companion form:

$$\begin{cases} \ddot{x}_1 = f_1(x_1^{(0\sim 1)}, \zeta, t) + G_1(x_1^{(0\sim 1)}, \zeta, t)\dot{x}_2 \\ \ddot{x}_2 = f_2(x_{1\sim 2}^{(0\sim 1)}, \zeta, t) + G_2(x_{1\sim 2}^{(0\sim 1)}, \zeta, t)\dot{x}_3 \\ \qquad \vdots \\ \ddot{x}_{n-1} = f_{n-1}(x_{1\sim n-1}^{(0\sim 1)}, \zeta, t) + G_{n-1}(x_{1\sim n-1}^{(0\sim 1)}, \zeta, t)\dot{x}_n \\ \ddot{x}_n = f_n(x_{1\sim n}^{(0\sim 1)}, \zeta, t) + G_n(x_{1\sim n}^{(0\sim 1)}, \zeta, t)u. \end{cases} \tag{6.24}$$

Following the same procedure, the above SFS (6.24) can also be converted into a FAS model. We remark that the derivation process is even simpler since for this companion form each updating only needs a first-order differential of each subsystem.

6.3.2 Explicit Solution

Let us consider again the generalized SFS (6.18), but with the following condition imposed:

Condition 6.2 *The coefficient matrices $G_i(x_{1\sim i}^{(0\sim 1)}, \zeta, t) = G_i \in \mathbb{R}^{r\times r}$, $i = 1, 2, \ldots, n-1$, are restricted to be constant nonsingular ones, and*

$$\det G_n(x_{1\sim n}^{(0\sim 1)}, \zeta, t) \neq 0, t \geq 0 \tag{6.25}$$

holds for arbitrary $x_k \in \mathbb{R}^r$, $k = 1, 2, \ldots, n$, and $\zeta \in \mathbb{R}^m$.

Similarly, using the above condition and Theorem 6.3, we can derive the following result, which provides a direct analytical explicit solution to the problem of converting the second-order SFS (6.18) into a FAS model (proof omitted).

Theorem 6.4 *Let Condition 6.2 be met, then, under the following transformation*

$$\begin{cases} x_1 = z \\ x_2 = G_1^{-1}[\ddot{z} - f_1(z^{(0\sim 1)}, \zeta, t)] \\ x_{k+1} = L_k^{-1}[z^{(2k)} - h_k(z^{(0\sim 2k-1)}, \zeta, t)], \\ \qquad\qquad k = 2, 3, \ldots, n-1, \end{cases} \tag{6.26}$$

SFS (6.18) can be equivalently transformed into the following FAS:

$$z^{(2n)} = h_n(z^{(0\sim 2n-1)}, \zeta, t) + L_n(z^{(0\sim 2n-1)}, \zeta, t)u, \tag{6.27}$$

with

$$\begin{cases} L_k = G_1 G_2 \cdots G_k, \ k = 1, 2, \ldots, n-1 \\ L_n(z^{(0\sim 2n-1)}, \zeta, t) = L_{n-1} G_n(x_{1\sim n}^{(0\sim 1)}, \zeta, t), \end{cases} \tag{6.28}$$

and

$$h_n(z^{(0\sim 2n-1)},\zeta,t)=L_{0\sim n-1}\begin{bmatrix} f_1^{(2(n-1))}(z^{(0\sim 1)},\zeta,t)\\ f_2^{(2(n-2))}(x_{1\sim 2}^{(0\sim 1)},\zeta,t)\\ \vdots\\ \ddot{f}_{n-1}(x_{1\sim n-1}^{(0\sim 1)},\zeta,t)\\ f_n(x_{1\sim n}^{(0\sim 1)},\zeta,t)\end{bmatrix}, \tag{6.29}$$

where $L_0=I_r$, and in (6.28)-(6.29) the x_k, $k=2,3,\ldots,n$, and their derivatives are given or determined by transformation (6.26).

For the most encountered case of $n=2$, the above result becomes the following.

Corollary 6.1 *If $\det G_1\neq 0$, and, for $x_1,x_2\in\mathbb{R}^r$, and $\zeta\in\mathbb{R}^m$, there holds*

$$\det G_2(x_{1\sim 2}^{(0\sim 1)},\zeta,t)\neq 0,$$

then, in the case of $n=2$, SFS (6.18) can be transformed into the following FAS:

$$z^{(4)}=h(z^{(0\sim 3)},\zeta,t)+L(z^{(0\sim 3)},\zeta,t)u, \tag{6.30}$$

with

$$L(z^{(0\sim 3)},\zeta,t) = G_1G_2(x_{1\sim 2}^{(0\sim 1)},\zeta,t), \tag{6.31}$$

$$h(z^{(0\sim 3)},\zeta,t) = [I_r\ G_1]\begin{bmatrix}\ddot{f}_1(z^{(0\sim 1)},\zeta,t)\\ f_2(x_{1\sim 2}^{(0\sim 1)},\zeta,t)\end{bmatrix}, \tag{6.32}$$

where $L_0=I_r$, and in (6.31)-(6.32) the x_2 and its derivative are given and determined by

$$x_2=G_1^{-1}[\ddot{z}-f_1(z^{(0\sim 1)},\zeta,t)].$$

6.4 Mixed-order SFSs

When modeling complex physical systems, there are systems governed by Newton's Law, Lagrangian Equation, or Theorem of Momentum, etc., while there may also be subsystems governed by other physical or chemical laws which produce the first-order or high-order differential equations. Therefore, there may be subsystems of first-, second- and high-order. Moreover, by the FAS approach, some of the subsystems may also be equivalently turned into high-order ones. Due to these observations, we have also introduced in Duan (2021b) the so-called high-order (mixed-order) SFSs.

6.4.1 Recursive Solution

The high-order (mixed-order) SFSs introduced in Duan (2021b) appear in the following form:

$$\begin{cases} x_1^{(m_1)} = f_1(x_1^{(0\sim m_1-1)}, \zeta, t) + G_1(x_1^{(0\sim m_1-1)}, \zeta, t)x_2 \\ x_2^{(m_2)} = f_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2}, \zeta, t) + G_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2}, \zeta, t)x_3 \\ \quad\quad\vdots \\ x_n^{(m_n)} = f_n(x_i^{(0\sim m_i-1)}|_{i=1\sim n}, \zeta, t) + G_n(x_i^{(0\sim m_i-1)}|_{i=1\sim n}, \zeta, t)u, \end{cases} \tag{6.33}$$

where $x_k \in \mathbb{R}^r$, $k = 1, 2, \ldots, n$, are the state vectors, $u \in \mathbb{R}^r$ is the control input vector, $\zeta \in \mathbb{R}^m$ is a parameter vector; m_k, $k = 1, 2, \ldots, n$, are a set of positive integers, which often take the values of 1 and 2 in practical applications, and $f_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k}, \zeta, t) \in \mathbb{R}^r, k = 1, 2, \ldots, n$, are a set of sufficiently smooth vector functions; $G_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k}, \zeta, t) \in \mathbb{R}^{r\times r}$, $k = 1, 2, \ldots, n$, are a set of sufficiently smooth matrix functions satisfying the following assumption:

Assumption 6.3 *For all $x_k \in \mathbb{R}^r$, $k = 1, 2, \ldots, n$, and $\zeta \in \mathbb{R}^m$, there holds*

$$\det G_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k}, \zeta, t) \neq 0, \ \forall t \geq 0.$$

For convenience, we further introduce the following notations:

$$\begin{cases} p_i = m_1 + m_2 + \cdots + m_i, \ i = 1, 2, \ldots, n \\ p = p_n. \end{cases} \tag{6.34}$$

Then, parallel to the first-order and second-order cases, we have the following conclusion.

Theorem 6.5 *Let Assumption 6.3 hold, then, under the following transformation*

$$\begin{cases} x_1 = z \\ x_2 = G_1^{-1}(z^{(0\sim m_1-1)}, \zeta, t)[z^{(m_1)} - f_1(z^{(0\sim m_1-1)}, \zeta, t)] \\ x_{i+1} = L_i^{-1}(z^{(0\sim p_i-1)}, \zeta, t)[z^{(p_i)} - h_i(z^{(0\sim p_i-1)}, \zeta, t)], \\ \qquad\qquad\qquad\qquad\qquad\qquad i = 2, 3, \ldots, n-1, \end{cases} \tag{6.35}$$

SFS (6.33) can be equivalently turned into the following FAS:

$$z^{(p)} = h_n(z^{(0\sim p-1)}, \zeta, t) + L_n(z^{(0\sim p-1)}, \zeta, t)u, \tag{6.36}$$

where the functions $L_k(z^{(0\sim p_k-1)},\zeta,t)$ and $h_k(z^{(0\sim p_k-1)},\zeta,t)$ are recursively given by

$$\begin{aligned} & L_k(z^{(0\sim p_k-1)},\zeta,t) \\ = \; & L_{k-1}(z^{(0\sim p_{k-1}-1)},\zeta,t)G_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k-1},\zeta,t), \qquad (6.37) \\ & h_k(z^{(0\sim p_k-1)},\zeta,t) \\ = \; & h_{k-1}^{(m_k)}(z^{(0\sim p_{k-1}-1)},\zeta,t) \\ & + \sum_{j=0}^{m_k-1} C(j,m_k)L_{k-1}^{(m_k-j)}(z^{(0\sim p_{k-1}-1)},\zeta,t)x_k^{(j)} \\ & + L_{k-1}(x_1^{(0\sim p_{k-1}-1)},\zeta,t)\, f_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k-1},\zeta,t), \qquad (6.38) \\ & \qquad k=1,2,\ldots,n, \end{aligned}$$

with the initial values

$$L_0(\zeta,t)=I_r,\; h_0(\zeta,t)=0, \qquad (6.39)$$

where in (6.37)-(6.38) the $x_k,\ k=2,3,\ldots,n$, and their derivatives are given or determined by transformation (6.35).

For a proof of the above result, please refer to Appendix B.

Parallel to the second-order SFS (6.24), the mixed-order SFS (6.33) also has the following generalized form:

$$\begin{cases} x_1^{(m_1)} = f_1(x_1^{(0\sim m_1-1)},\zeta,t)+G_1(x_1^{(0\sim m_1-1)},\zeta,t)x_2^{(m_1^\circ)} \\ x_2^{(m_2)} = f_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2},\zeta,t)+G_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2},\zeta,t)x_3^{(m_2^\circ)} \\ \qquad\vdots \\ x_{n-1}^{(m_{n-1})} = f_{n-1}(x_i^{(0\sim m_i-1)}|_{i=1\sim n-1},\zeta,t)+G_{n-1}(x_i^{(0\sim m_i-1)}|_{i=1\sim n-1},\zeta,t)x_n^{(m_{n-1}^\circ)} \\ x_n^{(m_n)} = f_n(x_i^{(0\sim m_i-1)}|_{i=1\sim n},\zeta,t)+G_n(\upsilon,\zeta,t)u, \end{cases} \qquad (6.40)$$

where $m_i^\circ < m_i,\ i=1,2,\ldots,n-1$, are another set of integers. Following the same procedure as shown in the proof of Theorem 6.5, the above mixed-order SFS can also be converted into a FAS model.

Remark 6.4 The proposed FAS approach is generally more effective than the method of backstepping because of the following two facts:

- the FAS approach can always produce a constant linear closed-loop system, but the method of backstepping can not;
- due to the well-known problem of "differential explosion", the method of backstepping is generally not applicable to an SFS with more than 3 or 4 subsystems, while the FAS approach is, noting that the proposed recursive solutions are easy to realize.

Remark 6.5 Although it is apparent that the first- and second-order SFSs (6.1) and (6.18) are the special cases of the high-order one (6.33), it is hard to tell a similar relation among the outcomes of the converted high-order systems of these systems.

6.4.2 Explicit Solution

For the high-order SFS (6.33), we also impose the following condition:

Condition 6.3 *The coefficient matrices* $G_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k},\zeta,t)=G_k\in\mathbb{R}^{r\times r}$, $k=1,2,\ldots,n-1$, *are restricted to be constant nonsingular ones, and*

$$\det G_n(x_i^{(0\sim m_i-1)}|_{i=1\sim n},\zeta,t)\neq 0,\ \forall t\geq 0 \tag{6.41}$$

holds for all $x_k\in\mathbb{R}^r$, $k=1,2,\ldots,n$, *and* $\zeta\in\mathbb{R}^m$.

For convenience, we also introduce the following notations:

$$\begin{array}{rcl} p_{i\sim j} & = & m_i+m_{i+1}+\cdots+m_j, \\ i,j & = & 1,2,\ldots,n, \quad j>i. \end{array} \tag{6.42}$$

Clearly, we have, for $i=1,2,\ldots,n$,

$$p_i=p_{1\sim i}=m_1+m_2+\cdots+m_i, \tag{6.43}$$

$$p=p_n=p_{1\sim n}. \tag{6.44}$$

Parallel to the first-order and second-order cases, we have the following conclusion which can be derived from Theorem 6.5 (proof omitted).

Theorem 6.6 *Let Condition 6.3 be met, then, under the following transformation*

$$\left\{\begin{array}{l} x_1=z \\ x_2=G_1^{-1}[z^{(m_1)}-f_1(z^{(0\sim m_1-1)},\zeta,t)] \\ x_{k+1}=L_k^{-1}[z^{(p_k)}-h_k(z^{(0\sim p_k-1)},\zeta,t)], \\ \qquad\qquad\qquad\qquad k=2,3,\ldots,n-1, \end{array}\right. \tag{6.45}$$

SFS (6.33) can be equivalently turned into the following FAS:

$$z^{(p)}=h_n(z^{(0\sim p-1)},\zeta,t)+L_n(z^{(0\sim p-1)},\zeta,t)u, \tag{6.46}$$

with

$$\left\{\begin{array}{l} L_k=G_1G_2\cdots G_k,\ k=1,2,\ldots,n-1 \\ L_n(z^{(0\sim p-1)},\zeta,t)=L_{n-1}G_n(x_i^{(0\sim m_i)}|_{i=1\sim n-1},x_n^{(0\sim m_n-1)},\zeta,t), \end{array}\right. \tag{6.47}$$

and

$$h_n(z^{(0\sim p_n-1)},\zeta,t)=L_{0\sim n-1}\left[\begin{array}{c} f_1^{(p_{2\sim n})}(z^{(0\sim m_1-1)},\zeta,t) \\ f_2^{(p_{3\sim n})}(x_1^{(0\sim m_1-1)},x_2^{(0\sim m_2-1)},\zeta,t) \\ \vdots \\ f_{n-1}^{(m_n)}(x_i^{(0\sim m_i-1)}|_{i=1\sim n-1},\zeta,t) \\ f_n(x_i^{(0\sim m_i-1)}|_{i=1\sim n},\zeta,t) \end{array}\right] \tag{6.48}$$

where $L_0=I_r$, *and in (6.47)-(6.48) the* x_k, $k=2,3,\ldots,n$, *and their derivatives are given or determined by transformation (6.45).*

For the often encountered case of $n = 3$, the above result reduces to the following.

Corollary 6.2 *If* $\det G_1 G_2 \neq 0$, *and for all* $x_k \in \mathbb{R}^r$, $k = 1, 2, 3$, *and* $\zeta \in \mathbb{R}^m$, *there holds*

$$\det G_3(x_i^{(0\sim m_i-1)}|_{i=1\sim 3}, \zeta, t) \neq 0, \ \forall t \geq 0,$$

then, for the case of $n = 3$, *SFS (6.33) can be equivalently turned into the following FAS:*

$$z^{(p)} = h(z^{(0\sim p-1)}, \zeta, t) + L(z^{(0\sim p-1)}, \zeta, t)u, \tag{6.49}$$

with $p = m_1 + m_2 + m_3$, *and*

$$L(z^{(0\sim p-1)}, \zeta, t) = G_1 G_2 G_3(x_i^{(0\sim m_i-1)}|_{i=1\sim 3}, \zeta, t), \tag{6.50}$$

and

$$\begin{aligned} & h(z^{(0\sim p-1)}, \zeta, t) \\ = & \begin{bmatrix} I_r & G_1 & G_1 G_2 \end{bmatrix} \begin{bmatrix} f_1^{(m_2+m_3)}(z^{(0\sim m_1-1)}, \zeta, t) \\ f_2^{(m_3)}(x_1^{(0\sim m_1-1)}, x_2^{(0\sim m_2-1)}, \zeta, t) \\ f_3(x_i^{(0\sim m_i-1)}|_{i=1\sim 3}, \zeta, t) \end{bmatrix}, \end{aligned} \tag{6.51}$$

where in (6.50)-(6.51) the x_2, x_3 *and their derivatives are given and determined by*

$$\begin{cases} x_2 = G_1^{-1}[z^{(m_1)} - f_1(z^{(0\sim m_1-1)}, \zeta, t)] \\ x_3 = G_2^{-1} G_1^{-1}[z^{(p_2)} - h_2(z^{(0\sim p_2-1)}, \zeta, t)], \end{cases} \tag{6.52}$$

with

$$h_2(z^{(0\sim p_2-1)}, \zeta, t) = \begin{bmatrix} I_r & G_1 \end{bmatrix} \begin{bmatrix} f_1^{(m_2)}(z^{(0\sim m_1-1)}, \zeta, t) \\ f_2(x_1^{(0\sim m_1-1)}, x_2^{(0\sim m_2-1)}, \zeta, t) \end{bmatrix}. \tag{6.53}$$

6.5 Examples

6.5.1 Example 1: Cascade Systems

Consider a system with two subsystems, the model of the first subsystem is

$$\ddot{x} = f(x, \dot{x}) + G_1 \tau + d, \tag{6.54}$$

where $x, \tau, d \in \mathbb{R}^r$ are state vector, the control vector and a constant disturbance vector, respectively; $G_1 \in \mathbb{R}^{r\times r}$ is a nonsingular constant matrix, $f(\cdot, \cdot)$ is a vector function that has second-order derivatives with respect to x and $\dot{x}$.

Assume that the second subsystem is purely rigid, then its model contains only the derivative term, that is,

$$\ddot{\tau} = \Phi(\dot{\tau}) + G_2 u, \tag{6.55}$$

where $u \in \mathbb{R}^r$ is the control vector of the second subsystem, $\Phi(\cdot) \in \mathbb{R}^r$ is a continuous vector function, and $G_2 \in \mathbb{R}^{r\times r}$ is nonsingular.

Denote $x_1 = x$, and $x_2 = \tau$, then the example system can be written in the following second-order SFS form:

$$\begin{cases} \ddot{x}_1 = f(x_1, \dot{x}_1) + G_1 x_2 + d \\ \ddot{x}_2 = \Phi(\dot{x}_2) + G_2 u. \end{cases} \tag{6.56}$$

By our notations with the second-order SFS (6.18), we have

$$f_1 = f(x_1, \dot{x}_1) + d, \quad f_2 = \Phi(\dot{x}_2).$$

It thus follows from formula (6.28) that

$$L_1 = G_1, \quad L_2 = G_1 G_2.$$

Recalling (6.26), we have the transformation

$$\begin{cases} x_1 = z \\ x_2 = G_1^{-1}[\ddot{z} - f(z, \dot{z}) + d], \end{cases} \tag{6.57}$$

from which we have

$$\dot{x}_2 = G_1^{-1}[\dddot{z} - \dot{f}(z, \dot{z})]. \tag{6.58}$$

This further implies

$$\Phi(\dot{x}_2) = \Phi\left(z^{(0\sim 3)}\right).$$

Thus it further follows from formula (6.29) that

$$h_1 = f_1 = f(z, \dot{z}) + d,$$

and

$$\begin{aligned} h_2 &= [I \quad L_1] \begin{bmatrix} \ddot{f}_1 \\ f_2 \end{bmatrix} \\ &= [I \quad G_1] \begin{bmatrix} \ddot{f}(z, \dot{z}) \\ \Phi(\dot{x}_2) \end{bmatrix} \\ &= \ddot{f}(z, \dot{z}) + G_1 \Phi\left(z^{(0\sim 3)}\right). \end{aligned}$$

Therefore, by Theorem 6.4 we obtain the equivalent fourth-order FAS:

$$z^{(4)} = \ddot{f}(z, \dot{z}) + G_1 \Phi\left(z^{(0\sim 3)}\right) + G_1 G_2 u. \tag{6.59}$$

It is clearly noted that the high-order FAS (6.59) is no longer affected by the constant disturbance signal d.

Regarding the control of high-order FAS (6.59), the parametric design method in Duan (2020a) (see also Duan (2021a), and Remark 6.3) can be used to obtain a linear time-invariant closed-loop system with a desired eigenstructure. In addition, this method also provides all degrees of freedom in the system design, which is suitable for further multi-objective comprehensive optimization design of the control system.

6.5.2 Example 2: Robot with Elastic Joints

Consider a robot system with elastic joints as shown in Fig. 6.2. The dynamic model is given as

$$\begin{cases} J_1\ddot{q}_1 + D_1(\dot{q}_1)\dot{q}_1 + kq_1 - kq_2 = d_1 \\ J_2\ddot{q}_2 + D_2\dot{q}_2 + kq_2 - kq_1 = \tau + d_2 \\ \dot{\tau} = \varphi(\tau) + u + d_3, \end{cases} \tag{6.60}$$

where the variables are defined as follows:

$q_i,\ i = 1,2$	angles of two joints;
$J_i,\ i = 1,2$	moments of inertia of joints;
$D_i,\ i = 1,2$	generalized joint damping;
k	spring stiffness;
τ	output torque of the motor;
u	voltage input of the motor;
$d_i,\ i = 1,2,3$	disturbance signals; and
$\varphi(t)$	piecewise continuous function.

Denote

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ \tau \end{bmatrix}, \tag{6.61}$$

$$\begin{cases} f_1 = -\frac{1}{J_1}[D_1(\dot{x}_1)\dot{x}_1 + kx_1] + \frac{1}{J_1}d_1 \\ f_2 = -\frac{1}{J_2}[D_2(\dot{x}_2)\dot{x}_2 + kx_2 - kx_1] + \frac{1}{J_2}d_2 \\ f_3 = \varphi(x_3) + d_3, \end{cases}$$

and

$$G_1 = \frac{k}{J_1},\ G_2 = \frac{1}{J_2},\ G_3 = 1.$$

Then system (6.60) can be readily arranged into a mixed-order SFS in the form of (6.33), which corresponds to the case of

$$n = 3, \text{ and } m_1 = m_2 = 2,\ m_3 = 1.$$

Thus we have

$$p_1 = 2,\ p_2 = 4, \text{ and } p = p_3 = 5.$$

Therefore, Theorem 6.5 or 6.6 can be readily applied to solve the problem. However, in order to support the point that many systems can really be physically modelled as a FAS model instead of a state-space one, in the following we directly operate on system (6.60) based on the idea shown in the proof of Theorem 6.5, rather than applying the formulas directly.

Taking derivatives of both sides of the first equation in (6.60), gives

$$J_1\dddot{q}_1 + D_1(\dot{q}_1)\ddot{q}_1 + \dot{D}_1(\dot{q}_1)\dot{q}_1 + k\dot{q}_1 - k\dot{q}_2 = \dot{d}_1,$$

from which we further have

$$J_1 q_1^{(4)} + D_1(\dot{q}_1)\dddot{q}_1 + [2\dot{D}_1(\dot{q}_1) + k]\ddot{q}_1 + \ddot{D}_1(\dot{q}_1)\dot{q}_1 - k\ddot{q}_2 = \ddot{d}_1. \tag{6.62}$$

It follows from the first equation in (6.60) that

$$q_2 = \frac{1}{k}[J_1\ddot{q}_1 + D_1(\dot{q}_1)\dot{q}_1 - d_1] + q_1, \tag{6.63}$$

which further gives

$$\dot{q}_2 = \frac{1}{k}\left[J_1\dddot{q}_1 + D_1(\dot{q}_1)\ddot{q}_1 + \dot{D}_1(\dot{q}_1)\dot{q}_1 - \dot{d}_1\right] + \dot{q}_1. \tag{6.64}$$

It follows from the second equation in (6.60) that

$$\ddot{q}_2 = -\frac{1}{J_2}[D_2\dot{q}_2 + kq_2 - kq_1 - \tau - d_2]. \tag{6.65}$$

Substituting (6.63) and (6.64) into the above equation (6.65), gives the following form of $\ddot{q}_2$ expressed only by q_1 and its derivatives:

$$\begin{aligned} \ddot{q}_2 \quad = \quad & -\frac{1}{kJ_2}D_2\left[J_1\dddot{q}_1 + D_1(\dot{q}_1)\ddot{q}_1 + \dot{D}_1(\dot{q}_1)\dot{q}_1 - \dot{d}_1\right] \\ & -\frac{1}{J_2}\left[D_2\dot{q}_1 + J_1\ddot{q}_1 + D_1(\dot{q}_1)\dot{q}_1 - \tau - d_1 - d_2\right]. \end{aligned} \tag{6.66}$$

Further, substituting the above equation into (6.62), yields the following fourth-order FAS:

$$J_2J_1q_1^{(4)} + \sum_{i=1}^{3} c_i\left(q_1^{(1\sim3)}\right)q_1^{(i)} = k\tau + \hat{d}, \tag{6.67}$$

where

$$\begin{cases} c_1\left(q_1^{(1\sim3)}\right) = J_2\ddot{D}_1(\dot{q}_1) + D_2\left(\dot{D}_1(\dot{q}_1) + k\right) + kD_1(\dot{q}_1) \\ c_2\left(q_1^{(1\sim3)}\right) = 2J_2\dot{D}_1(\dot{q}_1) + D_2D_1(\dot{q}_1) + (J_1 + J_2)k \\ c_3(\dot{q}_1) = J_2D_1(\dot{q}_1) + J_1D_2, \end{cases} \tag{6.68}$$

and

$$\hat{d} = k(d_1 + d_2) + D_2\dot{d}_1 + J_2\ddot{d}_1. \tag{6.69}$$

Finally, taking the derivatives of both sides of (6.67), and substituting the last equation in (6.60) into the result, give the following fifth-order FAS:

$$J_2J_1q_1^{(5)} + \sum_{i=0}^{4} a_i\left(q_1^{(1\sim4)}\right)q_1^{(i)} - k\varphi(\tau) = ku + kd, \tag{6.70}$$

where

$$\begin{cases} a_0 = 0 \\ a_1\left(q_1^{(1\sim4)}\right) = \dot{c}_1\left(q_1^{(1\sim3)}\right) \\ a_2\left(q_1^{(1\sim4)}\right) = \dot{c}_2\left(q_1^{(1\sim3)}\right) + c_1\left(q_1^{(1\sim3)}\right) \\ a_3\left(q_1^{(1\sim4)}\right) = \dot{c}_3(\dot{q}_1) + c_2\left(q_1^{(1\sim3)}\right) \\ a_4(\dot{q}_1) = c_3(\dot{q}_1), \end{cases} \tag{6.71}$$

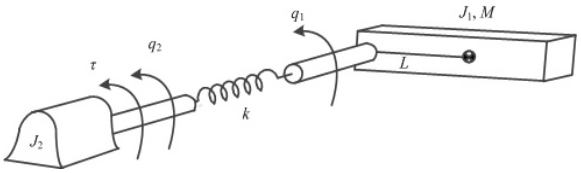


Figure 6.2: Robot with an elastic joint

and

$$d = k(\dot{d}_1 + \dot{d}_2 + d_3) + D_2\ddot{d}_1 + J_2\dddot{d}_1, \tag{6.72}$$

with $c_i\left(q_1^{(1\sim3)}\right)$, $i = 1, 2, 3$, being given by (6.68).

When the state variables $q_1, q_2, \dot{q}_1, \dot{q}_2$, and τ of all the subsystems of the original system (6.60) are all measurable, it is easy to obtain the state $q_1^{(0\sim4)}$ of system (6.70), thus we can design the following state feedback control law for system (6.70):

$$u = \frac{1}{k}\sum_{i=0}^{4}\left[a_i\left(q_1^{(1\sim4)}\right) - J_1J_2\alpha_i\right]q_1^{(i)} + \frac{1}{k}J_1J_2\upsilon - \varphi(\tau), \tag{6.73}$$

where α_i, $i = 0 \sim 4$, are a set of parameters, and υ is the reference input. Under the above control law, the following linear closed-loop system can be obtained:

$$q_1^{(5)} + \sum_{i=0}^{4}\alpha_i q_1^{(i)} = \upsilon + \frac{k}{J_1J_2}d. \tag{6.74}$$

Since the above system is a single variable one, the design parameters α_i, $i = 0 \sim 4$, are uniquely determined by the closed-loop poles of the system. Therefore, the design parameters α_i, $i = 0 \sim 4$, can be designed by making proper choice of the closed-loop poles.

When the disturbance signals d_i, $i = 1, 2$, are constant ones, it follows from (6.72) that the above design through the FAS model (6.74) has simultaneously achieved decoupling of the disturbance signals d_1 and d_2. When the disturbance d_1 is a ramp signal, only its slope has an affection on FAS (6.74). Even if for the case that the disturbances are all general signals, the problem still gets simplified since in the high-order system all the individual disturbance signals are assembled into a comprehensive one, and the problem is then turned into one of disturbance attenuation in the constant linear system (6.74).

Remark 6.6 It is clearly seen from the above two examples that the solution processes of the FAS approach are very simple. If the first example system is converted into a first-order system with 4 multivariable subsystems, and the second example system is converted into a first-order system with 5 subsystems, experience tells us that applications of the method of backstepping to the converted first-order systems should turn out to be much more complicated.

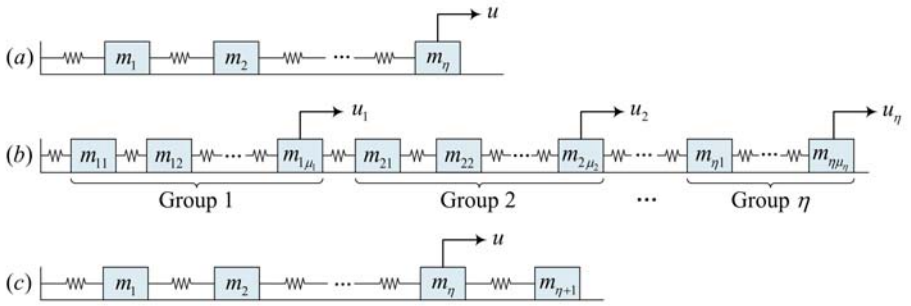


Figure 6.3: Multiple spring-mass systems

6.5.3 Example 3: Multiple Spring-mass Systems

Case A

Consider the system (a) in Fig. 6.3. By using the well-known Newton's Law, the set of basic equations of the system can be established as follows:

$$\begin{cases} m_1\ddot{x}_1 + c_1(\dot{x}_1) + k_1 x_1 = k_2(x_2 - x_1) \\ m_2\ddot{x}_2 + c_2(\dot{x}_2) + k_2(x_2 - x_1) = k_3(x_3 - x_2) \\ \quad\vdots \\ m_{r-1}\ddot{x}_{r-1} + c_{r-1}(\dot{x}_{r-1}) + k_{r-1}(x_{r-1} - x_{r-2}) = k_r(x_r - x_{r-1}) \\ m_r\ddot{x}_r + c_r(\dot{x}_r) + k_r(x_r - x_{r-1}) = u, \end{cases} \tag{6.75}$$

where $m_i > 0$ and x_i are respectively the mass and the position of the i-th moving object, $i = 1, 2, \ldots, r$; $c_i(\dot{x}_i)$, $i = 1, 2, \ldots, r$, are a set of differentiable functions representing the friction forces, and $k_i > 0$, $i = 1, 2, \ldots, r$, are the spring coefficients.

It is obvious that the set of basic equations in (6.75) form a typical second-order SFS proposed in Duan (2021b). According to the Theorem 3.2 in Duan (2021b), the above system (6.75) can be equivalently converted into a FAS in the form of

$$x_1^{(2r)} = \alpha_a\left(x_1^{(0\sim 2r-1)}\right) + \beta_a u, \tag{6.76}$$

where $\alpha_a\left(x_1^{(0\sim 2r-1)}\right)$ is a continuous function, and β_a is a nonzero scalar. According to the Theorem 3.2 in Duan (2021b), the other variables $x_i^{(0\sim 1)}$, $i = 2, 3, \ldots, r$, are determined by a differential homeomorphism between $x_1^{(0\sim 2r-1)}$ and $x_i^{(0\sim 1)}$, $i = 1, 2, \ldots, r$.

The above (6.76) is obviously a FAS model.

Case B

Based on the above fact, the spring-mass system (b) in Fig. 6.3, with all the mass and the spring coefficient of each subsystem being greater than zero, can be equivalently

converted into a FAS model in the form of

$$\begin{cases} x_{11}^{(2r_1)} = \alpha_1\left(x_{11}^{(0\sim 2r_1-1)}, x_{21}\right) + \beta_1 u_1 \\ x_{21}^{(2r_2)} = \alpha_2\left(x_{11}^{(0\sim 2r_1-1)}, x_{21}^{(0\sim 2r_2-1)}, x_{31}\right) + \beta_2 u_2 \\ \quad \vdots \\ x_{\eta 1}^{(2r_\eta)} = \alpha_\eta\left(x_{k,1}^{(0\sim 2r_k-1)}\Big|_{k=1\sim\eta}\right) + \beta_\eta u_\eta, \end{cases} \tag{6.77}$$

where $\alpha_i(\cdot)$ and β_i are a nonlinear continuous function and a nonzero scalar, respectively, $i = 1, 2, \ldots, \eta$.

The above FAS (6.77) is clearly a mixed-order global FAS under the condition of $\beta_i \neq 0$, $i = 1, 2, \ldots, \eta$.

Case C

Now, let us further consider the system (c) in Fig. 6.3. Again, the basic equations can be easily established by using Newton's Law as

$$\begin{cases} m_{11}\ddot{x}_{11} + c_{11}(\dot{x}_{11}) + k_{11}x_{11} = k_{12}(x_{12} - x_{11}) \\ m_{12}\ddot{x}_{12} + c_{12}(\dot{x}_{12}) + k_{12}(x_{12} - x_{11}) = k_{13}(x_{13} - x_{12}) \\ \quad \vdots \\ m_{1r_1}\ddot{x}_{1r_1} + c_{1r_1}\left(\dot{x}_{1r_1}\right) + k_{1r_1}\left(x_{1r_1} - x_{1,r_1-1}\right) = k_{21}\left(x_{21} - x_{1r_1}\right), \end{cases} \tag{6.78}$$

and

$$\begin{cases} m_{21}\ddot{x}_{21} + c_{21}(\dot{x}_{21}) + k_{21}\left(x_{21} - x_{1r_1}\right) = k_{22}(x_{22} - x_{21}) + u \\ m_{22}\ddot{x}_{22} + c_{22}(\dot{x}_{22}) + k_{22}(x_{22} - x_{21}) = k_{23}(x_{23} - x_{22}) \\ \quad \vdots \\ m_{2r_2}\ddot{x}_{2r_2} + c_{2r_2}\left(\dot{x}_{2r_2}\right) + k_{2r_2}\left(x_{2r_2} - x_{2,r_2-1}\right) = 0, \end{cases} \tag{6.79}$$

where $m_{ij} > 0$ and x_{ij} are the mass and the position of the (i, j)-th object, $j = 1, 2, \ldots, r_i$, $i = 1, 2$; $c_{ij}\left(\dot{x}_{ij}\right)$, $j = 1, 2, \ldots, r_i$, $i = 1, 2$, are a set of differential functions representing the friction forces, and $k_{ij} > 0$, $j = 1, 2, \ldots, r_i$, $i = 1, 2$, are the spring coefficients.

Rewriting the equations in (6.79) by reversing the orders of the equations, we obtain

$$\begin{cases} m_{2r_2}\ddot{x}_{2r_2} + c_{2r_2}\left(\dot{x}_{2r_2}\right) + k_{2r_2}x_{2r_2} = k_{2r_2}x_{2,r_2-1} \\ m_{2,r_2-1}\ddot{x}_{2,r_2-1} + c_{2,r_2-1}\left(\dot{x}_{2,r_2-1}\right) \\ \quad + k_{2,r_2-1}\left(x_{2,r_2-1} - x_{2,r_2-2}\right) = k_{2r_2}\left(x_{2r_2} - x_{2,r_2-1}\right) \\ \quad \vdots \\ m_{21}\ddot{x}_{21} + c_{21}(\dot{x}_{21}) + k_{21}\left(x_{21} - x_{1r_1}\right) = k_{22}(x_{22} - x_{21}) + u, \end{cases} \tag{6.80}$$

which also forms a typical second-order SFS proposed in Duan (2021b). Again, according to the Theorem 3.2 in Duan (2021b), the above system (6.80) can be equivalently converted into a FAS in the form of

$$x_{2,r_2}^{(2r_2)} = \alpha_0\left(x_{2,r_2}^{(0\sim 2r_2-1)}\right) + \beta_0\left(u + k_{21}x_{1,r_1}\right), \tag{6.81}$$

where $\alpha_0\left(x_{2,r_2}^{(0\sim 2r_2-1)}\right)$ is a continuous function, and β_0 is a nonzero scalar. Now, combining (6.78) with the above (6.81), gives the following set of equations describing the system (c) in Fig. 6.3:

$$\begin{cases} m_{11}\ddot{x}_{11}+c_{11}\left(\dot{x}_{11}\right)+k_{11}x_{11}=k_{12}\left(x_{12}-x_{11}\right) \\ m_{12}\ddot{x}_{12}+c_{12}\left(\dot{x}_{12}\right)+k_{12}\left(x_{12}-x_{11}\right)=k_{13}\left(x_{13}-x_{12}\right) \\ \quad\vdots \\ m_{1r_1}\ddot{x}_{1r_1}+c_{1r_1}\left(\dot{x}_{1r_1}\right)+k_{1r_1}\left(x_{1r_1}-x_{1,r_1-1}\right)=k_{21}\left(x_{21}-x_{1r_1}\right) \\ x_{2,r_2}^{(2r_2)}=\alpha_0\left(x_{2,r_2}^{(0\sim 2r_2-1)}\right)+\beta_0\left(u+k_{21}x_{1,r_1}\right). \end{cases} \tag{6.82}$$

It is easily recognized that the above system (6.82) is a mixed-order SFS, proposed in Duan (2021b). According to the Theorem 3.3 in Duan (2021b), the above system (6.82) can be equivalently converted into a FAS in the form of

$$x_{11}^{(2(r_1+r_2))}=\alpha_b\left(x_{11}^{(0\sim 2(r_1+r_2)-1)}\right)+\beta_b u,$$

where $\alpha_b\left(x_{11}^{(0\sim 2(r_1+r_2)-1)}\right)$ is a continuous function, and β_b is a nonzero scalar. Therefore, the system (c) in Fig. 6.3 can be also converted into a mixed-order global FAS.

Based on the above conclusions, we can further deduce the following conclusion: the system (b) in Fig. 6.3, with the mass and the spring coefficient of each subsystem being greater than zero, can be converted into a global FAS if and only if at least one of the forces u_i, $i=1,2,\ldots,\eta$, is not constantly zero.

6.6 Additional Principle

6.6.1 General Presentation

In this subsection, let us only focus on the general mixed-order SFS (6.33). Now we further assume

$$f_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k},\zeta,t) \;=\; \sum_{l=1}^{\omega} f_{kl}(x_i^{(0\sim m_i-1)}|_{i=1\sim k},\zeta,t), \qquad k=1,2,\ldots,n, \tag{6.83}$$

where $\omega\geq 1$ is an integer, $f_{kl}(x_i^{(0\sim m_i-1)}|_{i=1\sim k},\zeta,t)\in\mathbb{R}^r$, $k=1,2,\ldots,n$, $l=1,2,\ldots,\omega$, are a group of sufficiently smooth vector functions.

From the above Theorem 6.6, we can easily obtain the following result.

Theorem 6.7 *Let Condition 6.3 be met, then there exists an invertible transformation*

$$x_{1\sim n}=T(z^{(0\sim p-1)},\zeta,t), \tag{6.84}$$

such that the mixed-order SFS (6.33), with f_k, $k=1,2,\ldots,n$, given by (6.83), can be equivalently transformed into the following FAS:

$$z^{(p)}=\sum_{l=1}^{\omega} h_l(z^{(0\sim p-1)},\zeta,t)+L(z^{(0\sim p-1)},\zeta,t)u, \tag{6.85}$$

with

$$L(z^{(0\sim p-1)},\zeta,t)=G_1G_2\cdots G_{n-1}G_n(x_i^{(0\sim m_i-1)}|_{i=1\sim n},\zeta,t), \tag{6.86}$$

and

$$h_l(z^{(0\sim p-1)},\zeta,t)=L_{0\sim n-1}\begin{bmatrix} f_{1l}^{(p_{2\sim n})}(z^{(0\sim m_1-1)},\zeta,t)\\ f_{2l}^{(p_{3\sim n})}(x_1^{(0\sim m_1-1)},x_2^{(0\sim m_2-1)},\zeta,t)\\ \vdots\\ f_{n-1,l}^{(m_n)}(x_i^{(0\sim m_i-1)}|_{i=1\sim n-1},\zeta,t)\\ f_{n,l}(x_i^{(0\sim m_i-1)}|_{i=1\sim n},\zeta,t)\end{bmatrix}, \tag{6.87}$$

where $L_0=I$, *and*

$$L_k=G_1G_2\cdots G_k,\ k=1,2,\ldots,n-1, \tag{6.88}$$

and in (6.86)-(6.87) the $x_k,\ k=2,3,\ldots,n$, *and their derivatives are given by transformation (6.84).*

Proof. The result can be easily proven by applying Theorem 6.5. With relation (6.83), formula (6.48) becomes

$$\begin{aligned} h_k(z^{(0\sim p_k-1)},\zeta,t) &= L_{0\sim k-1}\begin{bmatrix} f_1^{(p_{2\sim k})}(z^{(0\sim m_1-1)},\zeta,t)\\ f_2^{(p_{3\sim k})}(x_1^{(0\sim m_1-1)},x_2^{(0\sim m_2-1)},\zeta,t)\\ \vdots\\ f_{k-1}^{(m_k)}(x_i^{(0\sim m_i-1)}|_{i=1\sim k-1},\zeta,t)\\ f_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k},\zeta,t)\end{bmatrix}\\ &= L_{0\sim k-1}\begin{bmatrix} \sum_{l=1}^{\omega} f_{1l}^{(p_{2\sim k})}(z^{(0\sim m_1-1)},\zeta,t)\\ \sum_{l=1}^{\omega} f_{2l}^{(p_{3\sim k})}(x_1^{(0\sim m_1-1)},x_2^{(0\sim m_2-1)},\zeta,t)\\ \vdots\\ \sum_{l=1}^{\omega} f_{k-1,l}^{(m_k)}(x_i^{(0\sim m_i-1)}|_{i=1\sim k-1},\zeta,t)\\ \sum_{l=1}^{\omega} f_{k,l}(x_i^{(0\sim m_i-1)}|_{i=1\sim k},\zeta,t)\end{bmatrix}\\ &= \sum_{l=1}^{\omega}\left(L_{0\sim k-1}\begin{bmatrix} f_{1l}^{(p_{2\sim k})}(z^{(0\sim m_1-1)},\zeta,t)\\ f_{2l}^{(p_{3\sim k})}(x_1^{(0\sim m_1-1)},x_2^{(0\sim m_2-1)},\zeta,t)\\ \vdots\\ f_{k-1,l}^{(m_k)}(x_i^{(0\sim m_i-1)}|_{i=1\sim k-1},\zeta,t)\\ f_{k,l}(x_i^{(0\sim m_i-1)}|_{i=1\sim k},\zeta,t)\end{bmatrix}\right) \end{aligned}$$

$$= \sum_{l=1}^{\omega} h_{k,l}(z^{(0\sim p_k-1)}, \zeta, t),$$

which gives (6.87). While the other relations hold obviously. ■

It is reminded that the notation of $p_{i\sim j}$ is given by (6.42).

The above result indicates an important fact: if the nonlinear term in each subsystem of SFS (6.33) is the sum of several individual nonlinear terms, then the nonlinear term in the obtained equivalent FAS model is also the sum of the same number of individual ones correspondingly.

6.6.2 Case of First-order SFSs

For the case of first-order SFSs, correspondingly, we assume

$$f_k(x_{1\sim k}, \zeta, t) = \sum_{l=1}^{\omega} f_{kl}(x_{1\sim k}, \zeta, t), \ k = 1, 2, \ldots, n, \tag{6.89}$$

where $\omega \geq 1$ is an integer, $f_{kl}(x_{1\sim i}, \zeta, t) \in \mathbb{R}^r$, $i = 1, 2, \ldots, n$, $l = 1, 2, \ldots, \omega$, are a group of sufficiently smooth vector functions. Applying Theorem 6.7, immediately yields the following corollary.

Corollary 6.3 *Let Condition 6.1 be met, then there exists an invertible transformation*

$$x_{1\sim n} = T(z^{(0\sim n-1)}, \zeta, t), \tag{6.90}$$

such that the first-order SFS (6.1), with f_k, $k = 1, 2, \ldots, n$, given by (6.89), can be equivalently transformed into the following FAS:

$$z^{(n)} = \sum_{l=1}^{\omega} h_l(z^{(0\sim n-1)}, \zeta, t) + L(z^{(0\sim n-1)}, \zeta, t)u, \tag{6.91}$$

with

$$L(z^{(0\sim n-1)}, \zeta, t) = G_1 G_2 \cdots G_{n-1} G_n(x_{1\sim n}, \zeta, t), \tag{6.92}$$

and

$$h_l(z^{(0\sim n-1)}, \zeta, t) = L_{0\sim n-1} \begin{bmatrix} f_{1l}^{(n-1)}(z, \zeta, t) \\ f_{2l}^{(n-2)}(z, x_2, \zeta, t) \\ \vdots \\ \dot{f}_{n-1,l}(x_{1\sim n-1}, \zeta, t) \\ f_{nl}(x_{1\sim n}, \zeta, t) \end{bmatrix}, \tag{6.93}$$

where $L_0 = I$, and L_k, $k = 1, 2, \ldots, n-1$, are given by (6.88), and in (6.92)-(6.93) the x_k, $k = 2, 3, \ldots, n$, and their derivatives are given by transformation (6.90).

Now let us consider the following SFS with uncertainties:

$$\begin{cases} \dot{x}_1 = f_1(x_1,\zeta,t) + \Delta f_1(x_1,\zeta,t) + G_1 x_2 \\ \dot{x}_2 = f_2(x_{1\sim 2},\zeta,t) + \Delta f_2(x_{1\sim 2},\zeta,t) + G_2 x_3 \\ \qquad \vdots \\ \dot{x}_{n-1} = f_{n-1}(x_{1\sim n-1},\zeta,t) + \Delta f_{n-1}(x_{1\sim n-1},\zeta,t) + G_{n-1} x_n \\ \dot{x}_n = f_n(x_{1\sim n},\zeta,t) + \Delta f_n(x_{1\sim n},\zeta,t) + G_n(x_{1\sim n},\zeta,t)u. \end{cases} \tag{6.94}$$

where $\Delta f_i(x_{1\sim i},\zeta,t) \in \mathbb{R}^r$, $i = 1,2,\ldots,n$, represent the uncertainties, and the other variables are as stated before.

Now, let us take $\omega = 2$, and

$$\begin{aligned} f_{k1}(x_{1\sim k},\zeta,t) &= f_k(x_{1\sim k},\zeta,t), \\ f_{k2}(x_{1\sim k},\zeta,t) &= \Delta f_k(x_{1\sim k},\zeta,t),\ k = 1,2,\ldots,n-1. \end{aligned}$$

Applying Corollary 6.3, gives the following result.

Proposition 6.1 *Let Condition 6.1 be satisfied, then the generalized SFS (6.94) can be transformed into the following FAS model*

$$z^{(n)} = h(z^{(0\sim n-1)},\zeta,t) + \Delta h(z^{(0\sim n-1)},\zeta,t) + L_n(z^{(0\sim n-1)},\zeta,t)u, \tag{6.95}$$

with

$$L_n(z^{(0\sim n-1)},\zeta,t) = G_1 G_2 \cdots G_{n-1} G_n(x_{1\sim n},\zeta,t) \tag{6.96}$$

and

$$h(z^{(0\sim n-1)},\zeta,t) = L_{0\sim n-1}\begin{bmatrix} f_1^{(n-1)}(z,\zeta,t) \\ f_2^{(n-2)}(z,x_2,\zeta,t) \\ \vdots \\ \dot{f}_{n-1}(x_{1\sim n-1},\zeta,t) \\ f_n(x_{1\sim n},\zeta,t) \end{bmatrix}, \tag{6.97}$$

$$\Delta h(z^{(0\sim n-1)},\zeta,t) = L_{0\sim n-1}\begin{bmatrix} \Delta f_1^{(n-1)}(z,\zeta,t) \\ \Delta f_2^{(n-2)}(z,x_2,\zeta,t) \\ \vdots \\ \Delta \dot{f}_{n-1}(x_{1\sim n-1},\zeta,t) \\ \Delta f_n(x_{1\sim n},\zeta,t) \end{bmatrix}, \tag{6.98}$$

where $L_0 = I$, and L_k, $k = 1,2,\ldots,n-1$, are given by (6.88).

The above proposition provides a basic result for further investigating robust control of uncertain SFS based on the FAS approach.

6.6.3 An Anti-disturbance Feature

For convenience, let us consider the following first-order SFS with polynomial disturbances:

$$\begin{cases} \dot{x}_1 = f_1(x_1, \zeta, t) + G_1 x_2 + p^{\{n-2\}}(t) \\ \dot{x}_2 = f_2(x_{1\sim 2}, \zeta, t) + G_2 x_3 + p^{\{n-3\}}(t) \\ \quad \vdots \\ \dot{x}_{n-1} = f_{n-1}(x_{1\sim n-1}, \zeta, t) + G_{n-1} x_n + p^{\{0\}}(t) \\ \dot{x}_n = f_n(x_{1\sim n}, \zeta, t) + G_n(x_{1\sim n}, \zeta, t)u, \end{cases} \tag{6.99}$$

where $x_i \in \mathbb{R}^r$, $i = 1, 2, \ldots, n$, are the state vectors, $u \in \mathbb{R}^r$ is the control input, $G_i \in \mathbb{R}^{r\times r}$, $i = 1, 2, \ldots, n-1$, are constant nonsingular matrices, $f_i(x_{1\sim i}, \zeta, t) \in \mathbb{R}^r$ and $G_n(x_{1\sim n}, \zeta, t) \in \mathbb{R}^{r\times r}$ are sufficiently smooth functions, and

$$p^{\{k\}}(t) = d_{k0} + d_{k1}t + \cdots + d_{kk}t^k,\ k = 0, 1, \ldots, n-2,$$

with the coefficients unknown, are a series of known polynomials representing the disturbances in system.

Now, let us take $\omega = 2$, and

$$\begin{aligned} f_{k1}(x_{1\sim k}, \zeta, t) &= f_k(x_{1\sim k}, \zeta, t), \\ f_{k2}(x_{1\sim k}, \zeta, t) &= p^{\{n-k-1\}}(t),\ k = 1, 2, \ldots, n-1. \end{aligned}$$

Then, noting that

$$h_2(z^{(0\sim n-1)}, \zeta, t) = L_{0\sim n-1}\begin{bmatrix} f_{12}^{(n-1)}(z, \zeta, t) \\ f_{22}^{(n-2)}(z, x_2, \zeta, t) \\ \vdots \\ \dot{f}_{n-1,2}(x_{1\sim n-1}, \zeta, t) \\ f_{n2}(x_{1\sim n}, \zeta, t) \end{bmatrix} \tag{6.100}$$

$$= L_{0\sim n-1}\begin{bmatrix} \frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}} p^{\{n-2\}}(t) \\ \frac{\mathrm{d}^{n-2}}{\mathrm{d}t^{n-2}} p^{\{n-3\}}(t) \\ \vdots \\ \frac{\mathrm{d}}{\mathrm{d}t} p^{\{0\}}(t) \\ 0 \end{bmatrix} \tag{6.101}$$

$$= 0, \tag{6.102}$$

and applying Corollary 6.3, give the following result.

Proposition 6.2 *Let Condition 6.1 be satisfied, then the generalized SFS (6.99) can be transformed into the following FAS model*

$$z^{(n)} = h_n(z^{(0\sim n-1)}, \zeta, t) + L_n(z^{(0\sim n-1)}, \zeta, t)u, \tag{6.103}$$

with

$$L_n(z^{(0\sim n-1)}, \zeta, t) = G_1 G_2 \cdots G_{n-1} G_n(x_{1\sim n}, \zeta, t) \tag{6.104}$$

and

$$h_n(z^{(0\sim n-1)},\zeta,t)=L_{0\sim n-1}\begin{bmatrix} f_1^{(n-1)}(z,\zeta,t)\\ f_2^{(n-2)}(z,x_2,\zeta,t)\\ \vdots\\ \dot{f}_{n-1}(x_{1\sim n-1},\zeta,t)\\ f_n(x_{1\sim n},\zeta,t)\end{bmatrix}, \tag{6.105}$$

where $L_0=I$, *and* L_k, $k=1,2,\ldots,n-1$, *are given by (6.88).*

The above result implies a very important fact, i.e., although the state-space model (6.99) is subject to polynomial disturbances, the derived FAS is free of disturbances. This becomes vitally important when we are only interested in $z=x_1$. In engineering, a constant disturbance is often encountered, which corresponds to the "drift" of certain values and is sometimes referred to as a step disturbance. The first-order disturbance $d(t)=d_1t+d_0$ is also commonly encountered and is referred to as a slope disturbance or a ramp disturbance. By Taylor series expansion theory, any function can be approximated by a polynomial, thus a general disturbance can also be approximated by a polynomial $p^{\{k\}}(t)$. Due to these facts, the above anti-disturbance feature of the FAS approach is truly crucial.

6.7 Notes and References

The FAS approach is very powerful in dealing with the control of nonlinear systems, since the full-actuation feature allows one to eliminate the nonlinearities and hence a constant linear closed-loop system can be obtained. To apply the FAS approach, the key step is to obtain a FAS model for a nonlinear system. Toward this goal, this chapter has shown the following:

1. the conventional SFS can be further generalized to contain parameter vectors and the derivatives of the state vectors, and a recursive solution exists to convert the generalized version of SFS equivalently into a FAS model;

2. due to the fact that second- and high-order subsystems are firstly obtained when modelling using physical laws such as Newton's Law, Lagrangian Equation, Theorem of Linear and Angular momentum, Kirchhoff's Law of Current and Voltage, etc., the proposed SFSs of second-order, and that with subsystems of mixed orders are much more commonly encountered than the first-order ones;

3. like the first-order case, effective recursive solutions also exist for converting the proposed second-order or mixed-order SFSs equivalently into FAS models; and

4. under certain common conditions, the recursive solutions to convert the proposed generalized SFSs equivalently into FAS models can be expressed in direct analytical explicit forms.

The above facts are very important since they partially provide a base of the effective FAS approach for control of generalized complicated SFSs, which may be no

longer effectively solved with the well-known method of backstepping due to more serious "differential explosion" problems caused by the increased complexity in the proposed generalized SFSs.

The contents of this chapter are restricted to global SFSs with uniform dimensions, and are closely related to Duan (2021b). The FAS treatment of global SFSs with nonuniform dimensions is examined in the next chapter. Also, the case of global non-affine SFSs is considered in Chapter 8. The FAS approach for sub-SFSs is discussed in the second volume of the book Duan (2025g).

Chapter 7

Strict Feedback Systems with Increasing Dimensions

This chapter further addresses the control problems of strict feedback systems (SFSs) with increasing dimensions. Compared with the commonly-considered SFSs treated in Chapter 6, where the subsystems have the same dimension, the control of SFSs with increasing dimensions is much more challenging. As in Chapter 6, first-order, second-order, and mixed-order cases have been all considered. We first considered SFSs whose gain matrices $G_i,\ i = 1,2,\ldots,n$, are all constant matrices and then extended the results to the more general cases where the gains $G_i,\ i = 1,2,\ldots,n$, are state-dependent matrix functions. A coordinate transformation has been introduced to obtain the high-order FAS (HOFAS) model. Based on the high-order FAS models, the exponentially stabilizing controllers have been directly given. Besides first-order SFSs, second- and high-order SFSs with increasing dimensions are parallelly considered.

7.1 First-order SFSs with Increasing Dimensions

In this first section, let us consider first-order SFSs with increasing dimensions.

7.1.1 Case of Constant Gains

Consider the following SFS:

$$\begin{cases} \dot{x}_1 = f_1(x_1) + G_1 x_2 \\ \dot{x}_2 = f_2(x_{1\sim 2}) + G_2 x_3 \\ \quad\vdots \\ \dot{x}_n = f_n(x_{1\sim n}) + G_n u, \end{cases} \tag{7.1}$$

where $x_i \in \mathbb{R}^{r_i},\ f_i(x_{1\sim i}) \in \mathbb{R}^{r_i},\ i = 1,2,\ldots,n, u \in \mathbb{R}^m, G_i \in \mathbb{R}^{r_i \times r_{i+1}},\ i = 1,2,\ldots,n-1$, and $G_n \in \mathbb{R}^{r_n \times m}$. In this chapter, it is assumed that $r_1 \le r_2 \le \cdots \le r_n \le m$. Hence,

the dimensions of subsystems are increasing. Moreover, the following assumption is needed for constant matrices $G_i,\ i = 1, 2, \ldots, n$.

Assumption 7.1 *The gain matrices $G_i,\ i = 1, 2, \ldots, n$, are all of full-row rank, i.e.,* $\text{rank}(G_i) = r_i,\ i = 1, 2, \ldots, n$.

Assumption 7.1 ensures that there exist nonsingular matrices Q_i and $\bar{G}_i,\ i = 1, 2, \ldots, n$, such that

$$G_i Q_i = \begin{bmatrix} \bar{G}_i & 0 \end{bmatrix}.$$

Based on this fact, let us define $x_{i+1,1} \in \mathbb{R}^{r_i}$, $x_{i+1,2} \in \mathbb{R}^{r_{i+1}-r_i}$, $f_{i1} \in \mathbb{R}^{r_i}$ and $f_{i2} \in \mathbb{R}^{r_{i+1}-r_i}$ by

$$Q_i^{-1} x_{i+1} = \begin{bmatrix} x_{i+1,1} \\ x_{i+1,2} \end{bmatrix}, \tag{7.2}$$

$$Q_i^{-1} f_{i+1}(x_1, x_2, \cdots, x_{i+1}) = \begin{bmatrix} f_{i+1,1}(x_1, x_{2\sim i+1,1}, x_{2\sim i+1,2}) \\ f_{i+1,2}(x_1, x_{2\sim i+1,1}, x_{2\sim i+1,2}) \end{bmatrix}, \quad i = 1, 2, \ldots, n-1. \tag{7.3}$$

For notation convenience, we also define that $x_{12} = x_1$. Meanwhile, define $u_1 \in \mathbb{R}^{r_n}$ and $u_2 \in \mathbb{R}^{m-r_n}$ such that

$$Q_n^{-1} u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{7.4}$$

For the first subsystem, it follows from the definition of x_{12} and x_{21} that

$$\begin{aligned} \dot{x}_{12} &= f_1(x_1) + G_1 Q_1 Q_1^{-1} x_2 \\ &= f_1(x_1) + \begin{bmatrix} \bar{G}_1 & 0 \end{bmatrix} Q_1^{-1} x_2 \\ &= f_1(x_1) + \begin{bmatrix} \bar{G}_1 & 0 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} \\ &= f_1(x_{12}) + \bar{G}_1 x_{21}. \end{aligned} \tag{7.5}$$

Then, for the second subsystem, it can be obtained that

$$\begin{aligned} \begin{bmatrix} \dot{x}_{21} \\ \dot{x}_{22} \end{bmatrix} &= Q_1^{-1} \left(f_2(x_1, x_2) + G_2 Q_2 Q_2^{-1} x_3 \right) \\ &= Q_1^{-1} \left(f_2(x_1, x_2) + \begin{bmatrix} \bar{G}_2 & 0 \end{bmatrix} Q_2^{-1} x_3 \right) \\ &= Q_1^{-1} \left(f_2(x_1, x_2) + \begin{bmatrix} \bar{G}_2 & 0 \end{bmatrix} \begin{bmatrix} x_{31} \\ x_{32} \end{bmatrix} \right) \\ &= Q_1^{-1} f_2(x_1, x_2) + Q_1^{-1} \bar{G}_2 x_{31} \\ &= \begin{bmatrix} f_{21}(x_{12}, x_{21}, x_{22}) \\ f_{22}(x_{12}, x_{21}, x_{22}) \end{bmatrix} + Q_1^{-1} \bar{G}_2 x_{31}. \end{aligned} \tag{7.6}$$

After an n-step induction, system (7.1) can be then transformed into

$$\begin{cases} \dot{x}_{12} = f_1(x_{12}) + \bar{G}_1 x_{21} \\ \dot{x}_{21} = f_{21}(x_{21}, x_{1\sim 2,2}) + \begin{bmatrix} I_{r_1} & 0 \end{bmatrix} Q_1^{-1} \bar{G}_2 x_{31} \\ \dot{x}_{22} = f_{22}(x_{21}, x_{1\sim 2,2}) + \begin{bmatrix} 0 & I_{r_2-r_1} \end{bmatrix} Q_1^{-1} \bar{G}_2 x_{31} \\ \quad \vdots \\ \dot{x}_{n1} = f_{n1}(x_{2\sim n,1}, x_{1\sim n,2}) + \begin{bmatrix} I_{r_{n-1}} & 0 \end{bmatrix} Q_{n-1}^{-1} \bar{G}_n u_1 \\ \dot{x}_{n2} = f_{n2}(x_{2\sim n,1}, x_{1\sim n,2}) + \begin{bmatrix} 0 & I_{r_n-r_{n-1}} \end{bmatrix} Q_{n-1}^{-1} \bar{G}_n u_1. \end{cases} \tag{7.7}$$

We use the newly obtained system (7.7) to derive a HOFAS model for the first-order SFS (7.1). In particular, the following theorem can be obtained.

Theorem 7.1 *Let Assumption 7.1 be satisfied, then under the transformation*

$$x_{i1} = \bar{G}_{1\sim i-1}^{-1} \left(x_{p2}^{(i-p)} \Big|_{p=1\sim i-1} - L_{i-1} \left(x_{p2}^{(0\sim i-1-p)} \Big|_{p=1\sim i-1} \right) \right), \tag{7.8}$$

$$i = 2, 3, \ldots, n, \tag{7.9}$$

SFS (7.7) can be equivalently transformed into the following HOFAS model:

$$x_{p2}^{(n+1-p)} \Big|_{p=1\sim n} = L_n \left(x_{p2}^{(0\sim n-p)} \Big|_{p=1\sim n} \right) + \bar{G}_{1\sim n} u_1, \tag{7.10}$$

where $L_i(\cdot)$ and $\bar{G}_{1\sim i}$ are given iteratively by

$$L_{i+1} \left(x_{p2}^{(0\sim i+1-p)} \Big|_{p=1\sim i+1} \right) = \begin{bmatrix} \dot{L}_i + \bar{G}_{1\sim i} f_{i+1,1}(x_{2\sim i+1,1}, x_{1\sim i+1,2}) \\ f_{i+1,2}(x_{2\sim i+1,1}, x_{1\sim i+1,2}) \end{bmatrix}, \tag{7.11}$$

$$\bar{G}_{1\sim i+1} = \begin{bmatrix} \bar{G}_{1\sim i} & 0 \\ 0 & I_{r_{i+1}-r_i} \end{bmatrix} Q_i^{-1} \bar{G}_{i+1}, \tag{7.12}$$

$$i = 1, 2, \ldots, n-1.$$

with $L_1(\cdot) = f_1(x_{12})$ and $\bar{G}_{1\sim 1} = \bar{G}_1$.

Proof. We prove the theorem via the n-step induction.

Step 1: For the first subsystem, we already have that

$$\begin{aligned} \dot{x}_{12} &= f_1(x_{12}) + \bar{G}_1 x_{21} \\ &= L_1(x_{12}) + \bar{G}_{1\sim 1} x_{21}. \end{aligned} \tag{7.13}$$

Step i ($2 \le i \le n-1$): Assume that

$$x_{p2}^{(i-p)} \Big|_{p=1\sim i-1} = L_{i-1} \left(x_{p2}^{(0\sim i-1-p)} \Big|_{p=1\sim i-1} \right) + \bar{G}_{1\sim i-1} x_{i1}, \tag{7.14}$$

then we prove that

$$x_{p2}^{(i+1-p)} \Big|_{p=1\sim i} = L_i \left(x_{p2}^{(0\sim i-p)} \Big|_{p=1\sim i} \right) + \bar{G}_{1\sim i} x_{i+1,1}.$$

It follows from (7.14) that

$$x_{i1} = \bar{G}_{1\sim i-1}^{-1}\left(x_{p2}^{(i-p)}\Big|_{p=1\sim i-1} - L_{i-1}\left(x_{p2}^{(0\sim i-1-p)}\Big|_{p=1\sim i-1}\right)\right). \tag{7.15}$$

We also have, by recalling system (7.7), the following equations:

$$\begin{cases} \dot{x}_{i1} = f_{i1}(x_{2\sim i,1}, x_{1\sim i,2}) + \left[\begin{array}{cc} I_{r_{i-1}} & 0 \end{array}\right] Q_{i-1}^{-1}\bar{G}_i x_{i+1,1} \\ \dot{x}_{i2} = f_{i2}(x_{2\sim i,1}, x_{1\sim i,2}) + \left[\begin{array}{cc} 0 & I_{r_i - r_{i-1}} \end{array}\right] Q_{i-1}^{-1}\bar{G}_i x_{i+1,1}. \end{cases} \tag{7.16}$$

Next, taking the derivatives of both sides of equation (7.14) and substituting the first equation of (7.16) into the result, one has that

$$\begin{aligned} & x_{p2}^{(i+1-p)}\Big|_{p=1\sim i-1} \\ = \;& \dot{L}_{i-1} + \bar{G}_{1\sim i-1}\dot{x}_{i1} \\ = \;& \dot{L}_{i-1} + \bar{G}_{1\sim i-1}\left(f_{i1}(x_{2\sim i,1}, x_{1\sim i,2}) + \left[\begin{array}{cc} I_{r_{i-1}} & 0 \end{array}\right] Q_{i-1}^{-1}\bar{G}_i x_{i+1,1}\right) \\ = \;& \dot{L}_{i-1} + \bar{G}_{1\sim i-1} f_{i1}(x_{2\sim i,1}, x_{1\sim i,2}) \\ & + \left[\begin{array}{cc} \bar{G}_{1\sim i-1} & 0 \end{array}\right] Q_{i-1}^{-1}\bar{G}_i x_{i+1,1}. \end{aligned} \tag{7.17}$$

By combining (7.17) with the last equation of (7.16), it can be obtained that

$$x_{p2}^{(i+1-p)}\Big|_{p=1\sim i} = L_i\left(x_{p2}^{(0\sim i-p)}\Big|_{p=1\sim i}\right) + \bar{G}_{1\sim i}x_{i+1,1},$$

where

$$L_i = \left[\begin{array}{c} \dot{L}_{i-1} + \bar{G}_{1\sim i-1} f_{i1}(x_{2\sim i,1}, x_{1\sim i,2}) \\ f_{i2}(x_{2\sim i,1}, x_{1\sim i,2}) \end{array}\right], \tag{7.18}$$

$$\bar{G}_{1\sim i} = \left[\begin{array}{cc} \bar{G}_{1\sim i-1} & 0 \\ 0 & I_{r_i - r_{i-1}} \end{array}\right] Q_{i-1}^{-1}\bar{G}_i. \tag{7.19}$$

Step n: Based on the previous $n-1$ steps, we can obtain that

$$x_{i1} = \bar{G}_{1\sim i-1}^{-1}\left(x_{p2}^{(i-p)}\Big|_{p=1\sim i-1} - L_{i-1}\left(x_{p2}^{(0\sim i-1-p)}\Big|_{p=1\sim i-1}\right)\right), \tag{7.20}$$
$$i = 2, 3, \ldots, n-1,$$

and

$$x_{p2}^{(n-p)}\Big|_{p=1\sim n-1} = L_{n-1}\left(x_{p2}^{(0\sim n-1-p)}\Big|_{p=1\sim n-1}\right) + \bar{G}_{1\sim n-1}x_{n1}, \tag{7.21}$$

where $L_i(\cdot)$ and $\bar{G}_{1\sim i}$ are given iteratively by

$$L_{i+1}\left(x_{p2}^{(0\sim i+1-p)}\Big|_{p=1\sim i+1}\right) = \left(\begin{array}{c} \dot{L}_i + \bar{G}_{1\sim i} f_{i+1,1}(x_{2\sim i+1,1}, x_{1\sim i+1,2}) \\ f_{i+1,2}(x_{2\sim i+1,1}, x_{1\sim i+1,2}) \end{array}\right), \tag{7.22}$$

$$\bar{G}_{1\sim i+1} = \left[\begin{array}{cc} \bar{G}_{1\sim i} & 0 \\ 0 & I_{r_{i+1} - r_i} \end{array}\right] Q_i^{-1}\bar{G}_{i+1}, \tag{7.23}$$
$$i = 1, 2, \ldots, n-1.$$

We recall the following equations from (7.7):

$$\begin{cases} \dot{x}_{n1} = f_{n1}(x_{2\sim n,1}, x_{1\sim n,2}) + \begin{bmatrix} I_{r_{n-1}} & 0 \end{bmatrix} Q_{n-1}^{-1} \bar{G}_n u_1 \\ \dot{x}_{n2} = f_{n2}(x_{2\sim n,1}, x_{1\sim n,2}) + \begin{bmatrix} 0 & I_{r_n - r_{n-1}} \end{bmatrix} Q_{n-1}^{-1} \bar{G}_n u_1. \end{cases} \tag{7.24}$$

Similar to the discussion in the previous steps, taking the derivative of both sides of (7.21) and substituting the first equation of (7.24) into the result, we obtain

$$\begin{aligned} & x_{p2}^{(n+1-p)}\Big|_{p=1\sim n-1} \\ = \;& \dot{L}_{n-1} + \bar{G}_{1\sim n-1}\left(f_{n1}(x_{2\sim n,1}, x_{1\sim n,2}) + \begin{bmatrix} I_{k_{n-1}} & 0 \end{bmatrix} Q_{n-1}^{-1} \bar{G}_n u_1\right) \\ = \;& \dot{L}_{n-1} + \bar{G}_{1\sim n-1} f_{n1}(x_{2\sim n,1}, x_{1\sim n,2}) + \begin{bmatrix} \bar{G}_{1\sim n-1} & 0 \end{bmatrix} Q_{n-1}^{-1} \bar{G}_n u_1. \end{aligned} \tag{7.25}$$

Combining (7.25) with the last equation of (7.24), gives

$$x_{p2}^{(n+1-p)}\Big|_{p=1\sim n} = L_n\left(x_{p2}^{(0\sim n-p)}\Big|_{p=1\sim n}\right) + \bar{G}_{1\sim n} u_1, \tag{7.26}$$

where

$$L_n = \begin{bmatrix} \dot{L}_{n-1} + \bar{G}_{1\sim n-1} f_{n1}(x_{2\sim n,1}, x_{1\sim n,2}) \\ f_{n2}(x_{2\sim n,1}, x_{1\sim n,2}) \end{bmatrix}, \tag{7.27}$$

$$\bar{G}_{1\sim n} = \begin{bmatrix} \bar{G}_{1\sim n-1} & 0 \\ 0 & I_{r_n - r_{n-1}} \end{bmatrix} Q_{n-1}^{-1} \bar{G}_n. \tag{7.28}$$

It follows from the definition of $\bar{G}_{1\sim n}$ that $\bar{G}_{1\sim n}$ is nonsingular. Therefore, system (7.26) is fully-actuated. ■

Remark 7.1 It is noted that in the HOFAS model, only part of the controller u, i.e., u_1, is used while the rest part u_2 can be removed. This is mainly because that system (7.1) is "over-actuated" when the dimension of controller is greater than the state. Due to this fact, we only need part of the controller to convert system (7.1) into a FAS. In fact, most of practical systems are not over-actuated, i.e., u_2 does not exist in those systems. However, there do exist "over-actuated" practical systems. For example, some real industrial systems are equipped with redundant actuators, such as spacecraft (Hu et al. (2021)), ships (Witkowska and Smierzchalski (2018)), and fusion systems (Khimani and Patil (2014)), for higher reliability and greater exibility. These practical systems are over-actuated.

As shown in Duan (2021b), based on the FAS model (7.10), the following exponentially stabilizing controller can be directly given:

$$u_1 = \bar{G}_{1\sim n}^{-1}\left(-L_n + A x_{p2}^{(0\sim n-p)}\Big|_{p=1\sim n}\right),$$

where $A \in \mathbb{R}^{\varpi\times\varpi}$, with

$$\varpi = \sum_{i=1}^{n}(n+1-i)r_i.$$

This controller clearly yields the following linear time-invariant (LTI) closed-loop system:

$$x_{p2}^{(n+1-p)}\Big|_{p=1\sim n} = Ax_{p2}^{(0\sim n-p)}\Big|_{p=1\sim n}.$$

In particular, if A is chosen as a block diagonal triangular matrix in the form of

$$\begin{bmatrix} A_{0\sim n-1,1} & 0 & \cdots & 0 \\ 0 & A_{0\sim n-2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{0n} \end{bmatrix},$$

where

$$A_{0\sim n-k,k} = \begin{bmatrix} A_{0k} & A_{1k} & \cdots & A_{n-k,k} \end{bmatrix},\ k = 1,2,\ldots,n,$$

with $A_{pi},\ i = 1,2,\ldots,n,\ 0 \le p \le n-i$, being matrices with appropriate dimensions, then the closed-loop system can be written as

$$\begin{cases} x_{12}^{(n)} = \sum_{p=0}^{n-1} A_{p1} x_{12}^{(p)} \\ x_{22}^{(n-1)} = \sum_{p=0}^{n-2} A_{p2} x_{22}^{(p)} \\ \quad\vdots \\ x_{n2}^{(1)} = A_{0n} x_{n2}, \end{cases}$$

Remark 7.2 As pointed out in Duan (2021b), the FAS approach has two distinctive advantages compared with the backstepping method. First, the FAS approach can always produce an LTI closed-loop system. Second, the FAS approach can avoid the "differential explosion" problem, which makes the method of backstepping generally not applicable to an SFS with more than 3 or 4 subsystems.

7.1.2 Case of State-dependent Gains

In the previous subsection, we only assume that $G_i,\ i = 1,2,\ldots,n$, are constant matrices. In this subsection, we consider the more general case of $G_i,\ i = 1,2,\ldots,n$, being matrix functions, which is described by the following dynamics:

$$\begin{cases} \dot{x}_1 = f_1(x_1) + G_1(x_1)x_2 \\ \dot{x}_2 = f_2(x_1,x_2) + G_2(x_1,x_2)x_3 \\ \quad\vdots \\ \dot{x}_n = f_n(x_1,x_2,\cdots,x_n) + G_n(x_1,x_2,\cdots,x_n)u, \end{cases} \tag{7.29}$$

where $x_i \in \mathbb{R}^{r_i},\ i = 1,2,\ldots,n$, and $u \in \mathbb{R}^m$. Moreover, $f_i(\cdot)$ and $G_i(\cdot)$ are sufficiently smooth functions. In this work, it is assumed that $r_1 \le r_2 \le \cdots \le r_n \le m$. and the following notation is introduced for convenience:

$$\pi_i = r_1 + r_2 + \cdots + r_i,\ i = 1,2,\ldots,n.$$

The following assumption is needed for matrix functions $G_i(\cdot),\ i = 1,2,\ldots,n$.

Assumption 7.2 *For any $x_{1\sim i} \in \mathbb{R}^{\pi_i}$, the matrix function $G_i(x_{1\sim i})$ is of full-row rank, i.e.,* $\text{rank}G_i(x_{1\sim i}) = r_i,\ i = 1, 2, \ldots, n$.

It follows from Assumption 7.2 that there exist sufficiently smooth functions $Q_i(\cdot) : \mathbb{R}^{\pi_i} \to \mathbb{R}^{r_{i+1}\times r_{i+1}}$ and $\bar{G}_i(\cdot) : \mathbb{R}^{\pi_i} \to \mathbb{R}^{r_i \times r_i}$ such that

$$G_i(\cdot)Q_i(\cdot) = \begin{bmatrix} \bar{G}_i(\cdot) & 0 \end{bmatrix},$$

where $Q_i(\cdot)$ and $\bar{G}_i(\cdot)$ are nonsingular for any $x_{1\sim i} \in \mathbb{R}^{\pi_i}$. Define a differential homeomorphism

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} : \mapsto \bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{bmatrix}$$

as follows:

$$\begin{cases} \bar{x}_1 = x_1 \\ \bar{x}_2 = Q_1^{-1}(x_1)x_2 \\ \bar{x}_3 = Q_2^{-1}(x_1, x_2)x_3 \\ \quad \vdots \\ \bar{x}_n = Q_{n-1}^{-1}(x_1, x_2, \cdots, x_{n-1})x_n. \end{cases} \tag{7.30}$$

Then the inverse mapping from $\bar{x}$ to x can be written as

$$\begin{cases} x_1 = \bar{x}_1 \\ x_2 = Q_1(\bar{x}_1)\bar{x}_2 \triangleq \bar{Q}_1(\bar{x}_1)\bar{x}_2 \\ x_3 = Q_2\left(\bar{x}_1, \bar{Q}_1(\bar{x}_1)\bar{x}_2\right)\bar{x}_3 \triangleq \bar{Q}_2(\bar{x}_1, \bar{x}_2)\bar{x}_3 \\ \quad \vdots \\ x_n = Q_{n-1}\left(\bar{x}_1, \bar{Q}_1(\bar{x_1})\bar{x}_2, \cdots, \bar{Q}_{n-2}(\bar{x}_1, \cdots, \bar{x}_{n-2})\bar{x}_{n-1}\right)\bar{x}_n \\ \quad \triangleq \bar{Q}_{n-1}(\bar{x}_1, \cdots, \bar{x}_{n-1})\bar{x}_n. \end{cases} \tag{7.31}$$

Furthermore, define $\bar{x}_{i1}$, $\bar{x}_{i2}$ and $\bar{u}_1$ as follows:

$$\begin{cases} \bar{x}_{i1} = \begin{bmatrix} I_{r_{i-1}} & 0 \end{bmatrix}\bar{x}_i,\ i = 2, 3, \ldots, n, \\ \bar{x}_{i2} = \begin{bmatrix} 0 & I_{r_i - r_{i-1}} \end{bmatrix}\bar{x}_i,\ i = 2, 3, \ldots, n, \\ \bar{u}_1 = \begin{bmatrix} I_{r_n} & 0 \end{bmatrix}\bar{Q}_n^{-1}u. \end{cases}$$

Denote

$$\bar{f}_1(\bar{x}_{12}) = f_1(\bar{x}_{12}),$$

and

$$\bar{f}_{i1}(\bar{x}_{2\sim i,1}, \bar{x}_{1\sim i,2}) = \begin{bmatrix} I_{r_{i-1}} & 0 \end{bmatrix}\bar{Q}_{i-1}^{-1}\left(-\dot{\bar{Q}}_{i-1}\bar{x}_i + f_i\left(\bar{x}_1, \bar{Q}_1\bar{x}_2, \cdots, \bar{Q}_{i-1}\bar{x}_i\right)\right), \tag{7.32}$$

$$\bar{f}_{i2}(\bar{x}_{2\sim i,1}, \bar{x}_{1\sim i,2}) = \begin{bmatrix} 0 & I_{r_i - r_{i-1}} \end{bmatrix}\bar{Q}_{i-1}^{-1}\left(-\dot{\bar{Q}}_{i-1}\bar{x}_i + f_i\left(\bar{x}_1, \bar{Q}_1\bar{x}_2, \cdots, \bar{Q}_{i-1}\bar{x}_i\right)\right), \tag{7.33}$$

$$i = 2, 3, \ldots, n.$$

Meanwhile, for notation convenience, we also define that $\bar{x}_{12} = \bar{x}_1$.

For the first subsystem of system (7.29), it follows from the definitions of $\bar{x}_{12}$ and $\bar{x}_{21}$ that

$$\begin{aligned}\dot{\bar{x}}_{12} &= f_1(x_1) + G_1(x_1)\bar{Q}_1\bar{Q}_1^{-1}x_2 \\ &= f_1(x_1) + \begin{bmatrix} \bar{G}_1(\bar{x}_{12}) & 0 \end{bmatrix}\bar{x}_2 \\ &= \bar{f}_1(\bar{x}_{12}) + \bar{G}_1(\bar{x}_{12})\bar{x}_{21}.\end{aligned}$$

Then, for the second subsystem of system (7.29), it can be obtained that

$$\begin{aligned}\begin{bmatrix} \dot{\bar{x}}_{21} \\ \dot{\bar{x}}_{22} \end{bmatrix} &= \bar{Q}_1^{-1}\left(-\dot{\bar{Q}}_1\bar{x}_2 + f_2(x_1,x_2) + G_2(x_1,x_2)x_3\right) \\ &= \bar{Q}_1^{-1}\left(-\dot{\bar{Q}}_1\bar{x}_2 + f_2(x_1,x_2) + G_2(x_1,x_2)\bar{Q}_2\bar{Q}_2^{-1}x_3\right) \\ &= \bar{Q}_1^{-1}\left(-\dot{\bar{Q}}_1\bar{x}_2 + f_2(x_1,x_2) + \begin{bmatrix} \bar{G}_2(\bar{x}_{21},\bar{x}_{1\sim 2,2}) & 0 \end{bmatrix}\bar{x}_3\right) \\ &= \bar{Q}_1^{-1}\left(-\dot{\bar{Q}}_1\bar{x}_2 + f_2(x_1,x_2)\right) + \bar{Q}_1^{-1}\bar{G}_2(\bar{x}_{21},\bar{x}_{1\sim 2,2})\bar{x}_{31} \\ &= \begin{bmatrix} \bar{f}_{21}(\bar{x}_{21},\bar{x}_{1\sim 2,2}) \\ \bar{f}_{22}(\bar{x}_{21},\bar{x}_{1\sim 2,2}) \end{bmatrix} + \bar{Q}_1^{-1}\bar{G}_2(\bar{x}_{21},\bar{x}_{1\sim 2,2})\bar{x}_{31}.\end{aligned}$$

After an n-step induction, system (7.29) can be then transformed into

$$\begin{cases} \dot{\bar{x}}_{12} = \bar{f}_1(\bar{x}_{12}) + \bar{G}_1(\bar{x}_{12})\bar{x}_{21} \\ \dot{\bar{x}}_{21} = \bar{f}_{21}(\bar{x}_{21},\bar{x}_{1\sim 2,2}) + \begin{bmatrix} I_{k_1} & 0 \end{bmatrix}\bar{Q}_1^{-1}\bar{G}_2(\bar{x}_{21},\bar{x}_{1\sim 2,2})\bar{x}_{31} \\ \dot{\bar{x}}_{22} = \bar{f}_{22}(\bar{x}_{21},\bar{x}_{1\sim 2,2}) + \begin{bmatrix} 0 & I_{k_2-k_1} \end{bmatrix}\bar{Q}_1^{-1}\bar{G}_2(\bar{x}_{21},\bar{x}_{1\sim 2,2})\bar{x}_{31} \\ \qquad \vdots \\ \dot{\bar{x}}_{n1} = \bar{f}_{n1}(\bar{x}_{2\sim n,1},\bar{x}_{1\sim n,2}) + \begin{bmatrix} I_{k_{n-1}} & 0 \end{bmatrix}\bar{Q}_{n-1}^{-1}\bar{G}_n(\bar{x}_{2\sim n,1},\bar{x}_{1\sim n,2})\bar{u}_1 \\ \dot{\bar{x}}_{n2} = \bar{f}_{n2}(\bar{x}_{2\sim n,1},\bar{x}_{1\sim n,2}) + \begin{bmatrix} 0 & I_{k_n-k_{n-1}} \end{bmatrix}\bar{Q}_{n-1}^{-1}\bar{G}_n(\bar{x}_{2\sim n,1},\bar{x}_{1\sim n,2})\bar{u}_1. \end{cases} \tag{7.34}$$

Based on the equivalent system (7.34), the following result can be derived.

Theorem 7.2 *Let Assumption 7.2 be satisfied, then under the following transformation*

$$\begin{aligned}\bar{x}_{i1} &= \bar{G}_{1\sim i-1}^{-1}\left(\bar{x}_{p2}^{(i-p)}\Big|_{p=1\sim i-1} - L_{i-1}\left(\bar{x}_{p2}^{(0\sim i-1-p)}\Big|_{p=1\sim i-1}\right)\right), \quad (7.35) \\ &\qquad i = 2,3,\ldots,n,\end{aligned}$$

SFS (7.34) can be equivalently transformed into the following HOFAS model:

$$\bar{x}_{p2}^{(0\sim n+1-p)}\Big|_{p=1\sim n} = L_n\left(\bar{x}_{p2}^{(0\sim n-p)}\Big|_{p=1\sim n}\right) + \bar{G}_{1\sim n}\left(\bar{x}_{p2}^{(0\sim n-p)}\Big|_{p=1\sim n}\right)\bar{u}_1, \tag{7.36}$$

where $L_i(\cdot)$ and $\bar{G}_{1\sim i}(\cdot)$ are given iteratively by

$$\begin{aligned} & L_{i+1}\left(\bar{x}_{p2}^{(0\sim i+1-p)}\Big|_{p=1\sim i+1}\right) \\ = & \begin{bmatrix} \dot{L}_i(\cdot)+\dot{\bar{G}}_{1\sim i}(\cdot)\bar{x}_{i+1,1}+\bar{G}_{1\sim i}(\cdot)\,\bar{f}_{i+1,1}(\cdot) \\ \bar{f}_{i+1,2}(\cdot) \end{bmatrix}, \\ & \bar{G}_{1\sim i+1}\left(\bar{x}_{p2}^{(0\sim i+1-p)}\Big|_{p=1\sim i+1}\right) \\ = & \begin{bmatrix} \bar{G}_{1\sim i}(\cdot) & 0 \\ 0 & I_{r_{i+1}-r_i} \end{bmatrix} \bar{Q}_i^{-1}(\cdot)\,\bar{G}_{i+1}(\cdot),\ i=1,2,\ldots,n-1, \end{aligned} \tag{7.37}$$

with $L_1=\bar{f}_1(\bar{x}_{12})$ and $\bar{G}_{1\sim 1}=\bar{G}_1(\bar{x}_{12})$.

For a proof of the above theorem, please refer to Appendix B.

7.2 Second-Order SFSs with Increasing Dimensions

In Duan (2021b), the author proposes more generalized second-order and high-order SFSs (see also the previous chapter). Meanwhile, it is shown that both the second-order and high-order SFSs can be converted into FAS models. In this subsection, we further extend the results in Duan (2021b) to second-order and high-order SFSs with increasing dimensions. Proofs of all theorems in this part are omitted as they can be easily established by applying the induction method introduced in Section 7.1 together with the results in Duan (2021b).

7.2.1 Case of Constant Gains

In this subsection, the following second-order SFS with increasing dimensions is first discussed:

$$\begin{cases} \ddot{x}_1 = f_1\left(x_1^{(0\sim 1)}\right)+G_1x_2 \\ \ddot{x}_2 = f_2\left(x_1^{(0\sim 2)},x_2^{(0\sim 1)}\right)+G_2x_3 \\ \qquad\vdots \\ \ddot{x}_{n-1} = f_n\left(x_i^{(0\sim 2)}\Big|_{i=1\sim n-2},x_{n-1}^{(0\sim 1)}\right)+G_{n-1}x_n \\ \ddot{x}_n = f_n\left(x_i^{(0\sim 2)}\Big|_{i=1\sim n-1},x_n^{(0\sim 1)}\right)+G_nu, \end{cases} \tag{7.38}$$

where $x_i\in\mathbb{R}^{r_i}$, $i=1,2,\ldots,n$, $u\in\mathbb{R}^m$, $G_i\in\mathbb{R}^{r_i\times r_{i+1}}$, $i=1,2,\ldots,n-1$, and $G_n\in\mathbb{R}^{r_n\times m}$. We adopt the similar definition for $x_{i+1,1}\in\mathbb{R}^{r_i}$,$x_{i+1,2}\in\mathbb{R}^{r_{i+1}-r_i}$, $f_{i1}\in\mathbb{R}^{r_i}$, $u_1\in\mathbb{R}^{r_n}$ and $u_2\in\mathbb{R}^{m-r_n}$ as (7.2)-(7.4).

For the first subsystem of system (7.38), it first follows from the definition of x_{12}

and x_{21} that

$$\begin{aligned}\ddot{x}_{12} &= f_1\left(x_1^{(0\sim1)}\right)+G_1Q_1Q_1^{-1}x_2\\ &= f_1\left(x_1^{(0\sim1)}\right)+\left[\begin{array}{cc}\bar{G}_1 & 0\end{array}\right]\left[\begin{array}{c}x_{21}\\ x_{22}\end{array}\right]\\ &= f_1\left(x_1^{(0\sim1)}\right)+\bar{G}_1x_{21}.\end{aligned}$$

Then, for the second subsystem of system (7.38), it can be obtained that

$$\begin{aligned}\left[\begin{array}{c}\ddot{x}_{21}\\ \ddot{x}_{22}\end{array}\right] &= Q_1^{-1}\left(f_2\left(x_1^{(0\sim2)},x_2^{(0\sim1)}\right)+G_2Q_2Q_2^{-1}x_3\right)\\ &= Q_1^{-1}\left(f_2\left(x_1^{(0\sim2)},x_2^{(0\sim1)}\right)+\left[\begin{array}{cc}\bar{G}_2 & 0\end{array}\right]\left[\begin{array}{c}x_{31}\\ x_{32}\end{array}\right]\right)\\ &= \left[\begin{array}{c}f_{21}\left(x_{21}^{(0\sim1)},x_{1\sim2,2}^{(0\sim1)}\right)\\ f_{22}\left(x_{21}^{(0\sim1)},x_{1\sim2,2}^{(0\sim1)}\right)\end{array}\right]+Q_1^{-1}\bar{G}_2x_{31}.\end{aligned}$$

After an n-step induction, system (7.38) can be then transformed into

$$\left\{\begin{array}{l}\ddot{x}_{12}=f_1\left(x_1^{(0\sim1)}\right)+\bar{G}_1x_{21}\\ \ddot{x}_{21}=f_{21}\left(x_{21}^{(0\sim1)},x_{1\sim2,2}^{(0\sim1)}\right)+\left[\begin{array}{cc}I_{r_1} & 0\end{array}\right]Q_1^{-1}\bar{G}_2x_{31}\\ \ddot{x}_{22}=f_{22}\left(x_{21}^{(0\sim1)},x_{1\sim2,2}^{(0\sim1)}\right)+\left[\begin{array}{cc}0 & I_{r_2-r_1}\end{array}\right]Q_1^{-1}\bar{G}_2x_{31}\\ \quad\vdots\\ \ddot{x}_{n1}=f_{n1}\left(x_{12}^{(0\sim2)},x_{ij}^{(0\sim2)}\Big|_{i=2\sim n-1,j=1\sim2},x_{nj}^{(0\sim1)}\Big|_{j=1\sim2}\right)\\ \qquad+\left[\begin{array}{cc}I_{r_{n-1}} & 0\end{array}\right]Q_{n-1}^{-1}\bar{G}_nu_1\\ \ddot{x}_{n2}=f_{n2}\left(x_{12}^{(0\sim2)},x_{ij}^{(0\sim2)}\Big|_{i=2\sim n-1,j=1\sim2},x_{nj}^{(0\sim1)}\Big|_{j=1\sim2}\right)\\ \qquad+\left[\begin{array}{cc}0 & I_{r_n-r_{n-1}}\end{array}\right]Q_{n-1}^{-1}\bar{G}_nu_1.\end{array}\right. \tag{7.39}$$

Based on system (7.39), we can derive the HOFAS model for the second-order SFSs with increasing dimensions. In particular, we present the following theorem.

Theorem 7.3 *Let Assumption 7.1 be satisfied, then under the following transformation*

$$\begin{aligned}x_{i1} &= \bar{G}_{1\sim i-1}^{-1}\left(x_{p2}^{(2i-2p)}\Big|_{p=1\sim i-1}-L_{i-1}\left(x_{p2}^{(0\sim2i-1-2p)}\Big|_{p=1\sim i-1}\right)\right), \qquad (7.40)\\ &\qquad i=2,3,\ldots,n,\end{aligned}$$

SFS (7.39) can be equivalently transformed into the following HOFAS model:

$$x_{p2}^{(2n+2-2p)}\Big|_{p=1\sim n}=L_n\left(x_{p2}^{(0\sim2n+1-2p)}\Big|_{p=1\sim n}\right)+\bar{G}_{1\sim n}u_1, \tag{7.41}$$

where $L_i(\cdot)$ and $G_{1\sim i}$ are given iteratively by

$$L_{i+1}(\cdot)=\begin{bmatrix}\ddot{L}_i(\cdot)+\bar{G}_{1\sim i}f_{i+1,1}(\cdot)\\ f_{i+1,2}(\cdot)\end{bmatrix}, \tag{7.42}$$

$$\bar{G}_{1\sim i+1}=\begin{bmatrix}\bar{G}_{1\sim i} & 0\\ 0 & I_{r_{i+1}-r_i}\end{bmatrix}Q_i^{-1}\bar{G}_{i+1},\ i=1,2,\ldots,n-1, \tag{7.43}$$

with $L_1=f_1\left(x_{12}^{(0\sim 1)}\right)$ and $\bar{G}_{1\sim 1}=\bar{G}_1$.

For a proof of the above theorem, please refer to the appendix.

7.2.2 Case of State-dependent Gains

Next, similar to the first-order SFSs, the more general case where $G_i,\ i=1,2,\ldots,n$, are assumed to be matrix functions can be also considered. In particular, the following second-order SFS is discussed:

$$\begin{cases}\ddot{x}_1=f_1\left(x_1^{(0\sim 1)}\right)+G_1\left(x_1^{(0\sim 1)}\right)x_2\\ \ddot{x}_2=f_2\left(x_1^{(0\sim 2)},x_2^{(0\sim 1)}\right)+G_2\left(x_1^{(0\sim 2)},x_2^{(0\sim 1)}\right)x_3\\ \quad\vdots\\ \ddot{x}_{n-1}=f_n\left(x_i^{(0\sim 2)}\Big|_{i=1\sim n-2},x_{n-1}^{(0\sim 1)}\right)+G_{n-1}\left(x_i^{(0\sim 2)}\Big|_{i=1\sim n-2},x_{n-1}^{(0\sim 1)}\right)x_n\\ \ddot{x}_n=f_n\left(x_i^{(0\sim 2)}\Big|_{i=1\sim n-1},x_n^{(0\sim 1)}\right)+G_n\left(x_i^{(0\sim 2)}\Big|_{i=1\sim n-1},x_n^{(0\sim 1)}\right)u,\end{cases} \tag{7.44}$$

where $x_i\in\mathbb{R}^{r_i},\ i=1,2,\ldots,n$, $u\in\mathbb{R}^m$. Moreover, f_i and G_i are sufficiently smooth functions. In this work, it is assumed that $r_1\le r_2\le\cdots\le r_n\le m$. The following assumption is needed for matrix functions $G_i,\ i=1,2,\ldots,n$.

Assumption 7.3 *For any $X_i\in\mathbb{R}^{3r_1+\cdots+3r_{i-1}}$ and $Y_i\in\mathbb{R}^{2r_i}$, the matrix function $G_i(X_i,Y_i)$ is of full-row rank, i.e.,* $\mathrm{rank}G_i(X_i,Y_i)=r_i,\ i=1,2,\ldots,n$.

It follows from Assumption 7.3 that there exist differentiable functions $Q_i(\cdot):\mathbb{R}^{3r_1+\cdots+3r_{i-1}+2r_i}\to\mathbb{R}^{r_{i+1}\times r_{i+1}}$ and $\bar{G}_i(\cdot):\mathbb{R}^{3r_1+\cdots+3r_{i-1}+2r_i}\to\mathbb{R}^{r_i\times r_i}$ such that

$$G_i(\cdot)Q_i(\cdot)=\begin{bmatrix}\bar{G}_i(\cdot) & 0\end{bmatrix},$$

where $Q_i(\cdot)$ and $\bar{G}_i(\cdot)$ are both of full rank. Define the mapping from

$$x=\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}:\ \mapsto\ \bar{x}=\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\\ \vdots\\ \bar{x}_n\end{bmatrix}$$

as follows:

$$\begin{cases} \bar{x}_1 = x_1 \\ \bar{x}_2 = Q_1^{-1}\left(x_1^{(0\sim 1)}\right) x_2 \\ \bar{x}_3 = Q_2^{-1}\left(x_1^{(0\sim 2)}, x_2^{(0\sim 1)}\right) x_3 \\ \quad \vdots \\ \bar{x}_n = Q_{n-1}^{-1}\left(x_i^{(0\sim 2)}\Big|_{i=1\sim n-1}, x_n^{(0\sim 1)}\right) x_n. \end{cases} \tag{7.45}$$

Then the inverse mapping from $\bar{x}$ to x can be written as

$$\begin{cases} x_1 = \bar{x}_1 \\ x_2 = Q_1\left(\bar{x}_1^{(0\sim 1)}\right)\bar{x}_2 \triangleq \bar{Q}_1\left(\bar{x}_1^{(0\sim 1)}\right)\bar{x}_2 \\ x_3 = Q_2\left(\bar{x}_1^{(0\sim 2)}, \left(\bar{Q}_1\left(\bar{x}_1^{(0\sim 1)}\right)\bar{x}_2\right)^{(0\sim 1)}\right)\bar{x}_3 \\ \quad \triangleq \bar{Q}_2\left(\bar{x}_1^{(0\sim 2)}, \bar{x}_2^{(0\sim 1)}\right)\bar{x}_3 \\ \quad \vdots \\ x_n = Q_{n-1}\left(\left(\bar{Q}_{i-1}\bar{x}_i\right)^{(0\sim 2)}\Big|_{i=1\sim n-2}, \left(\bar{Q}_{n-2}\bar{x}_{n-1}\right)^{(0\sim 1)}\right)\bar{x}_n \\ \quad \triangleq \bar{Q}_{n-1}\left(\bar{x}_i^{(0\sim n-i)}\Big|_{i=1\sim n-1}\right)\bar{x}_n. \end{cases} \tag{7.46}$$

Define $\bar{x}_{i1}$, $\bar{x}_{i2}$ and $\bar{u}_1$ as follows:

$$\begin{cases} \bar{x}_{i1} = \begin{bmatrix} I_{r_{i-1}} & 0 \end{bmatrix}\bar{x}_i,\ i = 2, 3, \ldots, n, \\ \bar{x}_{i2} = \begin{bmatrix} 0 & I_{r_i - r_{i-1}} \end{bmatrix}\bar{x}_i,\ i = 2, 3, \ldots, n, \\ \bar{u}_1 = \begin{bmatrix} 0 & I_{r_n} \end{bmatrix}\bar{Q}_n^{-1}u. \end{cases}$$

Furthermore, denote

$$\bar{f}_1\left(\bar{x}_{12}^{(0\sim 1)}\right) = f_1\left(\bar{x}_{12}^{(0\sim 1)}\right),$$

and

$$\bar{f}_{i1}(\cdot) = \begin{bmatrix} I_{k_{i-1}} & 0 \end{bmatrix}\bar{Q}_{i-1}^{-1}\left(-\ddot{\bar{Q}}_{i-1}\bar{x}_i - 2\dot{\bar{Q}}_{i-1}\dot{\bar{x}}_i + f_i\right),$$

$$\bar{f}_{i2}(\cdot) = \begin{bmatrix} 0 & I_{k_i - k_{i-1}} \end{bmatrix}\bar{Q}_{i-1}^{-1}\left(-\ddot{\bar{Q}}_{i-1}\bar{x}_i - 2\dot{\bar{Q}}_{i-1}\dot{\bar{x}}_i + f_i\right), \tag{7.47}$$

$$i = 2, 3, \ldots, n. \tag{7.48}$$

Meanwhile, for notation convenience, we also define that $\bar{x}_{12} = \bar{x}_1$.

For the first subsystem of system (7.44), it first follows from the definition of $\bar{x}_{12}$ and $\bar{x}_{21}$ that

$$\begin{aligned} \ddot{x}_{12} &= f_1\left(x_1^{(0\sim 1)}\right) + G_1\left(x_1^{(0\sim 1)}\right)\bar{Q}_1\bar{Q}_1^{-1}x_2 \\ &= f_1\left(x_1^{(0\sim 1)}\right) + \begin{bmatrix} \bar{G}_1\left(\bar{x}_{12}^{(0\sim 1)}\right) & 0 \end{bmatrix}\bar{x}_2 \\ &= \bar{f}_1\left(\bar{x}_{12}^{(0\sim 1)}\right) + \bar{G}_1\left(\bar{x}_{12}^{(0\sim 1)}\right)\bar{x}_{21}. \end{aligned}$$

Then, for the second subsystem of system (7.44), it can be obtained that

$$\begin{aligned}\begin{bmatrix}\ddot{\bar{x}}_{21}\\ \ddot{\bar{x}}_{22}\end{bmatrix} &= \bar{Q}_1^{-1}\left(-\ddot{\bar{Q}}_1\bar{x}_2-\dot{\bar{Q}}_1\dot{\bar{x}}_2-\dot{\bar{Q}}_1\dot{\bar{x}}_2+f_2\right)\\ &\quad+\bar{Q}_1^{-1}G_2\left(x_1^{(0\sim 2)},x_2^{(0\sim 1)}\right)\bar{Q}_2\bar{Q}_2^{-1}x_3\\ &= \bar{Q}_1^{-1}\left(-\ddot{\bar{Q}}_1\bar{x}_2-2\dot{\bar{Q}}_1\dot{\bar{x}}_2+f_2\right)\\ &\quad+\bar{Q}_1^{-1}\begin{bmatrix}\bar{G}_2\left(\bar{x}_{12}^{(0\sim 2)},\bar{x}_{21}^{(0\sim 1)},\bar{x}_{22}^{(0\sim 1)}\right) & 0\end{bmatrix}\bar{x}_3\\ &= \begin{bmatrix}\bar{f}_{21}\left(\bar{x}_{12}^{(0\sim 2)},\bar{x}_{21}^{(0\sim 1)},\bar{x}_{22}^{(0\sim 1)}\right)\\ \bar{f}_{22}\left(\bar{x}_{12}^{(0\sim 2)},\bar{x}_{21}^{(0\sim 1)},\bar{x}_{22}^{(0\sim 1)}\right)\end{bmatrix}\\ &\quad+\bar{Q}_1^{-1}\bar{G}_2\left(\bar{x}_{12}^{(0\sim 2)},\bar{x}_{21}^{(0\sim 1)},\bar{x}_{22}^{(0\sim 1)}\right)\bar{x}_{31}.\end{aligned}$$

After an n-step induction, system (7.29) can be then transformed into

$$\begin{cases}\ddot{\bar{x}}_1=\bar{f}_1\left(\bar{x}_{12}^{(0\sim 1)}\right)+\bar{G}_1\left(\bar{x}_{12}^{(0\sim 1)}\right)\bar{x}_{21}\\ \ddot{\bar{x}}_{21}=\bar{f}_{21}\left(\bar{x}_{12}^{(0\sim 2)},\bar{x}_{21}^{(0\sim 1)},\bar{x}_{22}^{(0\sim 1)}\right)\\ \qquad+\begin{bmatrix}I_{r_1} & 0\end{bmatrix}\bar{Q}_1^{-1}\bar{G}_2\left(\bar{x}_{12}^{(0\sim 2)},\bar{x}_{21}^{(0\sim 1)},\bar{x}_{22}^{(0\sim 1)}\right)\bar{x}_{31}\\ \ddot{\bar{x}}_{22}=\bar{f}_{22}\left(\bar{x}_{12}^{(0\sim 2)},\bar{x}_{21}^{(0\sim 1)},\bar{x}_{22}^{(0\sim 1)}\right)\\ \qquad+\begin{bmatrix}0 & I_{r_2-r_1}\end{bmatrix}\bar{Q}_1^{-1}\bar{G}_2\left(\bar{x}_{12}^{(0\sim 2)},\bar{x}_{21}^{(0\sim 1)},\bar{x}_{22}^{(0\sim 1)}\right)\bar{x}_{31}\\ \quad\vdots\\ \ddot{\bar{x}}_{n1}=\bar{f}_{n1}\left(\bar{x}_{12}^{(0\sim n+1)},\bar{x}_{ij}^{(0\sim n-i+2)}\Big|_{i=2\sim n-1,j=1\sim 2},\bar{x}_{nj}^{(0\sim 1)}\Big|_{j=1\sim 2}\right)\\ \qquad+\begin{bmatrix}I_{r_{n-1}} & 0\end{bmatrix}\bar{Q}_{n-1}^{-1}\\ \qquad\times\bar{G}_n\left(\bar{x}_{12}^{(0\sim n+1)},\bar{x}_{ij}^{(0\sim n-i+2)}\Big|_{i=2\sim n-1,j=1\sim 2},\bar{x}_{nj}^{(0\sim 1)}\Big|_{j=1\sim 2}\right)\bar{u}_1\\ \ddot{\bar{x}}_{n2}=\bar{f}_{n2}\left(\bar{x}_{12}^{(0\sim n+1)},\bar{x}_{ij}^{(0\sim n-i+2)}\Big|_{i=2\sim n-1,j=1\sim 2},\bar{x}_{nj}^{(0\sim 1)}\Big|_{j=1\sim 2}\right)\\ \qquad+\begin{bmatrix}0 & I_{r_n-r_{n-1}}\end{bmatrix}\bar{Q}_{n-1}^{-1}\\ \qquad\times\bar{G}_n\left(\bar{x}_{12}^{(0\sim n+1)},\bar{x}_{ij}^{(0\sim n-i+2)}\Big|_{i=2\sim n-1,j=1\sim 2},\bar{x}_{nj}^{(0\sim 1)}\Big|_{j=1\sim 2}\right)\bar{u}_1,\end{cases}\tag{7.49}$$

where

$$\bar{f}_1\left(\bar{x}_{12}^{(0\sim 1)}\right)=f_1\left(\bar{x}_{12}^{(0\sim 1)}\right),$$

and, for $i=2,3,\ldots,n$,

$$\bar{f}_{i1}(\cdot)=\begin{bmatrix}I_{r_{i-1}} & 0\end{bmatrix}\bar{Q}_{i-1}^{-1}\left(-\ddot{\bar{Q}}_{i-1}\bar{x}_i-2\dot{\bar{Q}}_{i-1}\dot{\bar{x}}_i+f_i\right),\tag{7.50}$$

$$\bar{f}_{i2}(\cdot)=\begin{bmatrix}0 & I_{r_i-r_{i-1}}\end{bmatrix}\bar{Q}_{i-1}^{-1}\left(-\ddot{\bar{Q}}_{i-1}\bar{x}_i-2\dot{\bar{Q}}_{i-1}\dot{\bar{x}}_i+f_i\right).\tag{7.51}$$

The FAS model for (7.49) can be illustrated via the following theorem.

Theorem 7.4 *Let Assumption 7.3 be satisfied, then under the following transformation*

$$\begin{aligned}\bar{x}_{i1} &= \bar{G}_{1\sim i-1}^{-1}\left(\bar{x}_{p2}^{(2i-2p)}\Big|_{p=1\sim i-1} - L_{i-1}\left(\bar{x}_{p2}^{(0\sim 2i-1-2p)}\Big|_{p=1\sim i-1}\right)\right), \\ & \qquad i=2,3,\ldots,n,\end{aligned} \tag{7.52}$$

SFS (7.49) can be equivalently transformed into the following HOFAS model:

$$\begin{aligned}\bar{x}_{p2}^{(2i+2-2p)}\Big|_{p=1\sim i} &= L_n\left(\bar{x}_{p2}^{(0\sim 2n+1-2p)}\Big|_{p=1\sim n}\right) \\ &\quad +\bar{G}_{1\sim n}\left(\bar{x}_{p2}^{(0\sim 2n+1-2p)}\Big|_{p=1\sim n}\right)\bar{u}_1,\end{aligned} \tag{7.53}$$

where $L_i(\cdot)$ and $\bar{G}_{1\sim i}(\cdot)$ are given iteratively by

$$\begin{aligned}&L_{i+1}(\cdot) \\ &= \begin{bmatrix} \ddot{L}_i(\cdot)+\ddot{\bar{G}}_{1\sim i}(\cdot)\bar{x}_{i+1,1}+2\dot{\bar{G}}_{1\sim i}(\cdot)\dot{\bar{x}}_{i+1,1}+\bar{G}_{1\sim i}(\cdot)\bar{f}_{i+1,1}(\cdot) \\ \bar{f}_{i+1,2}(\cdot)\end{bmatrix},\end{aligned} \tag{7.54}$$

$$\begin{aligned}&\bar{G}_{1\sim i+1}(\cdot) \\ &= \begin{bmatrix}\bar{G}_{1\sim i}(\cdot) & 0 \\ 0 & I_{r_{i+1}-r_i}\end{bmatrix}\bar{Q}_i^{-1}(\cdot)\bar{G}_{i+1}(\cdot),\ i=1,2,\ldots,n-1,\end{aligned} \tag{7.55}$$

with $L_1=\bar{f}_1\left(\bar{x}_{12}^{(0\sim 1)}\right)$ and $\bar{G}_{1\sim 1}(\cdot)=\bar{G}_1\left(\bar{x}_{12}^{(0\sim 1)}\right)$.

The above theorem can be similarly proven.

7.3 High-Order Case with Increasing Dimensions

In this subsection, a generalized form of SFSs, where each subsystem is high-order, is discussed. The high-order SFSs are firstly considered by Duan in Duan (2021b) (see also Chapter 6). However, the work Duan (2021b) only considers SFSs with uniform dimensions. In this section, the dimensions of all the subsystems are assumed to be increasing.

7.3.1 Case of Constant Gains

In particular, we consider the following system:

$$\begin{cases} x_1^{(m_1)} = f_1\left(x_1^{(0\sim m_1-1)}\right)+G_1x_2 \\ x_2^{(m_2)} = f_2\left(x_1^{(0\sim m_1)},x_2^{(0\sim m_2-1)}\right)+G_2x_2 \\ \quad\vdots \\ x_{n-1}^{(m_{n-1})} = f_{n-1}\left(x_i^{(0\sim m_i)}\Big|_{i=1\sim n-2},x_{n-1}^{(0\sim m_{n-1}-1)}\right)+G_{n-1}x_n \\ x_n^{(m_n)} = f_n\left(x_i^{(0\sim m_i)}\Big|_{i=1\sim n-1},x_n^{(0\sim m_n-1)}\right)+G_nu,\end{cases} \tag{7.56}$$

where $x_i \in \mathbb{R}^{r_i},\ i = 1, 2, \ldots, n$, $u \in \mathbb{R}^m$, $G_i \in \mathbb{R}^{r_i \times r_{i+1}},\ i = 1, 2, \ldots, n-1$, and $G_n \in \mathbb{R}^{r_n \times m}$. We adopt the similar definition for $x_{i+1,1} \in \mathbb{R}^{r_i}$, $x_{i+1,2} \in \mathbb{R}^{r_{i+1}-r_i}$, $f_{i+1,1} \in \mathbb{R}^{r_i}$, $u_1 \in \mathbb{R}^{r_n}$ and $u_2 \in \mathbb{R}^{m-r_n}$ as in (7.2)-(7.4).

For the first subsystem of system (7.56), it follows from the definitions of x_{12} and x_{21} that

$$\begin{aligned} x_{12}^{(m_1)} &= f_1\left(x_{12}^{(0\sim m_1-1)}\right) + G_1 Q_1 Q_1^{-1} x_2 \\ &= f_1\left(x_{12}^{(0\sim m_1-1)}\right) + \begin{bmatrix} \bar{G}_1 & 0 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} \\ &= f_1\left(x_{12}^{(0\sim m_1-1)}\right) + \bar{G}_1 x_{21}. \end{aligned}$$

Then, for the second subsystem of system (7.56), it can be obtained that

$$\begin{aligned} \begin{bmatrix} x_{21}^{(m_2)} \\ x_{22}^{(m_2)} \end{bmatrix} &= Q_1^{-1}\left(f_2\left(x_1^{(0\sim m_1)}, x_2^{(0\sim m_2-1)}\right) + G_2 x_2\right) \\ &= Q_1^{-1} f_2\left(x_1^{(0\sim m_1)}, x_2^{(0\sim m_2-1)}\right) + Q_1^{-1} G_2 Q_2 Q_2^{-1} x_3 \\ &= \begin{bmatrix} f_{21}\left(x_{12}^{(0\sim m_1)}, x_{21}^{(0\sim m_2-1)}, x_{22}^{(0\sim m_2-1)}\right) \\ f_{22}\left(x_{12}^{(0\sim m_1)}, x_{21}^{(0\sim m_2-1)}, x_{22}^{(0\sim m_2-1)}\right) \end{bmatrix} \\ &\quad + Q_1^{-1} \begin{bmatrix} \bar{G}_2 & 0 \end{bmatrix} \begin{bmatrix} x_{31} \\ x_{32} \end{bmatrix} \\ &= \begin{bmatrix} f_{21}\left(x_{12}^{(0\sim m_1)}, x_{21}^{(0\sim m_2-1)}, x_{22}^{(0\sim m_2-1)}\right) \\ f_{22}\left(x_{12}^{(0\sim m_1)}, x_{21}^{(0\sim m_2-1)}, x_{22}^{(0\sim m_2-1)}\right) \end{bmatrix} + Q_1^{-1} \bar{G}_2 x_{31}. \end{aligned}$$

After an n-step induction, system (7.56) can be then transformed into

$$\begin{cases} x_{12}^{(m_1)} = f_1\left(x_{12}^{(0\sim m_1-1)}\right) + \bar{G}_1 x_{21} \\ x_{21}^{(m_2)} = f_{21}\left(x_{12}^{(0\sim m_1)}, x_{21}^{(0\sim m_2-1)}, x_{22}^{(0\sim m_2-1)}\right) + \begin{bmatrix} I_{r_1} & 0 \end{bmatrix} Q_1^{-1} \bar{G}_2 x_{31} \\ x_{22}^{(m_2)} = f_{22}\left(x_{12}^{(0\sim m_1)}, x_{21}^{(0\sim m_2-1)}, x_{22}^{(0\sim m_2-1)}\right) + \begin{bmatrix} 0 & I_{r_2-r_1} \end{bmatrix} Q_1^{-1} \bar{G}_2 x_{31} \\ \quad \vdots \\ x_{n1}^{(m_n)} = f_{n1}\left(x_{12}^{(0\sim m_1)}, x_{ij}^{(0\sim m_i)}\Big|_{i=2\sim n-1, j=1\sim 2}, x_{nj}^{(0\sim m_n-1)}\Big|_{j=1\sim 2}\right) \\ \qquad + \begin{bmatrix} I_{r_n} & 0 \end{bmatrix} Q_{n-1}^{-1} \bar{G}_n u_1 \\ x_{n2}^{(m_n)} = f_{n2}\left(x_{12}^{(0\sim m_1)}, x_{ij}^{(0\sim m_i)}\Big|_{i=2\sim n-1, j=1\sim 2}, x_{nj}^{(0\sim m_n-1)}\Big|_{j=1\sim 2}\right) \\ \qquad + \begin{bmatrix} 0 & I_{r_n-r_{n-1}} \end{bmatrix} Q_{n-1}^{-1} \bar{G}_n u_1. \end{cases} \tag{7.57}$$

For notation convenience, we denote that for $j \geq i \geq 1$,

$$q_{i\sim j} = m_i + m_{i+1} + \cdots + m_j.$$

Based on the equivalent form (7.57), the HOFAS model for high-order SFS (7.56) can be obtained, which is given by the following theorem.

Theorem 7.5 *Let Assumption 7.1 be satisfied, then under the following transformation*

$$x_{i1} = \bar{G}_{1\sim i-1}^{-1}\left(x_{p2}^{(q_{p\sim i-1})}\Big|_{p=1\sim i-1} - L_{i-1}\left(x_{p2}^{(0\sim q_{p\sim i-1}-1)}\Big|_{p=1\sim i-1}\right)\right), \quad i=2,3,\ldots,n, \tag{7.58}$$

SFS (7.57) can be equivalently transformed into the following HOFAS model:

$$x_{p2}^{(q_{p\sim n})}\Big|_{p=1\sim n} = L_n\left(x_{p2}^{(0\sim q_{p\sim n}-1)}\Big|_{p=1\sim n}\right) + \bar{G}_{1\sim n}u_1, \tag{7.59}$$

where $L_i(\cdot)$ and $\bar{G}_{1\sim i}$ are given iteratively by

$$L_{i+1}(\cdot) = \begin{bmatrix} L_i^{(m_{i+1})}(\cdot) + \bar{G}_{1\sim i} f_{i+1,1}(\cdot) \\ f_{i+1,2}(\cdot) \end{bmatrix}, \tag{7.60}$$

$$\bar{G}_{1\sim i+1} = \begin{bmatrix} \bar{G}_{1\sim i} & 0 \\ 0 & I_{r_{i+1}-r_i} \end{bmatrix} Q_i^{-1}\bar{G}_{i+1},\ i=1,2,\ldots,n-1, \tag{7.61}$$

with $L_1 = f_1\left(x_{12}^{(0\sim m_1-1)}\right)$ and $\bar{G}_{1\sim 1} = \bar{G}_1$.

For a proof of the above theorem, please refer to Appendix B.

7.3.2 Case of State-dependent Gains

Similar to the first-order and second-order SFSs, we can also consider the following more general high-order SFS with $G_i,\ i=1,2,\ldots,n$, being matrix functions:

$$\begin{cases} x_1^{(m_1)} = f_1\left(x_1^{(0\sim m_1-1)}\right) + G_1\left(x_1^{(0\sim m_1-1)}\right)x_2 \\ x_2^{(m_2)} = f_2\left(x_1^{(0\sim m_1)}, x_2^{(0\sim m_2-1)}\right) + G_2\left(x_1^{(0\sim m_1)}, x_2^{(0\sim m_2-1)}\right)x_2 \\ \quad\vdots \\ x_{n-1}^{(m_{n-1})} = f_{n-1}\left(x_i^{(0\sim m_i)}\Big|_{i=1\sim n-2}, x_{n-1}^{(0\sim m_{n-1}-1)}\right) \\ \qquad + G_{n-1}\left(x_i^{(0\sim m_i)}\Big|_{i=1\sim n-2}, x_{n-1}^{(0\sim m_{n-1}-1)}\right)x_n \\ x_n^{(m_n)} = f_n\left(x_i^{(0\sim m_i)}\Big|_{i=1\sim n-1}, x_n^{(0\sim m_n-1)}\right) \\ \qquad + G_n\left(x_i^{(0\sim m_i)}\Big|_{i=1\sim n-1}, x_n^{(0\sim m_n-1)}\right)u, \end{cases} \tag{7.62}$$

where $x_i \in \mathbb{R}^{r_i},\ i=1,2,\ldots,n$, and $u \in \mathbb{R}^m$. Moreover, $f_i(\cdot)$ and $G_i(\cdot)$ are sufficiently smooth functions. In this work, we assume that $r_1 \le r_2 \le \cdots \le r_n \le m$. The following assumption is needed for system (7.62).

Assumption 7.4 *For any $X_i \in \mathbb{R}^{\varpi_i}$ with $\varpi_i = \sum_{p=1}^{i-1}(m_p+1)r_p$ and $Y_i \in \mathbb{R}^{m_i r_i}$, the gain matrix function $G_i(X_i, Y_i)$ is of full-row rank, i.e.,* $\mathrm{rank} G_i(X_i, Y_i) = r_i$, $i = 1, 2, \ldots, n$.

It follows from Assumption 7.4 that there exist sufficiently smooth functions $Q_i(\cdot)$ and $\bar{G}_i(\cdot)$ for $i = 1, 2, \ldots, n$, such that

$$G_i(\cdot) Q_i(\cdot) = \begin{bmatrix} \bar{G}_i(\cdot) & 0 \end{bmatrix},$$

where $Q_i(\cdot)$ and $\bar{G}_i(\cdot)$ are nonsingular. Then we define

$$\bar{x} = \begin{bmatrix} \bar{x}_1^{\mathrm{T}} & \bar{x}_2^{\mathrm{T}} & \cdots & \bar{x}_n^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$$

as follows:

$$\begin{cases} \bar{x}_1 = x_1 \\ \bar{x}_2 = Q_1^{-1}\left(x_1^{(0\sim m_1-1)}\right) x_2 \\ \bar{x}_3 = Q_2^{-1}\left(x_1^{(0\sim m_1)}, x_2^{(0\sim m_2-1)}\right) x_3 \\ \quad \vdots \\ \bar{x}_n = Q_{n-1}^{-1}\left(x_i^{(0\sim m_i)}\Big|_{i=1\sim n-2}, x_{n-1}^{(0\sim m_{n-1}-1)}\right) x_n. \end{cases} \tag{7.63}$$

The inverse mapping from $\bar{x}$ to x can be written as

$$\begin{cases} x_1 = \bar{x}_1 \\ x_2 = Q_1\left(\bar{x}_1^{(0\sim m_1-1)}\right)\bar{x}_2 \triangleq \bar{Q}_1\left(\bar{x}_1^{(0\sim m_1-1)}\right)\bar{x}_2 \\ x_3 = Q_2\left(\bar{x}_1^{(0\sim m_1)}, \left(\bar{Q}_1\left(\bar{x}_1^{(0\sim m_1-1)}\right)\bar{x}_2\right)^{(0\sim m_2-1)}\right)\bar{x}_3 \\ \quad \triangleq \bar{Q}_2\left(\bar{x}_1^{(0\sim m_1+m_2-2)}, \bar{x}_2^{(0\sim m_2-1)}\right)\bar{x}_3 \\ \quad \vdots \\ x_n = Q_{n-1}\left(\left(\bar{Q}_{i-1}\bar{x}_i\right)^{(0\sim m_i)}\Big|_{i=1\sim n-2}, \left(\bar{Q}_{n-2}\bar{x}_{n-1}\right)^{(0\sim m_{n-1}-1)}\right)\bar{x}_n \\ \quad \triangleq \bar{Q}_{n-1}\left(\bar{x}_i^{(0\sim q_{i\sim n-1}-n+i)}\Big|_{i=1\sim n-1}\right)\bar{x}_n. \end{cases} \tag{7.64}$$

Furthermore, define $\bar{x}_{i1}$, $\bar{x}_{i2}$ and $\bar{u}_1$ as follows:

$$\begin{cases} \bar{x}_{i1} = \begin{bmatrix} I_{r_{i-1}} & 0 \end{bmatrix}\bar{x}_i, \ i = 2, 3, \ldots, n \\ \bar{x}_{i2} = \begin{bmatrix} 0 & I_{r_i - r_{i-1}} \end{bmatrix}\bar{x}_i, \ i = 2, 3, \ldots, n \\ \bar{u}_1 = \begin{bmatrix} I_{r_n} & 0 \end{bmatrix}\bar{Q}_n^{-1} u. \end{cases}$$

Denote

$$\bar{f}_1 = f_1,$$

and

$$\bar{f}_{i1} = \begin{bmatrix} I_{r_{i-1}} & 0 \end{bmatrix} \bar{Q}_{i-1}^{-1} \left(-\sum_{k=1}^{m_i} C_{m_i}^k \bar{Q}_{i-1}^{(k)} \bar{x}_i^{(m_i-k)} + f_i \right), \tag{7.65}$$

$$\bar{f}_{i2} = \begin{bmatrix} 0 & I_{r_i - r_{i-1}} \end{bmatrix} \bar{Q}_{i-1}^{-1} \left(-\sum_{k=1}^{m_i} C_{m_i}^k \bar{Q}_{i-1}^{(k)} \bar{x}_i^{(m_i-k)} + f_i \right), \tag{7.66}$$

$$i = 2, 3, \ldots, n,$$

with $C_{m_i}^k$, $i = 2, 3, \ldots, n$, being the binomial coefficients. Meanwhile, for notation convenience, we also define $\bar{x}_{12} = \bar{x}_1$.

For the first subsystem of system (7.62), it first follows from the definitions of $\bar{x}_{12}$ and $\bar{x}_{21}$ that

$$\begin{aligned} \bar{x}_{12}^{(m_1)} &= \bar{f}_1 \left(\bar{x}_{12}^{(0\sim m_1-1)} \right) + G_1 \left(x_1^{(0\sim m_1-1)} \right) \bar{Q}_1 \bar{Q}_1^{-1} x_2 \\ &= \bar{f}_1 \left(\bar{x}_{12}^{(0\sim m_1-1)} \right) + \left[\bar{G}_1 \left(\bar{x}_1^{(0\sim m_1-1)} \right) \quad 0 \right] \bar{x}_2 \\ &= \bar{f}_1 \left(\bar{x}_{12}^{(0\sim m_1-1)} \right) + \bar{G}_1 \left(\bar{x}_1^{(0\sim m_1-1)} \right) \bar{x}_{21}. \end{aligned}$$

Then, for the second subsystem of system (7.62), it can be obtained that

$$\begin{aligned} \begin{bmatrix} \bar{x}_{21}^{(m_2)} \\ \bar{x}_{22}^{(m_2)} \end{bmatrix} &= \bar{Q}_1^{-1} \left(-\sum_{k=1}^{m_2} C_{m_2}^k \bar{Q}_1^{(k)} \bar{x}_2^{(m_2-k)} + f_2 \left(x_1^{(0\sim m_1)}, x_2^{(0\sim m_2-1)} \right) \right) \\ &\quad + \bar{Q}_1^{-1} G_2 \left(x_1^{(0\sim m_1)}, x_2^{(0\sim m_2-1)} \right) \bar{Q}_2 \bar{Q}_2^{-1} x_3 \\ &= \begin{bmatrix} \bar{f}_{21} \left(\bar{x}_{12}^{(0\sim m_1+m_2-1)}, \bar{x}_{21}^{(0\sim m_2-1)}, \bar{x}_{22}^{(0\sim m_2-1)} \right) \\ \bar{f}_{22} \left(\bar{x}_{12}^{(0\sim m_1+m_2-1)}, \bar{x}_{21}^{(0\sim m_2-1)}, \bar{x}_{22}^{(0\sim m_2-1)} \right) \end{bmatrix} \\ &\quad + \bar{Q}_1^{-1} \left[\bar{G}_2 \left(\bar{x}_{12}^{(0\sim m_1+m_2-1)}, \bar{x}_{21}^{(0\sim m_2-1)}, \bar{x}_{22}^{(0\sim m_2-1)} \right) \quad 0 \right] \bar{x}_3 \\ &= \begin{bmatrix} \bar{f}_{21} \left(\bar{x}_{12}^{(0\sim m_1+m_2-1)}, \bar{x}_{21}^{(0\sim m_2-1)}, \bar{x}_{22}^{(0\sim m_2-1)} \right) \\ \bar{f}_{22} \left(\bar{x}_{12}^{(0\sim m_1+m_2-1)}, \bar{x}_{21}^{(0\sim m_2-1)}, \bar{x}_{22}^{(0\sim m_2-1)} \right) \end{bmatrix} \\ &\quad + \bar{Q}_1^{-1} \bar{G}_2 \left(\bar{x}_{12}^{(0\sim m_1+m_2-1)}, \bar{x}_{21}^{(0\sim m_2-1)}, \bar{x}_{22}^{(0\sim m_2-1)} \right) \bar{x}_{31}. \end{aligned}$$

After an n-step induction, system (7.62) can be then transformed into

$$\begin{cases}
\bar{x}_{12}^{(m_1)} = \bar{f}_1\left(\bar{x}_{12}^{(0\sim m_1-1)}\right) + \bar{G}_1\left(\bar{x}_1^{(0\sim m_1-1)}\right)\bar{x}_{21} \\
\bar{x}_{21}^{(m_2)} = \bar{f}_{21}\left(\bar{x}_{12}^{(0\sim m_1+m_2-1)}, \bar{x}_{21}^{(0\sim m_2-1)}, \bar{x}_{22}^{(0\sim m_2-1)}\right) \\
\qquad + \left[\begin{array}{cc} I_{r_1} & 0 \end{array}\right]\bar{Q}_1^{-1}\bar{G}_2\left(\bar{x}_{12}^{(0\sim m_1+m_2-1)}, \bar{x}_{21}^{(0\sim m_2-1)}, \bar{x}_{22}^{(0\sim m_2-1)}\right)\bar{x}_{31} \\
\bar{x}_{22}^{(m_2)} = \bar{f}_{22}\left(\bar{x}_{12}^{(0\sim m_1+m_2-1)}, \bar{x}_{21}^{(0\sim m_2-1)}, \bar{x}_{22}^{(0\sim m_2-1)}\right) \\
\qquad + \left[\begin{array}{cc} 0 & I_{r_2-r_1} \end{array}\right]\bar{Q}_1^{-1}\bar{G}_2\left(\bar{x}_{12}^{(0\sim m_1+m_2-1)}, \bar{x}_{21}^{(0\sim m_2-1)}, \bar{x}_{22}^{(0\sim m_2-1)}\right)\bar{x}_{31} \\
\qquad \vdots \\
\bar{x}_{n1}^{(m_n)} = \bar{f}_{n1}\left(\bar{x}_{12}^{(0\sim q_{1\sim n}-n+1)}, \bar{x}_{ij}^{(0\sim q_{i\sim n}-n+i)}\Big|_{i=2\sim n-1, j=1\sim 2}, \bar{x}_{nj}^{(0\sim m_n-1)}\Big|_{j=1\sim 2}\right) \\
\qquad + \left[\begin{array}{cc} I_{r_{n-1}} & 0 \end{array}\right]\bar{Q}_{n-1}^{-1} \\
\qquad \times \bar{G}_n\left(\bar{x}_{12}^{(0\sim q_{1\sim n}-n+1)}, \bar{x}_{ij}^{(0\sim q_{i\sim n}-n+i)}\Big|_{i=2\sim n-1, j=1\sim 2}, \bar{x}_{nj}^{(0\sim m_n-1)}\Big|_{j=1\sim 2}\right)\bar{u}_1 \\
\bar{x}_{n2}^{(m_n)} = \bar{f}_{n2}\left(\bar{x}_{12}^{(0\sim q_{1\sim n}-n+1)}, \bar{x}_{ij}^{(0\sim q_{i\sim n}-n+i)}\Big|_{i=2\sim n-1, j=1\sim 2}, \bar{x}_{nj}^{(0\sim m_n-1)}\Big|_{j=1\sim 2}\right) \\
\qquad + \left[\begin{array}{cc} 0 & I_{r_n-r_{n-1}} \end{array}\right]\bar{Q}_{n-1}^{-1} \\
\qquad \times \bar{G}_n\left(\bar{x}_{12}^{(0\sim q_{1\sim n}-n+1)}, \bar{x}_{ij}^{(0\sim q_{i\sim n}-n+i)}\Big|_{i=2\sim n-1, j=1\sim 2}, \bar{x}_{nj}^{(0\sim m_n-1)}\Big|_{j=1\sim 2}\right)\bar{u}_1,
\end{cases} \tag{7.67}$$

where $\bar{f}_1 = f_1$,and, for $i = 2, 3, \ldots, n$,

$$\bar{f}_{i1} = \left[\begin{array}{cc} I_{r_{i-1}} & 0 \end{array}\right]\bar{Q}_{i-1}^{-1}\left(-\sum_{k=1}^{m_i} C_{m_i}^k \bar{Q}_{i-1}^{(k)}\bar{x}_i^{(m_i-k)} + f_i\right), \tag{7.68}$$

$$\bar{f}_{i2} = \left[\begin{array}{cc} 0 & I_{r_i-r_{i-1}} \end{array}\right]\bar{Q}_{i-1}^{-1}\left(-\sum_{k=1}^{m_i} C_{m_i}^k \bar{Q}_{i-1}^{(k)}\bar{x}_i^{(m_i-k)} + f_i\right), \tag{7.69}$$

with C being the binomial coefficient defined by

$$C_{m_i}^k = \frac{m_i!}{(m_i-k)!k!}.$$

Based on the equivalent system (7.67), the following result can be obtained for system (7.62).

Theorem 7.6 *Let Assumption 7.4 be satisfied, then under the transformation*

$$\bar{x}_{i1} = \bar{G}_{1\sim i-1}^{-1}\left(\bar{x}_{p2}^{(q_{p\sim i-1})}\Big|_{p=1\sim i-1} - L_{i-1}\left(\bar{x}_{p2}^{(0\sim q_{p\sim i-1}-1)}\Big|_{p=1\sim i-1}\right)\right), \tag{7.70}$$
$$i = 2, 3, \ldots, n,$$

SFS (7.67) can be equivalently transformed into the following HOFAS model:

$$\bar{x}_{p2}^{(q_{p\sim n})}\Big|_{p=1\sim n} = L_n(\cdot) + \bar{G}_{1\sim n}(\cdot)u_1, \tag{7.71}$$

where $L_i(\cdot)$ and $\bar{G}_{1\sim i}(\cdot)$ are given iteratively by

$$\begin{aligned} & L_{i+1}(\cdot) \\ = & \left[\begin{array}{c} L_i^{(m_{i+1})}(\cdot)+\sum_{k=1}^{m_{i+1}} C_{m_{i+1}}^k(\cdot)\bar{G}_{1\sim i}^{(k)}(\cdot)\bar{x}_{i1}^{(m_{i+1}-k)}+\bar{G}_{1\sim i}(\cdot)\, f_{i+1,1}(\cdot) \\ f_{i+1,2}(\cdot) \end{array}\right] \end{aligned} \tag{7.72}$$

$$\begin{aligned} & \bar{G}_{1\sim i+1}(\cdot) \\ = & \left[\begin{array}{cc} \bar{G}_{1\sim i}(\cdot) & 0 \\ 0 & I_{r_{i+1}-r_i} \end{array}\right] Q_i^{-1}(\cdot)\bar{G}_{i+1}(\cdot),\ i=1,2,\ldots,n-1, \end{aligned} \tag{7.73}$$

with $L_1=\bar{f}_1\left(\bar{x}_{12}^{(0\sim m_1-1)}\right)$ *and* $\bar{G}_{1\sim 1}=\bar{G}_1\left(\bar{x}_1^{(0\sim m_1-1)}\right)$.

Again, this theorem can be similarly proven.

Remark 7.3 Different from (7.30), the mapping (7.63) is not a differential homeomorphism from x to $\bar{x}$ as it is not one-to-one. However, it can be verified that if system (7.71) is stabilized under controller u_1, then system (7.62) is also stabilized. Therefore, the HOFAS model (7.71) is applicable for control of system (7.62).

7.4 Examples

In this section, we provide two numerical examples to illustrate the effectiveness of our results.

7.4.1 Example 1

Consider the following first-order system:

$$\left\{\begin{array}{l} \dot{x}_1=0.1x_1+\left[\begin{array}{cc} 1 & 1 \end{array}\right]x_2 \\ \dot{x}_2=x_1x_2+\left[\begin{array}{ccc} 0 & 1 & 1 \\ 1 & 2 & 2 \end{array}\right]u, \end{array}\right. \tag{7.74}$$

where

$$x=\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right]\in\mathbb{R}^3,\ x_2=\left[\begin{array}{c} x_{2a} \\ x_{2b} \end{array}\right]\in\mathbb{R}^2,\ u=\left[\begin{array}{c} u_a \\ u_b \\ u_c \end{array}\right]\in\mathbb{R}^3,$$

are the states and the control inputs.

Obviously, system (7.74) is a first-order SFS with increasing dimension. Specifically, we have $r_1=1$, $r_2=2$, and $m=3$. Noting that

$$G_1=\left[\begin{array}{cc} 1 & 1 \end{array}\right],\ G_2=\left[\begin{array}{ccc} 0 & 1 & 1 \\ 1 & 2 & 2 \end{array}\right],$$

we can obtain

$$Q_1=\left[\begin{array}{cc} 1 & -1 \\ 0 & 1 \end{array}\right],\ Q_2=\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{array}\right],$$

which give

$$\bar{G}_1 = 1,\ \bar{G}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

Further noting

$$f_1(x_1) = 0.1x_1,\ \ f_2(x_1, x_2) = x_1 x_2,$$

we can obtain, following (7.2)-(7.4),

$$\begin{bmatrix} x_{2,1} \\ x_{2,2} \end{bmatrix} = Q_1^{-1} x_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{2a} \\ x_{2b} \end{bmatrix}, \tag{7.75}$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = Q_2^{-1} u = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_a \\ u_b \\ u_c \end{bmatrix}, \tag{7.76}$$

and

$$\begin{aligned} \begin{bmatrix} f_{2,1}(x_1, x_{2,1}) \\ f_{2,2}(x_1, x_{2,2}) \end{bmatrix} &= Q_1^{-1} f_2(x_1, x_2) \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_1 \begin{bmatrix} x_{2a} \\ x_{2b} \end{bmatrix} \\ &= \begin{bmatrix} x_1 x_{2,1} \\ x_1 x_{2,2} \end{bmatrix}. \end{aligned} \tag{7.77}$$

Further, by (7.11) and (7.12), we obtain

$$\begin{aligned} L_2\left(x_{12}^{(0\sim 1)}, x_{21}, x_{2,2}\right) &= \begin{bmatrix} \dot{L}_1(x_{12}) + \bar{G}_{1\sim 1} f_{2,1}(x_1, x_{2,1}) \\ f_{2,2}(x_1, x_{2,2}) \end{bmatrix} \\ &= \begin{bmatrix} \dot{f}_1(x_{12}) + x_1 x_{2,1} \\ x_1 x_{2,2} \end{bmatrix} \\ &= \begin{bmatrix} 0.1\dot{x}_{12} + x_{12} x_{2,1} \\ x_{12} x_{2,2} \end{bmatrix}, \end{aligned} \tag{7.78}$$

and

$$\begin{aligned} \bar{G}_{1\sim 2} &= \begin{bmatrix} \bar{G}_{1\sim 1} & 0 \\ 0 & I_{r_2 - r_1} \end{bmatrix} Q_1^{-1} \bar{G}_2 \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}. \end{aligned}$$

Therefore, it follows from Theorem 7.1 that the needed transformation is

$$\begin{cases} x_1 = x_{12} \\ x_2 = \begin{bmatrix} \dot{x}_{12} - 0.1x_{12} - x_{22} \\ x_{22} \end{bmatrix}, \end{cases} \tag{7.79}$$

under which system (7.74) is equivalently transformed into the following multi-order FAS:

$$\begin{bmatrix} \ddot{x}_{12} \\ \dot{x}_{22} \end{bmatrix} = \begin{bmatrix} 0.1\dot{x}_{12} + x_{12}x_{21} \\ x_{12}x_{22} \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} u_1. \tag{7.80}$$

Then we can design for FAS (7.80) the following controller:

$$\begin{aligned} u_1 &= \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -2.1\dot{x}_{12} - x_{12} - x_{12}x_{21} \\ -x_{12}x_{22} - x_{22} \end{bmatrix} \\ &= \begin{bmatrix} 4.2\dot{x}_{12} + 2x_{12} + 2x_{12}x_{21} - 3x_{12}x_{22} - 3x_{22} \\ -2.1\dot{x}_{12} - x_{12} - x_{12}x_{21} + x_{12}x_{22} + x_{22} \end{bmatrix}. \end{aligned} \tag{7.81}$$

Correspondingly, it follows from (7.81) that the controller for the original system (7.74) is given by

$$\begin{aligned} \begin{bmatrix} u_a \\ u_b \\ u_c \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4.2\dot{x}_{12} + 2x_{12} + 2x_{12}x_{21} - 3x_{12}x_{22} - 3x_{22} \\ -2.1\dot{x}_{12} - x_{12} - x_{12}x_{21} + x_{12}x_{22} + x_{22} \\ u_2 \end{bmatrix} \\ &= \begin{bmatrix} 4.2\dot{x}_{12} + 2x_{12} + 2x_{12}x_{21} - 3x_{12}x_{22} - 3x_{22} \\ -2.1\dot{x}_{12} - x_{12} - x_{12}x_{21} + x_{12}x_{22} + x_{22} - u_2 \\ u_2 \end{bmatrix} \\ &= \begin{bmatrix} 2.42x_1 + 4.2x_{2a} + 1.2x_{2b} + 2x_1x_{2a} - x_1x_{2b} \\ -1.21x_1 - 2.1x_{2a} - 1.1x_{2b} - x_1x_{2a} - u_2 \\ u_2 \end{bmatrix}, \end{aligned} \tag{7.82}$$

where u_2 is an arbitrarily chosen parameter which can often be chosen to be zero.

The closed-loop system of FAS (7.80) under controller (7.81) is obviously given by

$$\begin{cases} \ddot{x}_{12} + 2\dot{x}_{12} + x_{12} = 0 \\ \dot{x}_{22} + x_{22} = 0, \end{cases}$$

which possesses the set of closed-loop poles $\{-1, -1, -1\}$.

Further, using transformation (7.75), the closed-loop system of the original system under controller (7.82) is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_{2a} \\ \dot{x}_{2b} \end{bmatrix} = \begin{bmatrix} 0.1 & 1 & 1 \\ -1.21 & -2.1 & -1.1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_{2a} \\ x_{2b} \end{bmatrix}.$$

Choose $u_2 = 0$, and the initial values for the original system (7.74) as

$$x_1(0) = -2,\ x_2(0) = \begin{bmatrix} -2 & 4 \end{bmatrix}^{\mathrm{T}},$$

which correspond, for FAS (7.80), to

$$x_{12}(0) = -2,\ \dot{x}_{12}(0) = 1.8,\ x_{22}(0) = 4.$$

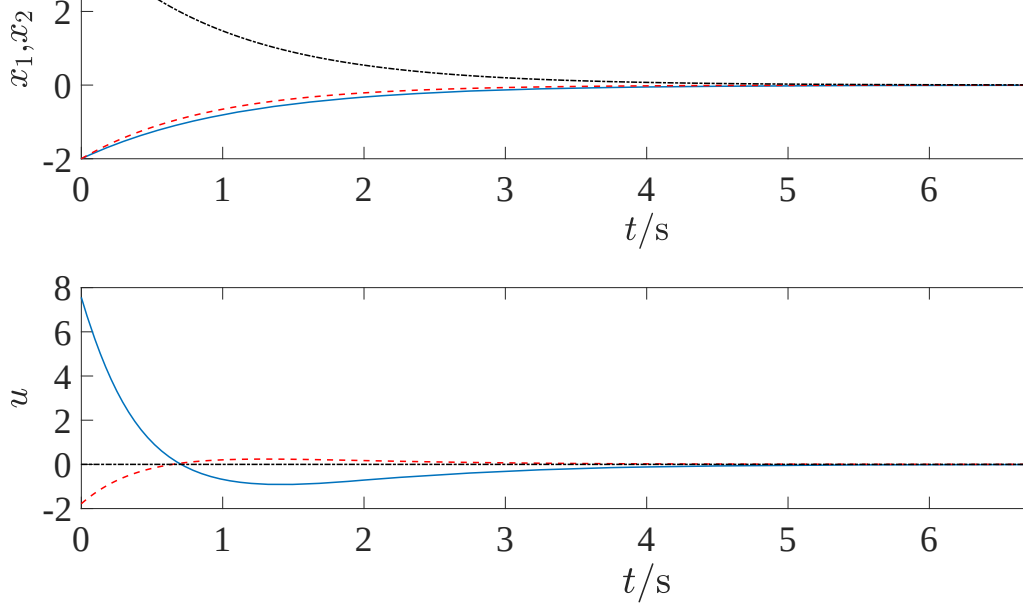


Figure 7.1: Simulation results, Example 1

Then, the simulation results are shown in Fig. 7.1, which shows that x_1 and x_2 converge smoothly to the origin with a settling time of about 6 seconds, and the absolute values of control inputs are all below 8. Therefore, the effectiveness of the FAS approach is well illustrated.

7.4.2 Example 2

Consider the following second-order system:

$$\begin{cases} \ddot{x}_1 = \dot{x}_1 \sin x_1 + \begin{bmatrix} 2 & 3 \end{bmatrix} x_2 \\ \ddot{x}_2 = \dot{x}_2 \sin x_1^2 + \begin{bmatrix} 0 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix} u, \end{cases} \tag{7.83}$$

where

$$x = \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix} \in \mathbb{R}^6,\ x_2 = \begin{bmatrix} x_{2a} \\ x_{2b} \end{bmatrix} \in \mathbb{R}^2,\ u = \begin{bmatrix} u_a \\ u_b \\ u_c \end{bmatrix} \in \mathbb{R}^3,$$

are the states and the control inputs.

Obviously, system (7.83) is a second-order SFS with increasing dimension. Specifically, we have $r_1 = 1$, $r_2 = 2$, and $m = 3$. Noting that

$$G_1 = \begin{bmatrix} 2 & 3 \end{bmatrix},\ G_2 = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix},$$

we can obtain

$$Q_1 = \begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & 1 \end{bmatrix},\ Q_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix},$$

which give

$$\bar{G}_1 = 2,\ \bar{G}_2 = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}.$$

Further noting

$$f_1(x_1,\dot{x}_1)=\dot{x}_1\sin x_1,\ \ f_2(x_1,\dot{x}_2)=\dot{x}_2\sin x_1^2,$$

we can obtain, following (7.2)-(7.4),

$$\begin{bmatrix} x_{2,1} \\ x_{2,2} \end{bmatrix} = Q_1^{-1}x_2 = \frac{1}{2}\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_{2a} \\ x_{2b} \end{bmatrix}, \tag{7.84}$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = Q_2^{-1}u = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} u_a \\ u_b \\ u_c \end{bmatrix}, \tag{7.85}$$

and

$$\begin{aligned} \begin{bmatrix} f_{2,1}(x_1,x_{2,1}) \\ f_{2,2}(x_1,x_{2,2}) \end{bmatrix} &= Q_1^{-1}f_2(\dot{x}_2) \\ &= \frac{1}{2}\sin x_1^2\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \dot{x}_{2a} \\ \dot{x}_{2b} \end{bmatrix} \\ &= \sin x_1^2\begin{bmatrix} \dot{x}_{2,1} \\ \dot{x}_{2,2} \end{bmatrix} \\ &= \begin{bmatrix} \dot{x}_{21}\sin x_{12}^2 \\ \dot{x}_{22}\sin x_{12}^2 \end{bmatrix}. \end{aligned} \tag{7.86}$$

Further, by (7.42) and (7.43), we obtain

$$\begin{aligned} & L_2\left(x_{12}^{(0\sim 3)},\dot{x}_{21},\dot{x}_{2,2}\right) \\ = & \begin{bmatrix} \ddot{L}_1+\bar{G}_{1\sim 1}f_{2,1} \\ f_{2,2} \end{bmatrix} \\ = & \begin{bmatrix} \ddot{f}_1(x_1,\dot{x}_1)+2\dot{x}_{21}\sin x_{12}^2 \\ \dot{x}_{22}\sin x_{12}^2 \end{bmatrix} \\ = & \begin{bmatrix} x_{12}^{(3)}\sin x_{12}+3\dot{x}_{12}\ddot{x}_{12}\cos x_{12}-\dot{x}_{12}^3\sin x_{12}+2\dot{x}_{21}\sin x_{12}^2 \\ \dot{x}_{22}\sin x_{12}^2 \end{bmatrix}, \end{aligned} \tag{7.87}$$

and

$$\begin{aligned} \bar{G}_{1\sim 2} &= \begin{bmatrix} \bar{G}_{1\sim 1} & 0 \\ 0 & I_{r_2-r_1} \end{bmatrix}Q_1^{-1}\bar{G}_2 \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 8 \\ 3 & 2 \end{bmatrix}. \end{aligned} \tag{7.88}$$

Therefore, it follows from Theorem 7.3 that the needed transformation is

$$\begin{cases} x_1 = x_{12} \\ x_2 = \begin{bmatrix} 0.5\ddot{x}_{12}-0.5\dot{x}_{12}\sin x_{12}-1.5x_{22} \\ x_{22} \end{bmatrix}, \end{cases} \tag{7.89}$$

under which system (7.83) is equivalently transformed into the following multi-order FAS:

$$\begin{bmatrix} x_{12}^{(4)} \\ \ddot{x}_{22} \end{bmatrix} = \begin{bmatrix} L_{21}\left(x_{12}^{(0\sim3)}, \dot{x}_{21}\right) \\ \dot{x}_{22}\sin x_{12}^2 \end{bmatrix} + \begin{bmatrix} 9 & 8 \\ 3 & 2 \end{bmatrix} u_1, \tag{7.90}$$

where

$$L_{21}\left(x_{12}^{(0\sim3)}, \dot{x}_{21}\right) = x_{12}^{(3)}\sin x_{12} + 3\dot{x}_{12}\ddot{x}_{12}\cos x_{12} - \dot{x}_{12}^3\sin x_{12} + 2\dot{x}_{21}\sin x_{12}^2.$$

Then we can design for FAS (7.90) the following controller:

$$\begin{aligned} u_1 &= -\begin{bmatrix} 9 & 8 \\ 3 & 2 \end{bmatrix}^{-1} \begin{bmatrix} L_{21}\left(x_{12}^{(0\sim3)}, \dot{x}_{21}\right) + 6x_{12}^{(3)} + 18\ddot{x}_{12} + 24\dot{x}_{12} + 16x_{12} \\ \dot{x}_{22}\sin x_{12}^2 + 2\dot{x}_{22} + x_{22} \end{bmatrix} \\ &\triangleq \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix}, \end{aligned} \tag{7.91}$$

where

$$\begin{aligned} u_{11} &= \frac{1}{3}L_{21}\left(x_{12}^{(0\sim3)}, \dot{x}_{21}\right) + 2x_{12}^{(3)} + 6\ddot{x}_{12} + 8\dot{x}_{12} + \frac{16}{3}x_{12} \\ &\quad -\frac{4}{3}\dot{x}_{22}\sin x_{12}^2 - \frac{8}{3}\dot{x}_{22} - \frac{4}{3}x_{22}, \end{aligned} \tag{7.92}$$

$$\begin{aligned} u_{12} &= -0.5L_{21}\left(x_{12}^{(0\sim3)}, \dot{x}_{21}\right) - 3x_{12}^{(3)} - 9\ddot{x}_{12} - 12\dot{x}_{12} - 8x_{12} \\ &\quad +1.5\dot{x}_{22}\sin x_{12}^2 + 3\dot{x}_{22} + 1.5x_{22}. \end{aligned} \tag{7.93}$$

Correspondingly, it follows from (7.81) that the controller for the original system (7.74) is given by

$$\begin{aligned} \begin{bmatrix} u_a \\ u_b \\ u_c \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \\ u_2 \end{bmatrix} \\ &= \begin{bmatrix} u_{11} + u_2 \\ u_{12} - 2u_2 \\ u_2 \end{bmatrix}, \end{aligned} \tag{7.94}$$

where u_{11} and u_{12} are given in (7.92) and (7.93), and u_2 is an arbitrarily chosen parameter which can often be chosen to be zero.

The closed-loop system of FAS (7.90) under controller (7.91)-(7.93) is obviously given by

$$\begin{cases} x_{12}^{(4)} + 6x_{12}^{(3)} + 18\ddot{x}_{12} + 24\dot{x}_{12} + 16x_{12} = 0 \\ \ddot{x}_{22} + 2\dot{x}_{22} + x_{22} = 0, \end{cases}$$

which possesses the set of closed-loop poles $\{-1,-1,-1\pm\mathrm{i},-2\pm\mathrm{i}\}$.

Further, using transformation (7.84), the closed-loop system of the original system (7.83) under controller (7.94) is given by

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_{2a} \\ \ddot{x}_{2b} \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \sin x_1 + 2x_{2a} + 3x_{2b} \\ \zeta\left(x_1^{(0\sim1)}, x_{2a}^{(0\sim1)}, x_{2b}^{(0\sim1)}\right) \\ -2\dot{x}_{2b} - x_{2b} \end{bmatrix},$$

where

$$\zeta\left(x_1^{(0\sim1)}, x_{2a}^{(0\sim1)}, x_{2b}^{(0\sim1)}\right) = \alpha\left(x_1^{(0\sim1)}\right) + \beta\left(x_1^{(0\sim1)}, x_{2a}^{(0\sim1)}\right) + \gamma\left(x_1^{(0\sim1)}, x_{2b}^{(0\sim1)}\right),$$

with

$$\begin{aligned}
\alpha\left(x_1^{(0\sim1)}\right) &= -0.5\dot{x}_1 \sin^3 x_1 - 2\dot{x}_1^2 \sin x_1 \cos x_1 + 0.5\dot{x}_1^3 \sin x_1 \\
&\quad -3\dot{x}_1 \sin^2 x_1 - 3\dot{x}_1^2 \cos x_1 - 9\dot{x}_1 \sin x_1 - 12\dot{x}_1 - 8x_1, \\
\beta\left(x_1^{(0\sim1)}, x_{2a}^{(0\sim1)}\right) &= -x_{2a} \sin^2 x_1 - \dot{x}_{2a} \sin x_1 - 3\dot{x}_1 x_{2a} \cos x_1 \\
&\quad -6x_{2a} \sin x_1 - 6\dot{x}_{2a} - 18x_{2a}, \\
\gamma\left(x_1^{(0\sim1)}, x_{2b}^{(0\sim1)}\right) &= -1.5x_{2b} \sin^2 x_1 - 1.5\dot{x}_{2b} \sin x_1 \\
&\quad -4.5\dot{x}_1 x_{2b} \cos x_1 - 9x_{2b} \sin x_1 - 6\dot{x}_{2b} - 25.5x_{2b}.
\end{aligned}$$

Choose $u_2 = 0$, and the initial values for FAS (7.90) as

$$x_{12}^{(0\sim3)}(0) = \begin{bmatrix} 0.5 & -0.5 & 0 & 0 \end{bmatrix}^{\mathrm{T}},\ x_{22}(0) = 1,\ \dot{x}_{22}(0) = 0,$$

which correspond, for the original system (7.83), to

$$x_1^{(0\sim1)}(0) = \begin{bmatrix} 0.5 & -0.5 \end{bmatrix}^{\mathrm{T}},\ x_2(0) = \begin{bmatrix} -1.38 & 1 \end{bmatrix}^{\mathrm{T}},\ \dot{x}_2(0) = \begin{bmatrix} -0.11 & 0 \end{bmatrix}^{\mathrm{T}}.$$

The simulation results are shown in Figs. 7.2 and 7.3 which show that all states converge smoothly to the origin with a settling time of about 6 seconds, and the absolute values of control inputs are all below 6.

7.5 Notes and References

SFSs refer to dynamic systems consisting of multiple subsystems in a triangular structure. Control of SFSs has been widely studied for many years with a quantity of interesting results obtained, to name a few, Krstic and Bement (2006); Pan and Basar (1998); Bekiaris-Liberis and Krstic (2010); Li et al. (2020, 2021). One of the most commonly-used methods for handling control problems of such a system is the well-known backstepping method.

The backstepping technique, pioneered in the 1990s by Petar V. Kokotovick, Miroslav Krstic, and others, originally aimed at crafting stabilizing controls for SFSs (Kokotovic (1992); Krstic et al. (1995)). Its fundamental concept involves initiating the design

7.5 Notes and F

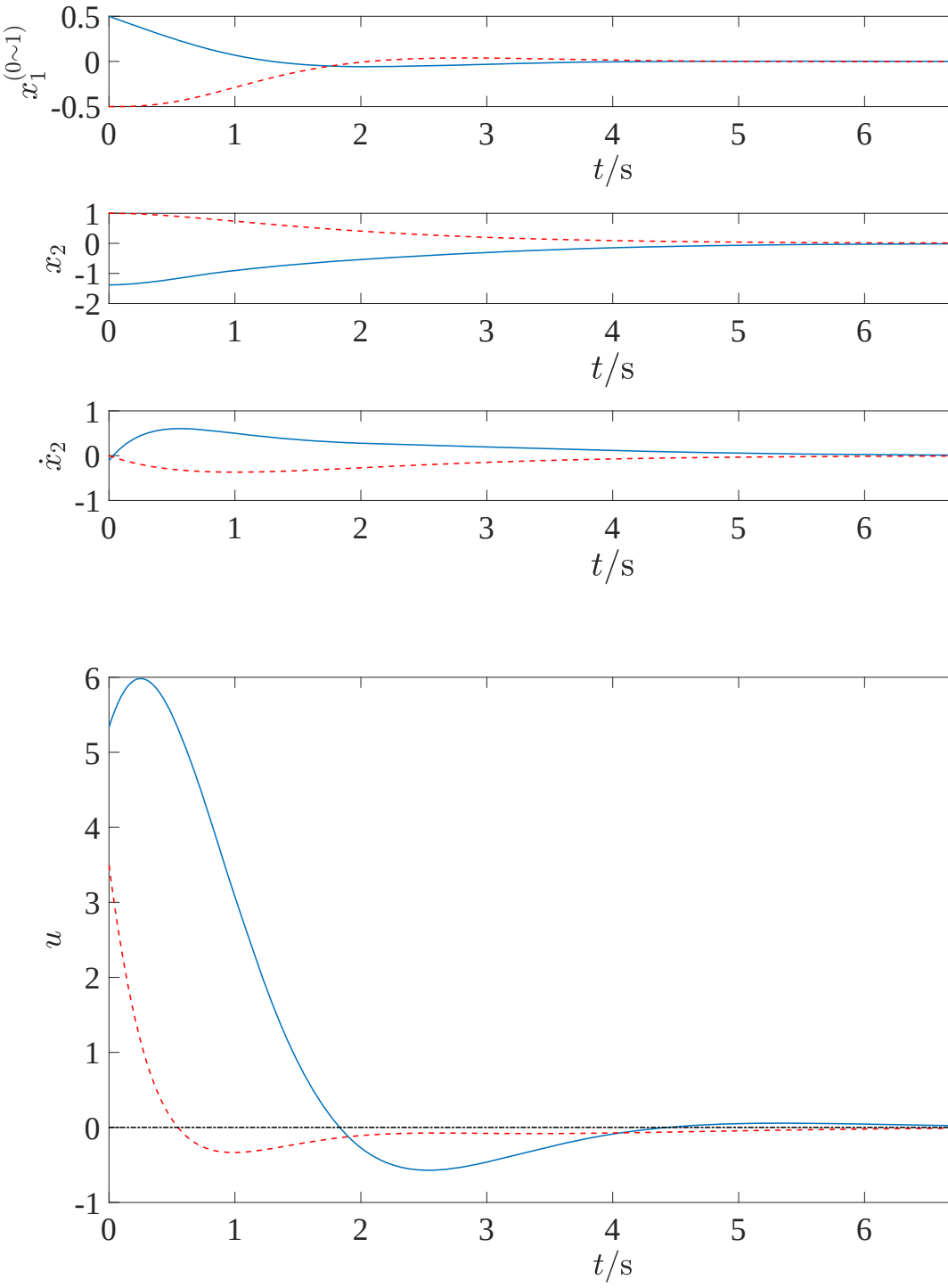


Figure 7.3: Control inputs, Example 2

process at the first subsystem already stabilized and then iteratively deriving new controllers to progressively stabilize subsequent subsystems until achieving control over the final external aspect. While this method assumes uniform dimensions for subsystems (i.e., $x_i,\ i = 1, 2, \ldots, n$), practical systems often feature nonuniform or differing dimensions. Examples include cascades of over-actuated (Meunier et al. (2009)) or underactuated systems (Olfati-Saber (2000)). Considering SFSs with nonuniform dimensions enables the development of more general theories, encompassing uniform cases as special instances. However, designing controllers for SFSs with nonuniform dimensions via the traditional backstepping approach proves challenging, if not infeasible. Therefore, addressing the more general scenario, where subsystems possess nonuniform dimensions, requires innovative approaches.

Fortunately, there are effective approaches available to address control problems in the context of the FAS approach, which was originally introduced in a series of papers Duan (2021a,b, 2023b). The HOFAS approach offers a distinct perspective for handling control problems by transforming a system into a HOFAS model, which allows for the direct derivation of an exponentially stabilizing controller. Duan has demonstrated that nearly all controllable systems, regardless of linearity or nonlinearity and time-invariance or time-varying nature, can be transformed into HOFASs. The high-order FAS theory has opened up new avenues in control theory and has attracted significant research attention from scholars who have obtained numerous results in this field (Zhang et al. (2022a); Yu et al. (2024a); Meng et al. (2022); Dong et al. (2023); Cai et al. (2023b); Gu and Wang (2022); Cai et al. (2023d); Zhang et al. (2023c, 2022b); Wu et al. (2022)).

This chapter investigates the control of SFSs with expanding dimensions, introducing HOFAS models tailored for such systems to facilitate the direct derivation of exponentially stabilizing controllers. To our best knowledge, this work is among the first few studies concerning the control of SFSs with increasing dimensions. The contributions of this study can be delineated into two main areas.

Firstly, we present HOFAS models specifically designed for SFSs with increasing dimensions, building upon the prior work wherein various SFSs were transformed into FAS models (Duan (2021b)). While previous research Duan (2021b) predominantly addressed scenarios where subsystem dimensions remained constant, we extend these findings to encompass more intricate cases, namely SFSs with expanding dimensions. A pivotal technique employed involves the decomposition of variables based on these increasing dimensions. Additionally, whereas existing literature typically focuses on first-order SFSs, we broaden our scope to include second- and high-order SFSs, enhancing the generality of our approach.

Secondly, we tackle the control problem associated with SFSs exhibiting increasing dimensions. Leveraging the FAS models tailored for such systems, we derive stabilized controllers directly. A notable advantage of our controllers over those designed using backstepping methods lies in the resulting closed-loop systems, which exhibit linear time-invariance. This feature underscores the efficacy of HOFAS approaches in addressing control challenges posed by more complex SFSs.

Chapter 8

Nonaffine Strict Feedback Systems

This chapter further studies the control synthesis problem of a class of generalized form of nonaffine strict feedback systems (SFSs) via the FAS approach. Compared with the commonly-considered affine SFS treated in Chapter 6 and Chapter 7, the control of nonaffine SFSs is much more challenging. Similar to the previous two chapters, where the first-order, second-order and high-order cases are all treated, this chapter first proposes a recursive solution for a first-order nonaffine SFS via a FAS approach treatment. Particularly, two special cases are studied, and their recursive results are directly obtained based on the generalized results. Then, the results for the first-order case are extended to the second-order and mixed-order (high-order) cases. It shows that under certain common assumptions, these generalized SFSs can be equivalently converted into high-order FAS models, which possess a huge advantage on the control design since the fully-actuated feature allows us to remove all the original dynamics to achieve a desired linear closed-loop system with an arbitrarily assigned eigenstructure.

8.1 First-order Nonaffine SFSs

A critical issue in modelling a system as a first-order representation is the loss of its full-actuation property. In the physical world, many systems inherently possess full-actuation characteristics. This property allows for the complete cancellation of the open-loop system dynamics, whether linear or nonlinear, while establishing entirely new and desired closed-loop dynamics. Especially for nonlinear systems, it is possible to design a desired steady-state linear closed-loop system. These FASs are considered to be a small part of systems, while recent results show that they can be universal one by transforming dynamical systems into high-order differential equations. However, in the existing state-space framework they are transformed into first-order systems, and the full-actuation property is lost in most cases.

Despite this, many control strategies mechanically convert these high-order systems into first-order representations merely to apply conventional first-order control

theories. This approach often leads to unnecessary complexity in solving otherwise straightforward problems. Most practical systems are governed by physical laws, such as, Newton's Law, Lagrangian Equation, Theorem of Linear and Angular Momentum, Kirchhoff's Laws of Current and Voltage, etc. When modelled by these physical laws, a set of subsystems of second- or high-order are firstly obtained. The full-actuation characteristic of the system becomes apparent either in its original modelling form or when transformed into a high-order differential equation representation. Based on this insight, a FAS approach has been proposed in recent years (Duan (2021a,b,c,d,e,f,g, 2022e,d)). By employing a variable elimination and order elevation technique, a high-order FAS model can be obtained. This approach demonstrates that FASs are not limited to a subset of physical systems but rather represent a general modelling framework, similar to state-space representations, that serves as a system description specifically tailored for control design (Cai et al. (2023a); Huang et al. (2024a); Ma et al. (2024b)).

The backstepping control is a systematic approach to study nonlinear systems in lower-triangular forms, namely, the strict-feedback systems (SFSs), and receives great attention due to its ability to avoid the cancellation of some beneficial nonlinearities. However, the backstepping control has notable disadvantages, including excessive recursive steps that make implementation challenging, as well as the well-known "explosion of complexity" problem. Recent studies have found that a recursive approach can transform SFSs into FAS models, allowing the direct derivation of control laws using the FAS approach (Duan (2021b,c,d); Liu et al. (2024d,e); Hou et al. (2023); Liu et al. (2023a)).

Even though the affine SFS has been thoroughly studied using the FAS approach, a particular class of more specialized nonaffine SFS has received little attention. From a mathematical modelling perspective, a nonaffine SFS is a more general lower-triangular nonlinear system compared to an SFS. However, in contrast to affine strict-feedback nonlinear systems, which have been extensively studied, research on the control of nonaffine nonlinear systems remains relatively limited. This is primarily because control variables or virtual control variables in nonaffine nonlinear systems are often implicitly embedded within nonaffine functions, making most control methods designed for strict-feedback nonlinear systems inapplicable to nonaffine SFS nonlinear system control. Recently, a class of pseudo pure-feedback nonlinear systems has been studied in Zhang et al. (2025a) via the FAS approach while we will show that the result in this section is a slightly generalized result.

8.1.1 Problem Description

Let us consider the following time-varying first-order nonaffine strict-feedback nonlinear system with a parameter vector $\zeta \in \mathbb{R}^m$:

$$\begin{cases} \dot{x}_1 = f_1(x_1, \zeta, t) + g_1(x_{1\sim 2}, \zeta, t) \\ \dot{x}_2 = f_2(x_{1\sim 2}, \zeta, t) + g_2(x_{1\sim 3}, \zeta, t) \\ \quad\quad \vdots \\ \dot{x}_{n-1} = f_{n-1}(x_{1\sim n-1}, \zeta, t) + g_{n-1}(x_{1\sim n}, \zeta, t) \\ \dot{x}_n = f_n(x_{1\sim n}, \zeta, t) + g_n(x_{1\sim n}, u, \zeta, t), \end{cases} \tag{8.1}$$

where $x_i \in \mathbb{R}^r, i = 1, 2, \ldots, n$, are the state vectors, $u \in \mathbb{R}^r$ is the control input, $f_i(x_{1\sim i}, \zeta, t)$, $i = 1, 2, \ldots, n$, $g_i(x_{1\sim i+1}, \zeta, t), i = 1, 2, \ldots, n-1$, and $g_n(x_{1\sim n}, \zeta, u, t)$ are sufficiently smooth r-dimensional vector functions.

The above system (8.1) is called a nonaffine SFS, if the following assumption is met:

Assumption 8.1 *For arbitrary $x_i \in \mathbb{R}^r$, $i = 1, 2, \ldots, n$, $\zeta \in \mathbb{R}^m$, and $t \geq 0$, there hold*

$$\begin{cases} \frac{\partial}{\partial x_{i+1}} g_i(x_{1\sim i+1}, \zeta, t) \neq 0, \ i = 1, 2, \ldots, n-1 \\ \frac{\partial}{\partial u} g_n(x_{1\sim n}, \zeta, u, t) \neq 0. \end{cases} \tag{8.2}$$

The meaning of the above assumption lies in the following facts:

- for each $i = 1, 2, \ldots, n-1$, the inverse function of $g_i(x_{1\sim i+1}, \zeta, t)$ with respect to x_{i+1} exists for all $x_{1\sim i} \in \mathbb{R}^{ir}$, $\zeta \in \mathbb{R}^m$, and $t \geq 0$;
- the inverse function of $g_n(x_{1\sim n}, \zeta, u, t)$ with respect to u exists for all $x_{1\sim n} \in \mathbb{R}^{nr}$, $\zeta \in \mathbb{R}^m$, and $t \geq 0$;
- for arbitrary $y_i \in \mathbb{R}^r$, $i = 1, 2, \ldots, n-1$, the following equations

$$g_i(x_{1\sim i+1}, \zeta, t) = y_i, \ i = 1, 2, \ldots, n-1, \ t \geq 0, \tag{8.3}$$

  have solutions with respect to $x_{i+1} \in \mathbb{R}^r$, $i = 0, 1, \ldots, n-1$, which can also be represented by

$$x_{i+1} = g_i^{-1}(x_{1\sim i}, y_i, \zeta, t), \ i = 1, 2, \ldots, n-1; \tag{8.4}$$

- for an arbitrary $y_n \in \mathbb{R}^r$, the following equation

$$g_n(x_{1\sim n}, u, \zeta, t) = y_n, \ \ \forall t \geq 0, \tag{8.5}$$

  has a solution with respect to $u \in \mathbb{R}^r$, which can also be represented by

$$u = g_n^{-1}(x_{1\sim n}, y_n, \zeta, t). \tag{8.6}$$

In this section, we assume that the inverse function can be explicitly obtained. In certain high-dimensional cases, the above Assumption 8.1 may be difficult to verify. It is recommended that symbolic computation techniques may be considered to help in such cases.

The purpose of this section is to find a homeomorphism under which the above nonaffine SFS (8.1) is equivalently converted into a nonaffine FAS.

8.1.2 Recursive Solution

For this generalized SFS (8.1), some existing control methods, such as the backstepping methods, may become extremely difficult or even impossible to apply when r and n are large due to the "differential explosion" problem. However, if we can transform it into a FAS model, its control problems can then be solved easily and conveniently (Duan (2020a)).

Theorem 8.1 *Let Assumption 8.1 be satisfied, then, under the following transformation:*

$$\begin{cases} x_1 = z \\ x_2 = g_1^{-1}(z, y_1, \zeta, t),\ y_1 = \dot{z} - f_1(z, \zeta, t) \\ x_{k+1} = l_k^{-1}(z^{(0\sim k-1)}, y_k, \zeta^{(0\sim k-2)}, t),\ y_k = z^{(k)} - h_k(z^{(0\sim k-1)}, \zeta^{(0\sim k-1)}, t), \\ \qquad\qquad k = 2, 3, \ldots, n-1, \end{cases} \tag{8.7}$$

the generalized strict-feedback nonlinear system (8.1) can be transformed into the following FAS:

$$z^{(n)} = h_n(z^{(0\sim n-1)}, \zeta^{(0\sim n-1)}, t) + l_n(z^{(0\sim n-1)}, u, \zeta^{(0\sim n-2)}, t), \tag{8.8}$$

where the vector functions $h_n(z^{(0\sim n-1)}, \zeta^{(0\sim n-1)}, t)$ and $l_n(z^{(0\sim n-1)}, u, \zeta^{(0\sim n-2)}, t)$ are recursively given by

$$\begin{cases} h_2(z^{(0\sim 1)}, \zeta^{(0\sim 1)}, t) = \dot{h}_1(z, \zeta, t) + \frac{\partial l_1(z, x_2, \zeta, t)}{\partial z}\dot{z} + \frac{\partial l_1(z, x_2, \zeta, t)}{\partial \zeta}\dot{\zeta} \\ \qquad\qquad + \frac{\partial l_1(z, x_2, \zeta, t)}{\partial t} + \frac{\partial l_1(z, x_2, \zeta, t)}{\partial x_2} f_2(x_{1\sim 2}, \zeta, t) \\ h_k(z^{(0\sim k-1)}, \zeta^{(0\sim k-1)}, t) = \dot{h}_{k-1}(z^{(0\sim k-2)}, \zeta^{(0\sim k-2)}, t) \\ \qquad\qquad + \frac{\partial l_{k-1}(z^{(0\sim k-2)}, x_k, \zeta^{(0\sim k-3)}, t)}{\partial z^{(0\sim k-2)}}\dot{z}^{(0\sim k-2)} \\ \qquad\qquad + \frac{\partial l_{k-1}(z^{(0\sim k-2)}, x_k, \zeta^{(0\sim k-3)}, t)}{\partial \zeta^{(0\sim k-3)}}\dot{\zeta}^{(0\sim k-3)} + \frac{\partial l_{k-1}(z^{(0\sim k-2)}, x_k, \zeta^{(0\sim k-3)}, t)}{\partial t} \\ \qquad\qquad + \frac{\partial l_{k-1}(z^{(0\sim k-2)}, x_k, \zeta^{(0\sim k-3)}, t)}{\partial x_k} f_k(x_{1\sim k}, \zeta, t), \\ \qquad\qquad k = 3, 4, \ldots, n, \end{cases} \tag{8.9}$$

and

$$\begin{cases} l_2(z^{(0\sim 1)}, x_3, \zeta, t) = \frac{\partial l_1(z, x_2, \zeta, t)}{\partial x_2} g_2(x_{1\sim 3}, \zeta, t) \\ l_k(z^{(0\sim k-1)}, x_{k+1}, \zeta^{(0\sim k-2)}, t) = \frac{\partial l_{k-1}(z^{(0\sim k-2)}, x_k, \zeta^{(0\sim k-3)}, t)}{\partial x_k} g_k(x_{1\sim k+1}, \zeta, t), \\ \qquad\qquad k = 3, 4, \ldots, n-1 \\ l_n(z^{(0\sim n-1)}, u, \zeta^{(0\sim n-2)}, t) = \frac{\partial l_{n-1}(z^{(0\sim n-2)}, x_n, \zeta^{(0\sim n-3)}, t)}{\partial x_n} g_n(x_{1\sim n}, u, t), \end{cases} \tag{8.10}$$

and the initial values are

$$h_1(z, \zeta, t) = f_1(x_1, \zeta, t),\ l_1(z, x_2, \zeta, t) = g_1(x_{1\sim 2}, \zeta, t), \tag{8.11}$$

and in equations (8.9)-(8.10), the x_i, $i = 1, 2, \ldots, n$, are all given or determined by transformation (8.7).

For a proof of the above theorem, please refer to Appendix B.

Remark 8.1 Assumption 8.1 is a global one since it requires (8.2) to hold for $x_i \in \mathbb{R}^r$, $i = 1, 2, \ldots, n$. Relaxation of this assumption to a local one can be considered. If this assumption is valid only on a certain set $\Omega \subset \mathbb{R}^{nr}$, then the derived high-order system (8.8) is a sub-FAS model on a certain set Ω' determined from Ω by transformation (8.7).

Remark 8.2 Once the FAS model (8.8) is obtained, the controller of the system can then be simply designed as (Duan (2020a), Duan (2021a))

$$\begin{cases} u = -l_n^{-1}\left(z^{(0\sim n-1)}, u^*, \zeta^{(0\sim n-2)}, t\right) \\ u^* = h_n(z^{(0\sim n-1)}, \zeta, t) + A_{0\sim n-1} z^{(0\sim n-1)} - \upsilon, \end{cases} \tag{8.12}$$

which gives the following constant linear closed-loop system:

$$z^{(n)} + A_{0\sim n-1} z^{(0\sim n-1)} = \upsilon,$$

where υ is some external signal, while $A_{0\sim n-1}$ is an arbitrary matrix which makes $\Phi(A_{0\sim n-1})$ stable. A complete parametric approach for solving the matrix $A_{0\sim n-1}$ is given in Duan (2020a) (see also the Proposition 2 in Duan (2021a)).

Through the proof of the above theorem, we can observe that transformation (8.7) is in fact the inverse of the following transformation $\left(x_{1\sim n} \longmapsto z^{(0\sim n-1)}\right)$:

$$z^{(0\sim n-1)} = \begin{bmatrix} x_1 \\ h_1(x_1, \zeta, t) + l_1(x_{1\sim 2}, \zeta, t) \\ h_2(x_{1\sim 2}, \zeta^{(0\sim 1)}, t) + l_2(x_{1\sim 3}, \zeta, t) \\ \vdots \\ h_{n-1}(x_{1\sim n-1}, \zeta^{(0\sim n-2)}, t) + l_{n-1}(x_{1\sim n}, \zeta^{(0\sim n-3)}, t) \end{bmatrix}. \tag{8.13}$$

For each $x_{1\sim n}$, (8.13) clearly gives a unique $z^{(0\sim n-1)}$. Conversely, let $z^{(0\sim n-1)}$ be given, then we have $x_{1\sim n}$. Thus, transformation (8.7), or (8.13), is globally one-to-one. Furthermore, it is clearly seen from (8.13) that $x_{1\sim n} = 0$ implies $z^{(0\sim n-1)} = 0$. Thus transformation (8.7), or (8.13), also keeps the origin unchanged.

8.1.3 Special Cases

Case of $g_i(\cdot)$ independent on $x_{1\sim i}$, $i = 1, 2, \ldots, n$ and ζ

In this subsection, let us consider the special case of

$$\begin{cases} g_i(x_{1\sim i+1}, \zeta, t) = g_i(x_{i+1}, t) \in \mathbb{R}^r, \ i = 1, 2, \ldots, n-1 \\ g_n(x_{1\sim n}, u, \zeta, t) = g_n(u, t). \end{cases} \tag{8.14}$$

In this case, the nonaffine SFS (8.1) becomes

$$\begin{cases} \dot{x}_1 = f_1(x_1, \zeta, t) + g_1(x_2, t) \\ \dot{x}_2 = f_2(x_{1\sim 2}, \zeta, t) + g_2(x_3, t) \\ \qquad \vdots \\ \dot{x}_{n-1} = f_{n-1}(x_{1\sim n-1}, \zeta, t) + g_{n-1}(x_n, t) \\ \dot{x}_n = f_n(x_{1\sim n}, \zeta, t) + g_n(u, t). \end{cases} \tag{8.15}$$

Assumption 8.1 becomes

Assumption 8.2 *For arbitrary $x_i \in \mathbb{R}^r$, $i = 1, 2, \ldots, n$, $\zeta \in \mathbb{R}^m$, and $t \geq 0$, there hold*

$$\begin{cases} \frac{\partial}{\partial x_{i+1}} g_i\left(x_{i+1}, t\right) \neq 0, \ i = 1, 2, \ldots, n-1 \\ \frac{\partial}{\partial u} g_n(u, t) \neq 0. \end{cases} \tag{8.16}$$

Due to (8.14), we have

$$\dot{g}_i(x_{1\sim i+1}, \zeta, t) = \frac{\partial g_i\left(x_{i+1}, t\right)}{\partial x_{i+1}} \dot{x}_{i+1} + \frac{\partial g_i\left(x_{i+1}, t\right)}{\partial t}, \ i = 1, 2, \ldots, n-1.$$

Setting, for $i = 1, 2, \ldots, n$,

$$\frac{\partial}{\partial x_{1\sim i}} g_i(x_{1\sim i+1}, \zeta, t) = 0,$$
$$\frac{\partial}{\partial \zeta} g_i(x_{1\sim i+1}, \zeta, t) = 0,$$

and

$$\frac{\partial g_i(x_{1\sim i+1}, \zeta, t)}{\partial x_{i+1}} = \frac{\partial g_i\left(x_{i+1}, t\right)}{\partial x_{i+1}},$$
$$\frac{\partial g_i(x_{1\sim i+1}, \zeta, t)}{\partial t} = \frac{\partial g_i\left(x_{i+1}, t\right)}{\partial t},$$

where $x_{n+1} = u$, immediately gives the following corollary.

Corollary 8.1 *Let Assumption 8.2 be satisfied, then, under the following transformation:*

$$\begin{cases} x_1 = z \\ x_2 = g_1^{-1}\left(y_1, t\right), \ y_1 = \dot{x}_1 - f_1(x_1, \zeta, t) \\ x_{k+1} = l_k^{-1}\left(z^{(0\sim k-1)}, y_k, \zeta^{(0\sim k-2)}, t\right), \ y_k = z^{(k)} - h_k(z^{(0\sim k-1)}, \zeta^{(0\sim k-1)}, t), \\ \qquad\qquad k = 2, 3, \ldots, n-1, \end{cases} \tag{8.17}$$

the generalized strict-feedback nonlinear system (8.15) can be equivalently transformed into the following FAS model:

$$z^{(n)} = h_n(z^{(0\sim n-1)}, \zeta^{(0\sim n-1)}, t) + l_n(z^{(0\sim n-1)}, u, \zeta^{(0\sim n-2)}, t), \tag{8.18}$$

with the vector functions $h_n(z^{(0\sim n-1)}, \zeta^{(0\sim n-1)}, t)$ and $l_n(z^{(0\sim n-1)}, u, \zeta^{(0\sim n-2)}, t)$ being recursively given by

$$\begin{cases} h_2(z^{(0\sim 1)}, \zeta^{(0\sim 1)}, t) = \dot{h}_1(z, \zeta, t) + \frac{\partial l_1(z, x_2, t)}{\partial z} \dot{z} + \frac{\partial l_1(z, x_2, t)}{\partial t} \\ \qquad\qquad + \frac{\partial l_1(z, x_2, t)}{\partial x_2} f_2(x_{1\sim 2}, \zeta, t) \\ h_k(z^{(0\sim k-1)}, \zeta^{(0\sim k-1)}, t) = \dot{h}_{k-1}(z^{(0\sim k-2)}, \zeta^{(0\sim k-2)}, t) \\ \qquad\qquad + \frac{\partial l_{k-1}(z^{(0\sim k-2)}, x_k, \zeta^{(0\sim k-3)}, t)}{\partial z^{(0\sim k-2)}} \dot{z}^{(0\sim k-2)} + \frac{\partial l_{k-1}(z^{(0\sim k-2)}, x_k, \zeta^{(0\sim k-3)}, t)}{\partial t} \\ \qquad\qquad + \frac{\partial l_{k-1}(z^{(0\sim k-2)}, x_k, \zeta^{(0\sim k-3)}, t)}{\partial x_k} f_k(x_{1\sim k}, \zeta, t), \\ \qquad\qquad k = 3, 4, \ldots, n, \end{cases} \tag{8.19}$$

and

$$\begin{cases} l_k\left(z^{(0\sim k-1)}, x_{k+1}, \zeta^{(0\sim k-2)}, t\right) = \frac{\partial g_1(x_2,t)}{\partial x_2}\frac{\partial g_2(x_3,t)}{\partial x_3}\cdots\frac{\partial g_{k-1}(x_k,t)}{\partial x_k} g_k\left(x_{k+1}, t\right), \\ \qquad\qquad k = 2, 3, \ldots, n-1 \\ l_n(z^{(0\sim n-1)}, u, \zeta^{(0\sim n-2)}, t) = \frac{\partial l_{n-1}(z^{(0\sim n-2)}, x_n, \zeta^{(0\sim n-3)}, t)}{\partial x_n} g_n(u, t), \end{cases} \tag{8.20}$$

where

$$h_1(z, \zeta, t) = f_1(x_1, \zeta, t),\ l_1(z, x_2, t) = g_1(x_2, t),$$

and in (8.19)-(8.20) the state vectors $x_k,\ k = 2, 3, \ldots, n$, *are all given or determined by transformation (8.17).*

Remark 8.3 Another way of treating the special nonaffine SFS (8.15) is to convert it into an affine SFS. In fact, applying the following transformation

$$\begin{cases} \hat{x}_1 = x_1 \\ \hat{x}_{i+1} = g_i(x_{i+1}, t),\ k = 1, 2, \ldots, n-1 \\ \hat{u} = g_n(u, t), \end{cases} \tag{8.21}$$

to SFS (8.15), immediately converts it into the following SFS:

$$\begin{cases} \dot{\hat{x}}_1 = f_1(\hat{x}_1, \zeta, t) + \hat{x}_2 \\ \dot{\hat{x}}_2 = \frac{\partial g_1(x_2,t)}{\partial x_2} f_2(x_{1\sim 2}, \zeta, t) + \frac{\partial g_1(x_2,t)}{\partial t} + G_2\left(x_2, t\right)\hat{x}_3 \\ \qquad\vdots \\ \dot{\hat{x}}_{n-1} = \frac{\partial g_{n-2}(x_{n-1},t)}{\partial x_{n-1}} f_{n-1}(x_{1\sim n-1}, \zeta, t) + \frac{\partial g_{n-2}(x_{n-1},t)}{\partial t} + G_{n-1}\left(x_{n-1}, t\right)\hat{x}_n \\ \dot{\hat{x}}_n = \frac{\partial g_{n-1}(x_n,t)}{\partial x_n} f_n(x_{1\sim n}, \zeta, t) + \frac{\partial g_{n-1}(x_n,t)}{\partial t} + G_n\left(x_n, t\right)\hat{u}, \end{cases} \tag{8.22}$$

where

$$G_i\left(x_i, t\right) = \frac{\partial g_{i-1}(x_i, t)}{\partial x_i},\ i = 2, 3, ..., n,$$

which has been treated in Duan (2021b).

Case of $f_i(x_{1\sim i}, \zeta, t) = 0,\ i = 1, 2, ..., n$

In this case, nonaffine SFS (8.1) becomes the following new type of nonaffine SFS:

$$\begin{cases} \dot{x}_1 = g_1(x_{1\sim 2}, \zeta, t) \\ \dot{x}_2 = g_2(x_{1\sim 3}, \zeta, t) \\ \qquad\vdots \\ \dot{x}_{n-1} = g_{n-1}(x_{1\sim n}, \zeta, t) \\ \dot{x}_n = g_n(x_{1\sim n}, u, \zeta, t). \end{cases} \tag{8.23}$$

The above system is similar to the pseudo pure-feedback system existing in the literature (see, e.g., Zhang et al. (2025a)).

Setting

$$f_i(x_{1\sim i}, \zeta, t) = 0,\ i = 1, 2, \ldots, n$$

in Theorem 8.1, gives the following corollary.

Corollary 8.2 *Let Assumption 8.1 be satisfied, then, under the following transformation*

$$\begin{cases} x_1 = z \\ x_2 = g_1^{-1}(z, y_1, \zeta, t),\ y_1 = \dot{z} \\ x_{k+1} = l_k^{-1}(z^{(0\sim k-1)}, y_k, \zeta^{(0\sim k-2)}, t),\ y_k = z^{(k)} - h_k(z^{(0\sim k-1)}, \zeta^{(0\sim k-1)}, t), \\ \qquad k = 2, 3, \ldots, n-1, \end{cases} \tag{8.24}$$

the generalized SFS (8.23) can be transformed into the following FAS:

$$z^{(n)} = h_n(z^{(0\sim n-1)}, \zeta^{(0\sim n-1)}, t) + l_n(z^{(0\sim n-1)}, u, \zeta^{(0\sim n-2)}, t), \tag{8.25}$$

where the vector functions $h_n(z^{(0\sim n-1)}, \zeta^{(0\sim n-1)}, t)$ and $l_n(z^{(0\sim n-1)}, u, \zeta^{(0\sim n-2)}, t)$ are recursively given by

$$\begin{cases} h_2(z^{(0\sim 1)}, \zeta^{(0\sim 1)}, t) = \dot{h}_1(z, \zeta, t) + \frac{\partial l_1(z, x_2, \zeta, t)}{\partial z}\dot{z} + \frac{\partial l_1(z, x_2, \zeta, t)}{\partial x_2} f_2(x_{1\sim 2}, \zeta, t) \\ \qquad\qquad + \frac{\partial l_1(z, x_2, \zeta, t)}{\partial \zeta}\dot{\zeta} + \frac{\partial l_1(z, x_2, \zeta, t)}{\partial t} \\ h_k(z^{(0\sim k-1)}, \zeta^{(0\sim k-1)}, t) = \dot{h}_{k-1}(z^{(0\sim k-2)}, \zeta^{(0\sim k-2)}, t) \\ \qquad\qquad + \frac{\partial l_{k-1}(z^{(0\sim k-2)}, x_k, \zeta^{(0\sim k-3)}, t)}{\partial z^{(0\sim k-2)}}\dot{z}^{(0\sim k-2)} \\ \qquad\qquad + \frac{\partial l_{k-1}(z^{(0\sim k-2)}, x_k, \zeta^{(0\sim k-3)}, t)}{\partial \zeta^{(0\sim k-3)}}\dot{\zeta}^{(0\sim k-3)} + \frac{\partial l_{k-1}(z^{(0\sim k-2)}, x_k, \zeta^{(0\sim k-3)}, t)}{\partial t}, \\ \qquad\qquad\qquad k = 3, 4, \ldots, n, \end{cases} \tag{8.26}$$

and

$$\begin{cases} l_2(z^{(0\sim 1)}, x_3, \zeta, t) = \frac{\partial l_1(z, x_2, \zeta, t)}{\partial x_2} g_2(x_{1\sim 3}, \zeta, t), \\ l_k(z^{(0\sim k-1)}, x_{k+1}, \zeta, t) = \frac{\partial l_{k-1}(z^{(0\sim k-2)}, x_k, \zeta^{(0\sim k-3)}, t)}{\partial x_k} g_k(x_{1\sim k+1}, \zeta, t), \\ \qquad\qquad k = 3, 4, \ldots, n-1, \\ l_n(z^{(0\sim n-1)}, u, \zeta, t) = \frac{\partial l_{n-1}(z^{(0\sim n-2)}, x_n, \zeta^{(0\sim n-3)}, t)}{\partial x_n} g_n(x_{1\sim n}, u, \zeta, t), \end{cases} \tag{8.27}$$

and the initial values are

$$h_1(z, \zeta, t) = 0,\ l_1(z, x_2, \zeta, t) = g_1(x_{1\sim 2}, \zeta, t), \tag{8.28}$$

where in (8.26)-(8.27) the $x_i,\ i = 2, 3, \ldots, n$, are all given or determined by transformation (8.24).

Obviously, this is a slightly generalized result of that in Zhang et al. (2025a).

8.2 Second-order Nonaffine SFSs

In many engineering applications, nonlinear behaviors are widespread, making the control of nonlinear systems a critical and challenging research topic (Krstic et al. (1995)). Unlike linear systems, nonlinear systems offer greater modelling exibility, allowing them to capture complex physical phenomena. However, this significantly increases the complexity of analyzing and designing controllers for nonlinear systems compared

to linear ones, and to date, no unified framework exists for the general nonlinear systems. To address these challenges, researchers have focused on nonlinear systems with specific structural properties, where theoretical development becomes more tractable. Among them, the class of SFSs has attracted significant attention due to its hierarchical structure and its ability to model a wide range of physical systems, including robotic systems (Khalil (2002a)), missile and satellite control systems (Kim et al. (2004)) , ship fin roll stabilization systems (Yang et al. (2004)) and so on.

A typical recursive method for designing controllers in SFSs is the backstepping originally proposed in Tsinias (1989); Byrnes and Isidori (1989); Sontag and Sussmann (1988). This method has become one of the most widely used tools in nonlinear control theory and has achieved numerous interesting results (Wen et al. (2018); Zuo and Wang (2023); Wang et al. (2025a)) over the past decades. However, as the study of nonlinear systems deepens, the backstepping suffers from a well-known drawback called the "differentiation explosion". Specifically, as the system order increases, the recursive design requires repeated differentiation of the virtual control laws, leading to increasing complexity that is difficult to implement and may cause numerical instability in practical systems. To mitigate this issue, researchers have developed several improved or enhanced approaches. For instance, dynamic surface control introduces low-pass filters to approximate virtual control derivatives, thereby avoiding the need for repeated differentiation (Swaroop et al. (2000)). Similarly, command-filter-based backstepping employs tracking filters to simplify the control law implementation (Dong et al. (2012)). While these techniques alleviate the burden of repeated differentiation to some extent, they still rely on first-order state-space representations of the system. This poses a significant limitation, as many physical systems are naturally modeled by second-order or high-order dynamics form Duan (2021a). Applying backstepping or its variants to such systems typically requires transforming them into an equivalent first-order state-space form. Such transformations not only increase modelling complexity but also make controller design more difficult, particularly due to the need for constructing suitable Lyapunov functions in the transformed coordinates, which is often analytically intractable for general nonlinear systems of second- or high-order (Duan (2021e)).

Recently, Duan proposed a novel FAS method in Duan (2021a,e,g), which allows for direct controller design for the high-order system by leveraging the full-actuation characteristics, thereby avoiding the need for complex state transformations. Following this advantage, the FAS method opens a new direction for exploration in nonlinear control theory, attracting widespread attention from scholars (Duan (2024e)). In particular, the FAS method provides a new perspective for handling the stabilization problem of generalized SFSs in Duan (2021b), which is completely different from the state-space based framework. In detail, in the work of Duan (2021b), the generalized forms of conventional first-, second- and high-order SFSs are first introduced, and then recursive solutions are proposed for their transformation into equivalent HOFAS models. Building on this foundation, a growing body of work has successfully applied the HOFAS framework to address various SFSs, demonstrating its effectiveness in simplifying controller design and avoiding differentiation explosion. For instance, the problems of robust control and adaptive control for generalized SFSs with unknown parameters were investigated in Duan (2021c) and Duan (2021d), respectively. In Meng et al.

(2022), the HOFAS framework was integrated with adaptive event-triggered control to handle uncertain HOFASs. Furthermore, an active fault-tolerant control strategy was developed in Cai et al. (2023a). Additionally, to further extend the applicability of the FAS method, the recent work Zhang et al. (2025a) provided a set of diffeomorphism transformations for the first-, second- and mixed-order pseudo pure-feedback systems, respectively, thereby deriving a corresponding HOFAS model.

It is worth noting that a common feature of the aforementioned results lies in their reliance on the affine structure of the control input, where the input appears explicitly and linearly in the system dynamics. However, in many practical systems, the control input may enter the system in a nonaffine and even highly nonlinear manner. For such nonlinear nonaffine SFSs, the stabilization problem remains a challenging and unsolved issue, as even the classical backstepping tools are no longer applicable. Motivated by the above discussions, this section aims to investigate the control synthesis problem for a class of generalized second-order nonaffine SFSs within the framework of FAS method. By constructing a novel set of diffeomorphic transformations and employing the idea of ascending order and descending dimension, the considered generalized nonaffine second-order SFS is equivalently transformed into a HOFAS model, thereby enabling a tractable controller design.

8.2.1 Problem Description

Consider the following time-varying second-order nonaffine SFS with a parameter vector $\zeta \in \mathbb{R}^m$:

$$\begin{cases} \ddot{x}_1 = f_1(x_1^{(0\sim1)}, \zeta, t) + g_1(x_1^{(0\sim1)}, x_2, \zeta, t) \\ \ddot{x}_2 = f_2(x_{1\sim2}^{(0\sim1)}, \zeta, t) + g_2(x_{1\sim2}^{(0\sim1)}, x_3, \zeta, t) \\ \quad\vdots \\ \ddot{x}_{n-1} = f_{n-1}(x_{1\sim n-1}^{(0\sim1)}, \zeta, t) + g_{n-1}(x_{1\sim n-1}^{(0\sim1)}, x_n, \zeta, t) \\ \ddot{x}_n = f_n(x_{1\sim n}^{(0\sim1)}, \zeta, t) + g_n(x_{1\sim n}^{(0\sim1)}, u, \zeta, t), \end{cases} \tag{8.29}$$

where $x_i \in \mathbb{R}^r$, $i = 1, 2, \ldots, n$, are the state vectors, $u \in \mathbb{R}^r$ is the control input, $f_i(x_{1\sim i}^{(0\sim1)}, \zeta, t)$, $i = 1, 2, \ldots, n$, $g_i(x_{1\sim i}^{(0\sim1)}, x_{i+1}, \zeta, t)$, $i = 1, 2, \ldots, n-1$, and $g_n(x_{1\sim n}^{(0\sim1)}, u, \zeta, t)$ are sufficiently smooth r-dimensional vector functions.

The above system (8.29) is called a nonaffine SFS, if the following assumption is met:

Assumption 8.3 *For arbitrary $x_i \in \mathbb{R}^r$, $i = 1, 2, \ldots, n$, $\zeta \in \mathbb{R}^m$, and $\forall t \geq 0$, there hold*

$$\begin{cases} \frac{\partial}{\partial x_{i+1}} g_i(x_{1\sim i}^{(0\sim1)}, x_{i+1}, \zeta, t) \neq 0, \ i = 1, 2, \ldots, n-1 \\ \frac{\partial}{\partial u} g_n(x_{1\sim n}^{(0\sim1)}, u, \zeta, t) \neq 0. \end{cases} \tag{8.30}$$

The meaning of the above assumption lies in the following facts:

- for each $i = 1, 2, \ldots, n-1$, the inverse function of $g_i(x_{1\sim i}^{(0\sim1)}, x_{i+1}, \zeta, t)$ with respect to x_{i+1} exists for all $x_{1\sim i}^{(0\sim1)} \in \mathbb{R}^{2ir}$, $\zeta \in \mathbb{R}^m$, and $t \geq 0$;

- the inverse function of $g_n(x_{1\sim n}^{(0\sim 1)}, u, \zeta, t)$ with respect to u exists for all $x_{1\sim n}^{(0\sim 1)} \in \mathbb{R}^{2nr}$, $\zeta \in \mathbb{R}^m$, and $t \geq 0$;
- for arbitrary $y_i \in \mathbb{R}^r$, $i = 1, 2, \ldots, n-1$, the following equations

$$g_i(x_{1\sim i}^{(0\sim 1)}, x_{i+1}, \zeta, t) = y_i,\ t \geq 0, \tag{8.31}$$

  have solutions with respect to $x_{i+1} \in \mathbb{R}^r$, which can also be represented by

$$x_{i+1} = g_i^{-1}(x_{1\sim i}^{(0\sim 1)}, y_i, \zeta, t); \tag{8.32}$$

- for an arbitrary $y_n \in \mathbb{R}^r$, the following equation

$$g_n(x_{1\sim n}^{(0\sim 1)}, u, \zeta, t) = y_n,\ \ \forall t \geq 0, \tag{8.33}$$

  has a solution with respect to $u \in \mathbb{R}^r$, which can also be represented by

$$u = g_n^{-1}(x_{1\sim n}^{(0\sim 1)}, y_n, \zeta, t). \tag{8.34}$$

In certain high-dimensional cases, the above Assumption 8.3 may be difficult to verify. It is recommended that symbolic computation techniques may be considered to help in such cases.

The purpose of this section is to find a homeomorphism under which the above nonaffine SFS (8.29) is equivalently converted into a nonaffine FAS.

8.2.2 Recursive Solution

For this generalized form (8.29), some existing control methods, such as the backstepping methods, may become extremely difficult or even impossible to apply when r and n are large due to the "differential explosion" problem. However, if we can transform it into a FAS model, its control problems can then be solved easily and conveniently (Duan (2021a)).

Theorem 8.2 *Let Assumption 8.3 be satisfied, then, under the following transformations*

$$\begin{cases} x_1 = z \\ x_2 = g_1^{-1}(z^{(0\sim 1)}, y_1, \zeta, t),\ y_1 = \ddot{z} - f_1(z^{(0\sim 1)}, \zeta, t) \\ x_{k+1} = l_k^{-1}(z^{(0\sim 2k-1)}, y_k, \zeta^{(0\sim 2k-4)}, t),\ y_k = z^{(2k)} - h_k(z^{(0\sim 2k-1)}, \zeta^{(0\sim 2k-2)}, t), \\ \qquad\qquad k = 2, 3, \ldots, n-1, \end{cases} \tag{8.35}$$

and

$$\begin{cases} \dot{x}_1 = \dot{z} \\ \dot{x}_2 = \frac{dg_1^{-1}(z^{(0\sim 1)}, y_1, \zeta, t)}{dt} \\ \dot{x}_{k+1} = \frac{dl_k^{-1}(z^{(0\sim 2k-1)}, y_k, \zeta^{(0\sim 2k-4)}, t)}{dt},\ k = 2, 3, \ldots, n-1, \end{cases} \tag{8.36}$$

the generalized nonaffine strict-feedback nonlinear system (8.29) can be transformed into the following FAS:

$$z^{(2n)} = h_n(z^{(0\sim 2n-1)}, \zeta^{(0\sim 2n-2)}, t) + l_n(z^{(0\sim 2n-1)}, u, \zeta^{(0\sim 2n-4)}, t), \tag{8.37}$$

where the vector functions $h_n(z^{(0\sim 2n-1)}, \zeta^{(0\sim 2n-2)}, t)$ and $l_n(z^{(0\sim 2n-1)}, u, \zeta^{(0\sim 2n-4)}, t)$ are recursively given by

$$\begin{aligned} & h_k(z^{(0\sim 2k-1)}, \zeta^{(0\sim 2k-2)}, t) \\ = \;& \dot{\tilde{f}}_{k-1}(z^{(0\sim 2k-2)}, \zeta^{(0\sim 2k-3)}, t) + \dot{\tilde{g}}_{k-1}(z^{(0\sim 2k-2)}, \zeta^{(0\sim 2k-4)}, t)\dot{x}_k \\ & + \tilde{g}_{k-1}(z^{(0\sim 2k-2)}, \zeta^{(0\sim 2k-4)}, t)\, f_k(x_{1\sim k}^{(0\sim 1)}, \zeta, t), \end{aligned} \tag{8.38}$$

$$\begin{aligned} & l_k(z^{(0\sim 2k-1)}, x_{k+1}, \zeta^{(0\sim 2k-4)}, t) \\ = \;& \tilde{g}_{k-1}(z^{(0\sim 2k-2)}, \zeta^{(0\sim 2k-4)}, t) g_k(x_{1\sim k}^{(0\sim 1)}, x_{k+1}, \zeta, t), \end{aligned} \tag{8.39}$$

for $k = 2, 3, \ldots, n$, with $x_{n+1} = u$, and

$$\begin{aligned} & \tilde{f}_1(z^{(0\sim 2)}, \zeta^{(0\sim 1)}, t) \\ = \;& \dot{h}_1(z^{(0\sim 1)}, \zeta, t) + \frac{\partial l_1(z^{(0\sim 1)}, x_2, \zeta, t)}{\partial t} \\ & + \frac{\partial l_1(z^{(0\sim 1)}, x_2, \zeta, t)}{\partial z^{(0\sim 1)}} \dot{z}^{(0\sim 1)} + \frac{\partial l_1(z^{(0\sim 1)}, x_2, \zeta, t)}{\partial \zeta} \dot{\zeta}, \end{aligned} \tag{8.40}$$

$$\tilde{g}_1(z^{(0\sim 2)}, \zeta, t) = \frac{\partial l_1(z^{(0\sim 1)}, x_2, \zeta, t)}{\partial x_2}, \tag{8.41}$$

and

$$\begin{aligned} \tilde{f}_{k-1}(z^{(0\sim 2k-2)}, \zeta^{(0\sim 2k-3)}, t) = \;& \dot{h}_{k-1}(z^{(0\sim 2k-3)}, \zeta^{(0\sim 2k-4)}, t) \\ & + \frac{\partial l_{k-1}(z^{(0\sim 2k-3)}, x_k, \zeta^{(0\sim 2k-6)}, t)}{\partial t} \\ & + \frac{\partial l_{k-1}(z^{(0\sim 2k-3)}, x_k, \zeta^{(0\sim 2k-6)}, t)}{\partial z^{(0\sim 2k-3)}} \dot{z}^{(0\sim 2k-3)} \\ & + \frac{\partial l_{k-1}(z^{(0\sim 2k-3)}, x_k, \zeta^{(0\sim 2k-6)}, t)}{\partial \zeta^{(0\sim 2k-6)}} \dot{\zeta}^{(0\sim 2k-6)} \end{aligned} \tag{8.42}$$

$$\tilde{g}_{k-1}(z^{(0\sim 2k-2)}, \zeta^{(0\sim 2k-4)}, t) = \frac{\partial l_{k-1}(z^{(0\sim 2k-3)}, x_k, \zeta^{(0\sim 2k-6)}, t)}{\partial x_k}, \tag{8.43}$$

for $k = 3, 4, \ldots, n$, and the initial values are

$$\begin{cases} h_1(z^{(0\sim 1)}, \zeta, t) = f_1(x_1^{(0\sim 1)}, \zeta, t) \\ l_1(z^{(0\sim 1)}, x_2, \zeta, t) = g_1(x_1^{(0\sim 1)}, x_2, \zeta, t), \end{cases} \tag{8.44}$$

and in (8.38)-(8.39) the states $x_i^{(0\sim 1)}$, $i = 1, 2, \ldots, n$, are all given by the transformations (8.35) and (8.36).

For a proof of the above result, please refer to Appendix B.
SFS (8.29) clearly has the following companion form:

$$\begin{cases} \ddot{x}_1 = f_1(x_1^{(0\sim 1)}, \zeta, t) + g_1(x_1^{(0\sim 1)}, \dot{x}_2, \zeta, t) \\ \ddot{x}_2 = f_2(x_{1\sim 2}^{(0\sim 1)}, \zeta, t) + g_2(x_{1\sim 2}^{(0\sim 1)}, \dot{x}_3, \zeta, t) \\ \quad\quad \vdots \\ \ddot{x}_{n-1} = f_{n-1}(x_{1\sim n-1}^{(0\sim 1)}, \zeta, t) + g_{n-1}(x_{1\sim n-1}^{(0\sim 1)}, \dot{x}_n, \zeta, t) \\ \ddot{x}_n = f_n(x_{1\sim n}^{(0\sim 1)}, \zeta, t) + g_n(x_{1\sim n}^{(0\sim 1)}, u, \zeta, t). \end{cases} \tag{8.45}$$

Following the same procedure, the above SFS (8.45) can also be converted into a FAS model. We remark that the derivation process is even simpler since for this companion for each updating only needs a first-order differential of each subsystem. Furthermore, this companion form (8.45) suggests that while $\dot{x}_{i+1}$ can be determined, the position signals remain inaccessible. Consequently, for companion form (8.45), the overall system is stable only when the position signal is bounded.

Remark 8.4 Once the FAS model (8.37) is obtained, the controller of the system can then be simply designed as

$$\begin{cases} u = -l_n^{-1}\left(A_{0\sim 2n-1} z^{(0\sim 2n-1)} + u^*\right) \\ u^* = h_n(z^{(0\sim 2n-1)}, \zeta^{(0\sim 2n-2)}, t) - \upsilon, \end{cases} \tag{8.46}$$

which gives the following constant linear closed-loop system:

$$z^{(2n)} + A_{0\sim 2n-1} z^{(0\sim 2n-1)} = \upsilon,$$

where υ is some external signal, while $A_{0\sim 2n-1}$ is an arbitrary matrix which makes $\Phi(A_{0\sim 2n-1})$ stable. A complete parametric approach for solving the matrix $A_{0\sim 2n-1}$ is given in the Proposition 2 of Duan (2021a).

Through the proof of the above theorem, it can be observed that the transformations (8.35) and (8.36) form a homeomorphism between $x_{1\sim n}^{(0\sim 1)}$ and $z^{(0\sim 2n-1)}$, whose inverse is actually given by the following transformation

$$z^{(0\sim 2n-1)} = \begin{bmatrix} x_1^{(0\sim 1)} \\ h_1^{(0\sim 1)}\left(x_1^{(0\sim 1)}, \zeta, t\right) + l_1^{(0\sim 1)}\left(x_1^{(0\sim 1)}, x_2, \zeta, t\right) \\ h_2^{(0\sim 1)}(x_{1\sim 2}^{(0\sim 1)}, \zeta, t) + l_2^{(0\sim 1)}(x_{1\sim 2}^{(0\sim 1)}, x_3, \zeta, t) \\ \vdots \\ h_{n-1}^{(0\sim 1)}(x_{1\sim n-1}^{(0\sim 1)}, \zeta, t) + l_{n-1}^{(0\sim 1)}(x_{1\sim n-1}^{(0\sim 1)}, x_n, \zeta, t) \end{bmatrix}. \tag{8.47}$$

For each $x_{1\sim n}^{(0\sim 1)}$, (8.47) clearly gives a unique $z^{(0\sim 2n-1)}$. Conversely, let $z^{(0\sim 2n-1)}$ be given, then we have $x_{1\sim n}^{(0\sim 1)}$. Thus, transformation (8.35)-(8.36), or (8.47), is globally one-to-one. Furthermore, it is clearly seen from (8.47) that $x_{1\sim n}^{(0\sim 1)} = 0$ implies $z^{(0\sim 2n-1)} = 0$. Thus transformations (8.35)-(8.36), or (8.47), also keep the origin unchanged.

8.2.3 Special Cases

Case of $g_i(\cdot)$ independent on $x_{1\sim i}$, $i=1,2,\ldots,n$, and ζ

In this subsection, let us consider the special case of

$$\begin{cases} g_i(x_{1\sim i}^{(0\sim 1)}, x_{i+1}, \zeta, t) = g_i(x_{i+1}, t) \in \mathbb{R}^r,\ i=1,2,\ldots,n-1 \\ g_n(x_{1\sim n}, u, \zeta, t) = g_n(u, t). \end{cases} \tag{8.48}$$

In this case, the nonaffine SFS (8.29) becomes

$$\begin{cases} \dot{x}_1 = f_1(x_1^{(0\sim 1)}, \zeta, t) + g_1(x_2, t) \\ \dot{x}_2 = f_2(x_{1\sim 2}^{(0\sim 1)}, \zeta, t) + g_2(x_3, t) \\ \quad\vdots \\ \dot{x}_{n-1} = f_{n-1}(x_{1\sim n-1}^{(0\sim 1)}, \zeta, t) + g_{n-1}(x_n, t) \\ \dot{x}_n = f_n(x_{1\sim n}^{(0\sim 1)}, \zeta, t) + g_n(u, t), \end{cases} \tag{8.49}$$

and Assumption 8.3 becomes

Assumption 8.4 *For arbitrary $x_i \in \mathbb{R}^r$, $i=1,2,\ldots,n$, and $\forall t \geq 0$, there hold*

$$\begin{cases} \frac{\partial}{\partial x_{i+1}} g_i(x_{i+1}, t) \neq 0,\ i=1,2,\ldots,n-1, \\ \frac{\partial}{\partial u} g_n(u, t) \neq 0. \end{cases} \tag{8.50}$$

Due to (8.48), we have

$$\begin{aligned} \dot{g}_i(x_{1\sim i}^{(0\sim 1)}, x_{i+1}, \zeta, t) &= \dot{g}_i(x_{i+1}, t) \\ &= \frac{\partial}{\partial x_{i+1}} g_i(x_{i+1}, t)\dot{x}_{i+1} + \frac{\partial}{\partial t} g_i(x_{i+1}, t),\ i=1,2,\cdots,n-1. \end{aligned}$$

Hence, setting for $i=1,2,\ldots,n-1$,

$$\begin{aligned} \frac{\partial}{\partial x_{1\sim i}} g_i(x_{1\sim i}^{(0\sim 1)},\ x_{i+1}, \zeta, t) &= \frac{\partial}{\partial x_{1\sim i}} g_i(x_{i+1}, t) = 0, \\ \frac{\partial}{\partial \zeta} g_i(x_{1\sim i}^{(0\sim 1)}, x_{i+1}, \zeta, t) &= \frac{\partial}{\partial \zeta} g_i(x_{i+1}, t) = 0, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial x_{i+1}} g_i(x_{1\sim i}^{(0\sim 1)}, x_{i+1}, \zeta, t) &= \frac{\partial}{\partial x_{i+1}} g_i(x_{i+1}, t), \\ \frac{\partial}{\partial t} g_i(x_{1\sim i}^{(0\sim 1)}, x_{i+1}, \zeta, t) &= \frac{\partial}{\partial t} g_i(x_{i+1}, t), \end{aligned}$$

in Theorem 8.2, immediately gives the following corollary.

Corollary 8.3 *Let condition (8.48) and Assumption 8.4 be satisfied, then, under the following transformation*

$$\begin{cases} x_1 = z \\ x_2 = g_1^{-1}(y_1, t),\ y_1 = \ddot{z} - f_1(z^{(0\sim 1)}, \zeta, t), \\ x_{k+1} = l_k^{-1}\left(z^{(0\sim 2k-1)}, y_k, \zeta^{(0\sim 2k-4)}, t\right),\ y_k = z^{(2k)} - h_k(z^{(0\sim 2k-1)}, \zeta^{(0\sim 2k-2)}, t), \\ \qquad\qquad k = 2, 3, \ldots, n-1, \end{cases} \tag{8.51}$$

and

$$\begin{cases} \dot{x}_1 = \dot{z} \\ \dot{x}_2 = \frac{dg_1^{-1}(y_1,t)}{dt} \\ \dot{x}_{k+1} = \frac{dl_k^{-1}(z^{(0\sim 2k-1)}, y_k, \zeta^{(0\sim 2k-4)}, t)}{dt},\ k = 2, 3, \ldots, n-1, \end{cases} \tag{8.52}$$

the generalized strict-feedback nonlinear system (8.29) can be equivalently transformed into the following FAS model:

$$z^{(2n)} = h_n(z^{(0\sim 2n-1)}, \zeta^{(0\sim 2n-2)}, t) + l_n(z^{(0\sim 2n-1)}, u, \zeta^{(0\sim 2n-4)}, t), \tag{8.53}$$

where for $k = 2, 3, \ldots, n-1$,

$$\begin{cases} l_k(z^{(0\sim 2k-1)}, x_{k+1}, \zeta^{(0\sim 2k-4)}, t) = \frac{\partial g_1(x_2,t)}{\partial x_2}\frac{\partial g_2(x_3,t)}{\partial x_3}\cdots\frac{\partial g_{k-1}(x_k,t)}{\partial x_k} g_k(x_{k+1}, t) \\ l_n(z^{(0\sim 2n-1)}, u, \zeta^{(0\sim 2n-4)}, t) = \frac{\partial l_{n-1}(z^{(0\sim 2n-3)}, x_n, \zeta^{(0\sim 2n-6)}, t)}{\partial x_n} g_n(u, t), \end{cases} \tag{8.54}$$

and for $k = 2, 3, \ldots, n$,

$$\begin{aligned} h_k(z^{(0\sim 2k-1)}, \zeta^{(0\sim 2k-2)}, t) &= \dot{\tilde{f}}_{k-1}(z^{(0\sim 2k-2)}, \zeta^{(0\sim 2k-3)}, t) \\ &\quad + \tilde{g}_{k-1}(z^{(0\sim 2k-2)}, \zeta^{(0\sim 2k-4)}, t)\, f_k(x_{1\sim k}^{(0\sim 1)}, \zeta, t) \\ &\quad + \dot{\tilde{g}}_{k-1}(z^{(0\sim 2k-2)}, \zeta^{(0\sim 2k-4)}, t)\dot{x}_k, \end{aligned} \tag{8.55}$$

with

$$\begin{aligned} \tilde{f}_1(z^{(0\sim 2)}, \zeta^{(0\sim 1)}, t) &= \dot{h}_1(z^{(0\sim 1)}, \zeta, t) + \frac{\partial l_1(z^{(0\sim 1)}, x_2, \zeta, t)}{\partial t} \\ &\quad + \frac{\partial l_1(z^{(0\sim 1)}, x_2, \zeta, t)}{\partial z^{(0\sim 1)}}\dot{z}^{(0\sim 1)} + \frac{\partial l_1(z^{(0\sim 1)}, x_2, \zeta, t)}{\partial \zeta}\dot{\zeta}, \\ \tilde{g}_1(z^{(0\sim 2)}, \zeta, t) &= \frac{\partial l_1(z^{(0\sim 1)}, x_2, \zeta, t)}{\partial x_2}, \end{aligned}$$

and for $k = 3, 4, \ldots, n$,

$$\begin{aligned} \tilde{f}_{k-1}(z^{(0\sim 2k-2)}, \zeta^{(0\sim 2k-3)}, t) &= \dot{h}_{k-1}(z^{(0\sim 2k-3)}, \zeta^{(0\sim 2k-4)}, t) \\ &\quad + \frac{\partial l_{k-1}(z^{(0\sim 2k-3)}, x_k, \zeta^{(0\sim 2k-6)}, t)}{\partial t} \\ &\quad + \frac{\partial l_{k-1}(z^{(0\sim 2k-3)}, x_k, \zeta^{(0\sim 2k-6)}, t)}{\partial z^{(0\sim 2k-3)}}\dot{z}^{(0\sim 2k-3)} \end{aligned}$$

$$+\frac{\partial l_{k-1}(z^{(0\sim 2k-3)}, x_k, \zeta^{(0\sim 2k-6)}, t)}{\partial \zeta^{(0\sim 2k-6)}}\dot{\zeta}^{(0\sim 2k-6)},$$

$$\tilde{g}_{k-1}(z^{(0\sim 2k-2)}, \zeta^{(0\sim 2k-4)}, t) = \frac{\partial l_{k-1}(z^{(0\sim 2k-3)}, x_k, \zeta^{(0\sim 2k-6)}, t)}{\partial x_k},$$

and the initial values

$$h_1(z^{(0\sim 1)}, \zeta, t) = f_1(x_1^{(0\sim 1)}, \zeta, t),\ l_1(z^{(0\sim 1)}, x_2, \zeta, t) = g_1(x_2, t), \tag{8.56}$$

and in (8.54)-(8.55) the state vectors $x_k,\ k = 2, 3, \ldots, n$, and their derivatives are all given by transformations (8.51) and (8.52).

Remark 8.5 Another way of treating the special SFS (8.49) is to convert it into an affine SFS. In fact, by applying the following transformation

$$\begin{cases} \hat{x}_1 = x_1 \\ \hat{x}_{k+1} = g_k(x_{k+1}, t),\ k = 1, 2, \ldots, n-1 \\ \hat{u} = g_n(u, t), \end{cases} \tag{8.57}$$

to SFS (8.49), the following equations can be easily derived:

$$\begin{aligned}
\dot{\hat{x}}_1 &= \dot{x}_1 = f_1(\hat{x}_1^{(0\sim 1)}, \zeta, t) + \hat{x}_2, \\
\ddot{\hat{x}}_k &= \left(\frac{\partial g_{k-1}(x_k, t)}{\partial x_k}\dot{x}_k + \frac{\partial g_{k-1}(x_k, t)}{\partial t}\right)' \\
&= \frac{\partial^2 g_{k-1}(x_k, t)}{\partial x_k^2}(\dot{x}_k)^2 + \frac{\partial g_{k-1}(x_k, t)}{\partial x_k}\ddot{x}_k + 2\frac{\partial^2 g_{k-1}(x_k, t)}{\partial t \partial x_k}\dot{x}_k + \frac{\partial^2 g_{k-1}(x_k, t)}{\partial t^2} \\
&= \frac{\partial^2 g_{k-1}(x_k, t)}{\partial x_k^2}(\dot{x}_k)^2 + \frac{\partial g_{k-1}(x_k, t)}{\partial x_k} f_k(x_{1\sim k}^{(0\sim 1)}, \zeta, t) \\
&\quad + \frac{\partial g_{k-1}(x_k, t)}{\partial x_k}\hat{x}_{k+1} + \frac{\partial^2 g_{k-1}(x_k, t)}{\partial t^2} + 2\frac{\partial^2 g_{k-1}(x_k, t)}{\partial t \partial x_k}\dot{x}_k \\
&\triangleq h_k(\hat{x}_{1\sim k}^{(0\sim 1)}, \zeta, t) + l_k(\hat{x}_k, t)\hat{x}_{k+1},\ k = 2, 3, \ldots, n,
\end{aligned}$$

where $\hat{x}_{k+1}$ denotes $\hat{u}$, and the vector functions $h_k(\hat{x}_{1\sim k}^{(0\sim 1)}, \zeta, t)$ and $l_k(\hat{x}_k, t)$ for $k = 2, 3, \ldots, n$, are

$$\begin{aligned}
h_k(\hat{x}_{1\sim k}^{(0\sim 1)}, \zeta, t) &= \frac{\partial^2 g_{k-1}(x_k)}{\partial x_k^2}(\dot{x}_k)^2 + \frac{\partial g_{k-1}(x_k)}{\partial x_k} f_k(x_{1\sim k}^{(0\sim 1)}, \zeta, t) \\
&\quad + \frac{\partial^2 g_{k-1}(x_k, t)}{\partial t^2} + 2\frac{\partial^2 g_{k-1}(x_k, t)}{\partial t \partial x_k}\dot{x}_k, \\
l_k(\hat{x}_k, t) &= \frac{\partial g_{k-1}(x_k, t)}{\partial x_k}.
\end{aligned} \tag{8.58}$$

Let $h_1(\hat{x}_1^{(0\sim 1)},\zeta,t)=f_1(x_1^{(0\sim 1)},\zeta,t)$. Then, under transformation (8.57), system (8.49) can be immediately converted into the following SFS:

$$\begin{cases}\dot{\hat{x}}_1=h_1(\hat{x}_1^{(0\sim 1)},\zeta,t)+\hat{x}_2\\ \dot{\hat{x}}_2=h_2(\hat{x}_{1\sim 2}^{(0\sim 1)},\zeta,t)+l_2(\hat{x}_2,t)\hat{x}_3\\ \dot{\hat{x}}_3=h_3(\hat{x}_{1\sim 3}^{(0\sim 1)},\zeta,t)+l_3(\hat{x}_3,t)\hat{x}_4\\ \qquad\vdots\\ \dot{\hat{x}}_{n-1}=h_{n-1}(\hat{x}_{1\sim n-1}^{(0\sim 1)},\zeta,t)+l_{n-1}(\hat{x}_{n-1},t)\hat{x}_n\\ \dot{\hat{x}}_n=h_n(\hat{x}_{1\sim n}^{(0\sim 1)},\zeta,t)+l_n(\hat{x}_n,t)\hat{u},\end{cases}\tag{8.59}$$

which has been treated in Duan (2021b).

Case of $f_i(x_{1\sim i}^{(0\sim 1)},\zeta,t)=0,\ i=1,2,\cdots,n$

In this case, SFS (8.29) becomes the following nonaffine-SFS:

$$\begin{cases}\ddot{x}_1=g_1(x_1^{(0\sim 1)},x_2,\zeta,t)\\ \ddot{x}_2=g_2(x_{1\sim 2}^{(0\sim 1)},x_3,\zeta,t)\\ \qquad\vdots\\ \ddot{x}_{n-1}=g_{n-1}(x_{1\sim n-1}^{(0\sim 1)},x_n,\zeta,t)\\ \ddot{x}_n=g_n(x_{1\sim n}^{(0\sim 1)},u,\zeta,t).\end{cases}\tag{8.60}$$

The above system is also referred as second-order pseudo SFS in the literature (Zhang et al. (2025a)).

Setting

$$f_i(x_{1\sim i}^{(0\sim 1)},\zeta,t)=0,\ i=1,2,\ldots,n,$$

in Theorem 8.2, gives the following corollary.

Corollary 8.4 *Let Assumption 8.4 and condition* $f_i(x_{1\sim i}^{(0\sim 1)},\zeta,t)=0,\ i=1,2,\cdots,n,$ *be satisfied, then, under the following transformation:*

$$\begin{cases}x_1=z\\ x_2=g_1^{-1}(y_1,t),\ y_1=\ddot{z},\\ x_{k+1}=l_k^{-1}\left(z^{(0\sim 2k-1)},y_k,\zeta^{(0\sim 2k-4)},t\right),\ y_k=z^{(2k)}-h_k(z^{(0\sim 2k-1)},\zeta^{(0\sim 2k-2)},t),\\ \qquad\qquad k=2,3,\ldots,n-1,\end{cases}\tag{8.61}$$

and

$$\begin{cases}\dot{x}_1=\dot{z}\\ \dot{x}_2=\frac{dg_1^{-1}(y_1,t)}{dt}\\ \dot{x}_{k+1}=\frac{dl_k^{-1}(z^{(0\sim 2k-1)},y_k,\zeta^{(0\sim 2k-4)},t)}{dt},k=2,3,\ldots,n-1,\end{cases}\tag{8.62}$$

the generalized strict-feedback nonlinear system (8.29) can be equivalently transformed into the following FAS model:

$$z^{(2n)}=h_n(z^{(0\sim 2n-1)},\zeta^{0\sim 2n-2},t)+l_n(\,z^{(0\sim 2n-1)},u,\zeta^{(0\sim 2n-4)},t),\tag{8.63}$$

where the vector functions $h_n(z^{(0\sim 2n-1)}, \zeta^{(0\sim 2n-2)}, t)$ *and* $l_n(z^{(0\sim 2n-1)}, u, \zeta^{(0\sim 2n-4)}, t)$ *are recursively given by*

$$\begin{aligned} h_k(z^{(0\sim 2k-1)}, \zeta^{(0\sim 2k-2)}, t) &= \dot{\tilde{f}}_{k-1}(z^{(0\sim 2k-2)}, \zeta^{(0\sim 2k-3)}, t) \\ &\quad + \dot{\tilde{g}}_{k-1}(z^{(0\sim 2k-2)}, \zeta^{(0\sim 2k-4)}, t)\dot{x}_k \\ &\quad + \tilde{g}_{k-1}(z^{(0\sim 2k-2)}, \zeta^{(0\sim 2k-4)}, t)\, f_k(x_{1\sim k}^{(0\sim 1)}, \zeta, t) \end{aligned} \tag{8.64}$$

$$l_k(z^{(0\sim 2k-1)}, x_{k+1}, \zeta^{(0\sim 2k-4)}, t) = \tilde{g}_{k-1}(z^{(0\sim 2k-2)}, \zeta^{(0\sim 2k-4)}, t) g_k(x_{1\sim k}^{(0\sim 1)}, x_{k+1}, \zeta, t) \tag{8.65}$$

for $k = 2, 3, \ldots, n$, *with* $x_{n+1} = u$, *and*

$$\begin{aligned} \tilde{f}_1(z^{(0\sim 2)}, \zeta^{(0\sim 1)}, t) &= \frac{\partial l_1(\, z^{(0\sim 1)}, x_2, \zeta, t)}{\partial t} \\ &\quad + \frac{\partial l_1(\, z^{(0\sim 1)}, x_2, \zeta, t)}{\partial z^{(0\sim 1)}} \dot{z}^{(0\sim 1)} \\ &\quad + \frac{\partial l_1(\, z^{(0\sim 1)}, x_2, \zeta, t)}{\partial \zeta} \dot{\zeta}, \end{aligned} \tag{8.66}$$

$$\tilde{g}_1(z^{(0\sim 2)}, \zeta, t) = \frac{\partial l_1(\, z^{(0\sim 1)}, x_2, \zeta, t)}{\partial x_2}, \tag{8.67}$$

and for $k = 3, 4, \ldots, n$,

$$\begin{aligned} \tilde{f}_{k-1}(z^{(0\sim 2k-2)}, \zeta^{(0\sim 2k-3)}, t) &= \dot{h}_{k-1}(z^{(0\sim 2k-3)}, \zeta^{(0\sim 2k-4)}, t) \\ &\quad + \frac{\partial l_{k-1}(\, z^{(0\sim 2k-3)}, x_k, \zeta^{(0\sim 2k-6)}, t)}{\partial t} \\ &\quad + \frac{\partial l_{k-1}(\, z^{(0\sim 2k-3)}, x_k, \zeta^{(0\sim 2k-6)}, t)}{\partial z^{(0\sim 2k-3)}} \dot{z}^{(0\sim 2k-3)} \\ &\quad + \frac{\partial l_{k-1}(\, z^{(0\sim 2k-3)}, x_k, \zeta^{(0\sim 2k-6)}, t)}{\partial \zeta^{(0\sim 2k-6)}} \dot{\zeta}^{(0\sim 2k-6)}, \\ \tilde{g}_{k-1}(z^{(0\sim 2k-2)}, \zeta^{(0\sim 2k-4)}, t) &= \frac{\partial l_{k-1}(\, z^{(0\sim 2k-3)}, x_k, \zeta^{(0\sim 2k-6)}, t)}{\partial x_k}, \end{aligned}$$

and the initial values are

$$h_1(z^{(0\sim 1)}, \zeta, t) = 0, \quad l_1(z^{(0\sim 1)}, x_2, \zeta, t) = g_1(x_1^{(0\sim 1)}, x_2, \zeta, t). \tag{8.68}$$

Obviously, this is a slightly generalized result of that in Zhang et al. (2025a).

8.3 Mixed-order Nonaffine SFSs

In the real physical world, most practical systems are essentially nonlinear and complex. When modelling complex physical systems, there are systems governed by Newton's Law, Lagrangian Equation, or Theorem of Momentum, etc., while there may also be subsystems governed by other physical or chemical laws which produce the first-order or high-order differential equations (Duan (2024e)). Therefore, there may be

subsystems of first-, second- and high-order. The control of these mixed-order systems is very difficult and challenging.

It is well-known that SFSs play a vital role in the control theory of nonlinear systems. In the control community, many scholars have proposed a large number of solutions to handle the synthesis problems of SFSs, in which these second- or high-order SFSs are almost always converted into first-order systems and treated in the first-order system framework for the control systems analysis and design, which is called as the well-known first-order state-space approach (Zhang et al. (2025a)). However, these methods suffer from the limitations of the first-order state-space approach for that the state-space model emphasizes the state vector and is probably the best choice for solving the state response but not the best one for the control design (Duan (2021a)). Converting the second- or high-order SFSs into state-space models undoubtedly makes the control systems design much more complicated, as the state-space description of mixed-order SFSs requires complex state transformations, and control design of nonlinear SFSs in the state-space framework usually requires finding appropriate Lyapunov functions, which may be difficult.

To overcome the dilemma of first-order system approach, a novel FAS approach is originally proposed by Duan as a general control framework for dynamical systems (Duan (2021a,b,c,d,e,f,g, 2022e,d)). It represents a nuanced frontier in control system theory, bridging complex dynamics with profound control capabilities. Its core idea is to select a proper state transformation to convert the original system into a FAS model, which can establish a balanced ratio of control inputs to degrees of freedom, and ensure a more direct and effective control strategy for the desired system response. Due to its great advantages and power in dealing with complicated nonlinear systems, FAS approach has already attracted increasing attention in the field of control theory and applications immediately since its birth, and its effectiveness has been illustrated by many practical applications such as spacecraft, quadrotors, and robotics.

Based on the novel FAS approach framework, a generalized form of the conventional SFSs is firstly proposed in Duan (2021b), and a recursive solution is proposed to convert equivalently the generalized SFS into a high-order FAS model. Following this pioneering work, FAS-based adaptive command filtered backstepping is proposed by Liu et al. (2023d) for high-order SFSs, which directly utilizes the fully-actuated feature to design controllers without turning the original second and high-order SFSs into the first-order state-space models to reduce backstepping steps, and circumvents the complexity arising from repeatedly differentiating the virtual control. Additionally, the control problem of SFSs with increasing dimensions is addressed in Xu and Duan (2024) via high-order FAS approach, in which the subsystems in the considered SFSs are assumed to have increasing dimensions, and the exponentially stabilizing controllers have been directly given. Particularly, the high-order FAS approaches are utilized in Zhang et al. (2025a) to solve the synthesis problem of a class of generalized first-order, second-order, and mixed-order pseudo pure-feedback nonlinear control systems, which overcomes the problem that traditional backstepping design methods are not applicable to such systems without any restrictions.

In this section, a more generalized form of mixed-order nonaffine SFSs is investigated via the high-order FAS approach. Parallel to the generalized first-order and second-order cases, a recursive solution is firstly proposed to convert the generalized

mixed-order nonaffine SFS equivalently into a high-order FAS model under certain common assumptions. Using the feature of FAS model, control of the system can then be easily realized and better solved to achieve a constant and linear closed-loop system with a desired eigenstructure. Furthermore, two special cases are studied, and their recursive results are directly obtained based on the generalized results. Finally, an example of a mixed-order nonaffine SFS with three dimensions is employed to demonstrate the effectiveness of the proposed approach.

8.3.1 Problem Description

In this section, the following high-order (mixed-order) SFS is considered:

$$\begin{cases} x_1^{(m_1)} = f_1(x_1^{(0\sim m_1-1)}, \zeta, t) + g_1(x_1^{(0\sim m_1-1)}, x_2, \zeta, t) \\ x_2^{(m_2)} = f_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2}, \zeta, t) + g_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2}, x_3, \zeta, t) \\ \quad\vdots \\ x_{n-1}^{(m_{n-1})} = f_{n-1}(x_i^{(0\sim m_i-1)}|_{i=1\sim n-1}, \zeta, t) + g_{n-1}(x_i^{(0\sim m_i-1)}|_{i=1\sim n-1}, x_n, \zeta, t) \\ x_n^{(m_n)} = f_n(x_i^{(0\sim m_i-1)}|_{i=1\sim n}, \zeta, t) + g_n(x_i^{(0\sim m_i-1)}|_{i=1\sim n}, u, \zeta, t), \end{cases} \tag{8.69}$$

where $x_i \in \mathbb{R}^r$, $i = 1, 2, \ldots, n$, are state vectors, $u \in \mathbb{R}^r$ is the vector of control input, $\zeta \in \mathbb{R}^m$ is a parameter vector, $m_k, k = 1, 2, \ldots, n$, are a set of positive integers, which often take the values of 1 and 2 in practical applications, $f_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k}, \zeta, t)$, $k = 1, 2, \ldots, n$, $g_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k}, x_{k+1}, \zeta, t)$, $k = 1, 2, \ldots, n-1$, and $g_n(x_i^{(0\sim m_i-1)}|_{i=1\sim n}, u, \zeta, t)$ are a set of sufficiently smooth r-dimensional vector functions. For convenience, we set ζ as a constant parameter vector and further introduce the following notations:

$$\begin{cases} p_k = m_1 + m_2 + \cdots + m_k, \ k = 1, 2, \ldots, n, \\ p = p_n. \end{cases} \tag{8.70}$$

The above system (8.69) is called a mixed-order nonaffine SFS, if the following assumption is met:

Assumption 8.5 *For arbitrary $x_k \in \mathbb{R}^r$, $k = 1, 2, \ldots, n-1$, $\zeta \in \mathbb{R}^m$, and $\forall t \geq 0$, there hold*

$$\begin{cases} \frac{\partial}{\partial x_{k+1}} g_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k}, x_{k+1}, \zeta, t) \neq 0, \quad k = 1, 2, \ldots, n-1 \\ \frac{\partial}{\partial u} g_n(x_i^{(0\sim m_i-1)}|_{i=1\sim n}, u, \zeta, t) \neq 0, \end{cases} \tag{8.71}$$

and the inverse functions of $g_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k}, x_{k+1}, \zeta, t)$ with respect to x_{k+1} and $g_n(x_i^{(0\sim m_i-1)}|_{i=1\sim n}, u, \zeta, t)$ with respect to u can be explicitly solved.

The meaning of the above assumption lies in the following facts:

- for each $k = 1, 2, \ldots, n-1$, the inverse function of $g_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k}, x_{k+1}, \zeta, t)$ with respect to x_{k+1} exists for all $x_i^{(0\sim m_i-1)}|_{i=1\sim k} \in \mathbb{R}^{r\,p_k}$, $\zeta \in \mathbb{R}^m$, and $t \geq 0$;

- the inverse function of $g_n(x_i^{(0\sim m_i-1)}|_{i=1\sim n}, u, \zeta, t)$ with respect to u exists for all $x_i^{(0\sim m_i-1)}|_{i=1\sim n} \in \mathbb{R}^{rp}$, $\zeta \in \mathbb{R}^m$, and $t \geq 0$;
- for arbitrary $y_k \in \mathbb{R}^r$, $k = 1, 2, \ldots, n-1$, the following equations

$$g_k(x_i^{(0\sim m_k-1)}|_{i=1\sim k}, x_{k+1}, \zeta, t) = y_k,\ k = 1, 2, \ldots, n-1,\ \ \forall t \geq 0, \tag{8.72}$$

have solutions with respect to $x_{k+1} \in \mathbb{R}^r$, $k = 1, 2, \ldots, n-1$, which can also be represented by

$$x_{k+1} = g_k^{-1}(x_i^{(0\sim m_k-1)}|_{i=1\sim k}, y_k, \zeta, t),\ k = 1, 2, \ldots, n-1; \tag{8.73}$$

- for an arbitrary $y_n \in \mathbb{R}^r$, the following equation

$$g_n(x_i^{(0\sim m_i-1)}|_{i=1\sim n}, u, \zeta, t) = y_n,\ \ \forall t \geq 0, \tag{8.74}$$

has a solution with respect to $u \in \mathbb{R}^r$, which can also be represented by

$$u = g_n^{-1}(x_i^{(0\sim m_i-1)}|_{i=1\sim n}, y_n, \zeta, t). \tag{8.75}$$

In certain high-dimensional cases, the above Assumption 8.5 may be difficult to verify. It is recommended that symbolic computation techniques may be considered to help in such cases.

The purpose of this section is to find a homeomorphism under which the above nonaffine SFS (8.69) is equivalently converted into a nonaffine FAS. Moreover, a discrete-time version of similar results can be found in Duan (2022f), and generalization of the results to SFSs with time-varying delays can be found in Duan (2023b,c).

8.3.2 Recursive Solution

Parallel to the first-order and second-order cases, we have the following conclusion.

Theorem 8.3 *Let Assumption 8.5 hold, then, under the transformations*

$$\begin{cases} x_1 = z, \\ x_2 = l_1^{-1}(z^{(0\sim m_1-1)}, y_1, \zeta, t),\ y_1 = z^{(m_1)} - h_1(z^{(0\sim m_1-1)}, \zeta, t), \\ x_{k+1} = l_k^{-1}(z^{(0\sim p_k-1)}, y_k, \zeta, t),\ y_k = z^{(p_k)} - h_k(z^{(0\sim p_k-1)}, \zeta, t), \\ \qquad\qquad k = 2, 3, \ldots, n-1, \end{cases} \tag{8.76}$$

and

$$\begin{cases} x_1^{(i)} = z^{(i)}, \quad i = 1, 2, \ldots, m_1 - 1 \\ x_{k+1}^{(i)} = \frac{\mathrm{d}^i}{\mathrm{d}t^i} l_k^{-1}(z^{(0\sim p_k-1)}, y_k, \zeta, t), \quad y_k = z^{(p_k)} - h_k(z^{(0\sim p_k-1)}, \zeta, t), \\ \qquad i = 1, 2, \ldots, m_{k+1} - 1,\ k = 1, 2, \ldots, n-1. \end{cases} \tag{8.77}$$

the generalized SFS (8.69) can be equivalently turned into the following FAS:

$$z^{(p)} = h_n(z^{(0\sim p-1)}, \zeta, t) + l_n(z^{(0\sim p-1)}, u, \zeta, t), \tag{8.78}$$

where the vector functions $h_n(z^{(0\sim p-1)},\zeta,t)$ and $l_n(z^{(0\sim p-1)},u,\zeta,t)$ are recursively given by

$$\begin{aligned} h_k(z^{(0\sim p_k-1)},\zeta,t) = F_{k-1}^{(m_k-1)}(z^{(0\sim p_{k-1})},x_k,\zeta,t) + \sum_{j=0}^{m_k-2} C(j,m_k-1) \\ \times \left(\frac{\partial l_{k-1}(z^{(0\sim p_{k-1}-1)},x_k,\zeta,t)}{\partial x_k}\right)^{(m_k-1-j)} x_k^{(j+1)} \\ + \frac{\partial l_{k-1}(z^{(0\sim p_{k-1}-1)},x_k,\zeta,t)}{\partial x_k} \\ \times f_k(x_i^{(0\sim m_i)}|_{i=1\sim k},\zeta,t), \quad k=2,3,\ldots,n, \end{aligned} \tag{8.79}$$

and

$$\begin{cases} l_k(z^{(0\sim p_k-1)},x_{k+1},\zeta,t) = \frac{\partial l_{k-1}(z^{(0\sim p_{k-1}-1)},x_k,\zeta,t)}{\partial x_k} g_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k},x_{k+1},\zeta,t), \\ \qquad\qquad k=2,3,\ldots,n-1 \\ l_n(z^{(0\sim p-1)},u,\zeta,t) = \frac{\partial l_{n-1}(z^{(0\sim p_{k-1}-1)},x_n,\zeta,t)}{\partial x_n} g_n(u,\zeta,t), \end{cases} \tag{8.80}$$

with

$$\begin{aligned} F_{k-1}(z^{(0\sim p_{k-1})},\zeta,t) = \dot{h}_{k-1}(z^{(0\sim p_{k-1}-1)},\zeta,t) \\ + \frac{\partial l_{k-1}(z^{(0\sim p_{k-1}-1)},x_k,\zeta,t)}{\partial t} \\ + \frac{\partial l_{k-1}(z^{(0\sim p_{k-1}-1)},x_k,\zeta,t)}{\partial z^{(0\sim p_{k-1}-1)}} z^{(1\sim p_{k-1})}, \\ k=2,3,\ldots,n, \end{aligned} \tag{8.81}$$

and the initial values

$$\begin{cases} h_1(z^{(0\sim m_1-1)},\zeta,t) = f_1(x_1^{(0\sim m_1-1)},\zeta,t) \\ l_1(z^{(0\sim m_1-1)},x_2,\zeta,t) = g_1(x_1^{(0\sim m_1-1)},x_2,\zeta,t), \end{cases} \tag{8.82}$$

and in (8.79)-(8.82) the states $x_k^{(0\sim m_k-1)}$, $k=1,2,\ldots,n$, are given by the transformations (8.76) and (8.77).

A proof of the above result can be found in Appendix B.

Parallel to the second-order SFS, the mixed-order SFS (8.69) also has the following generalized form:

$$\begin{cases} x_1^{(m_1)} = f_1(x_1^{(0\sim m_1-1)},\zeta,t) + g_1(x_1^{(0\sim m_1-1)},x_2^{(m_1^\circ)},\zeta,t) \\ x_2^{(m_2)} = f_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2},\zeta,t) + g_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2},x_3^{(m_2^\circ)},\zeta,t) \\ \qquad\vdots \\ x_{n-1}^{(m_{n-1})} = f_{n-1}(x_i^{(0\sim m_i-1)}|_{i=1\sim n-1},\zeta,t) \\ \qquad\qquad + g_{n-1}(x_i^{(0\sim m_i-1)}|_{i=1\sim n-1},x_n^{(m_{n-1}^\circ)},\zeta,t) \\ x_n^{(m_n)} = f_n(x_i^{(0\sim m_i-1)}|_{i=1\sim n},\zeta,t) + g_n(x_i^{(0\sim m_i-1)}|_{i=1\sim n},u,\zeta,t), \end{cases} \tag{8.83}$$

where $m_i^\circ < m_i$, $i = 1, 2, \ldots, n-1$, are another set of integers. Following the same procedure as shown in the proof of Theorem 8.3, the above mixed-order SFS can also be converted into a FAS model.

Remark 8.6 The proposed FAS approach is generally more effective than the method of backstepping because of the following two facts:

- The FAS approach can always produce a constant linear closed-loop system, but the method of backstepping can not.
- Due to the well-known problem of "differential explosion", the method of backstepping is generally not applicable to an SFS with more than 3 or 4 subsystems, while the FAS approach is, noting that the proposed recursive solutions are easy to realize.

Remark 8.7 Although it is apparent that the first- and second-order SFSs are the special cases of the high-order one (8.69), it is hard to tell a similar relation among the outcomes of the converted high-order systems of these systems.

Remark 8.8 Once the FAS model (8.78) is obtained, the controller of the system can then be simply designed as

$$\begin{cases} u = -l_n^{-1}\left(A_{0\sim p-1} z^{(0\sim p-1)} + u^*\right) \\ u^* = h_n(z^{(0\sim p-1)}, \zeta, t) - \upsilon, \end{cases} \tag{8.84}$$

which gives the following constant linear closed-loop system:

$$z^{(p)} + A_{0\sim p-1} z^{(0\sim p-1)} = \upsilon,$$

where υ is some external signal, while $A_{0\sim p-1}$ is an arbitrary matrix which makes $\Phi(A_{0\sim p-1})$ stable. A complete parametric approach for solving the matrix $A_{0\sim p-1}$ is given in Duan (2020a) (see also the Proposition 2 in Duan (2021a)).

Remark 8.9 Through the proof of the above theorem, it can be observed that, as a matter of fact, the two transformations (8.76) and (8.77) form a homeomorphism between $x_i^{(0\sim m_i-1)}|_{i=1\sim n}$ and $z^{(0\sim p-1)}$, whose inverse is in fact the following:

$$z^{(0\sim p-1)} = \begin{bmatrix} x_1^{(0\sim m_1-1)} \\ h_1^{(0\sim m_2-1)}(x_1^{(0\sim m_1-1)}, \zeta, t) + l_1^{(0\sim m_2-1)}(x_1^{(0\sim m_1-1)}, x_2, \zeta, t) \\ h_2^{(0\sim m_3-1)}(x_i^{(0\sim m_i-1)}|_{i=1\sim 2}, \zeta, t) + l_2^{(0\sim m_3-1)}(x_i^{(0\sim m_i-1)}|_{i=1\sim 2}, x_3, \zeta, t) \\ \vdots \\ h_{n-1}^{(0\sim m_n-1)}(x_i^{(0\sim m_i-1)}|_{i=1\sim n-1}, \zeta, t) + l_{n-1}^{(0\sim m_n-1)}(x_i^{(0\sim m_i-1)}|_{i=1\sim n-1}, x_n, \zeta, t) \end{bmatrix}. \tag{8.85}$$

Furthermore, it is clearly seen from (8.85) that $x_i^{(0\sim m_i-1)}|_{i=1\sim n} = 0$ implies $z^{(0\sim p-1)} = 0$. Thus transformation (8.76)-(8.77), or (8.85), also keeps the origin unchanged.

8.3.3 Special Case

Case of $g_k(\cdot)$ independent on $x_{1\sim k}$, $k = 1, 2, \ldots, n$

In this subsection, let us consider the special case of

$$\begin{cases} g_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k}, x_{k+1}, \zeta, t) = g_k(x_{k+1}, \zeta, t) \in \mathbb{R}^r, \quad k = 1, 2, \ldots, n-1 \\ g_n(x_i^{(0\sim m_i-1)}|_{i=1\sim n}, u, \zeta, t) = g_n(u, \zeta, t) \in \mathbb{R}^r. \end{cases} \tag{8.86}$$

In this case, the nonaffine SFS (8.69) becomes

$$\begin{cases} x_1^{(m_1)} = f_1(x_1^{(0\sim m_1-1)}, \zeta, t) + g_1(x_2, \zeta, t) \\ x_2^{(m_2)} = f_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2}, \zeta, t) + g_2(x_3, \zeta, t) \\ \qquad \vdots \\ x_{n-1}^{(m_{n-1})} = f_{n-1}(x_i^{(0\sim m_i-1)}|_{i=1\sim n-1}, \zeta, t) + g_{n-1}(x_n, \zeta, t) \\ x_n^{(m_n)} = f_n(x_i^{(0\sim m_i-1)}|_{i=1\sim n}, \zeta, t) + g_n(u, \zeta, t). \end{cases} \tag{8.87}$$

Assumption 8.5 becomes

Assumption 8.6 *For arbitrary $x_k \in \mathbb{R}^r$, $k = 1, 2, \ldots, n$, $\zeta \in \mathbb{R}^m$, and $t \geq 0$, there hold*

$$\begin{cases} \frac{\partial}{\partial x_{k+1}} g_k(x_{k+1}, \zeta, t) \neq 0, \quad k = 1, 2, \ldots, n-1 \\ \frac{\partial}{\partial u} g_n(u, \zeta, t) \neq 0. \end{cases} \tag{8.88}$$

Due to (8.86), we have

$$\begin{aligned} \dot{g}_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k}, x_{k+1}, \zeta, t) &= \dot{g}_k(x_{k+1}, \zeta, t) \\ &= \frac{\partial g_k(x_{k+1}, \zeta, t)\, \dot{x}_{k+1}}{\partial x_{k+1}} + \frac{\partial g_k(x_{k+1}, \zeta, t)}{\partial t}, \\ & \qquad k = 1, 2, \ldots, n-1. \end{aligned}$$

Hence, setting, for $k = 1, 2, \ldots, n-1$,

$$\frac{\partial g_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k}, x_{k+1}, \zeta, t)}{\partial x_{1\sim k}} = 0,$$

and

$$\begin{cases} \frac{\partial g_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k}, x_{k+1}, \zeta, t)}{\partial x_{k+1}} = \frac{\partial g_k(x_{k+1}, \zeta, t)}{\partial x_{k+1}} \\ \frac{\partial g_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k}, x_{k+1}, \zeta, t)}{\partial t} = \frac{\partial g_k(x_{k+1}, \zeta, t)}{\partial t}, \end{cases}$$

in Theorem 8.3, immediately gives the following corollary.

Corollary 8.5 *Let Assumption 8.6 be satisfied, then, under the following transformations*

$$\begin{cases} x_1 = z \\ x_2 = l_1^{-1}(z^{(0\sim m_1-1)}, y_1, \zeta, t), \ y_1 = z^{(m_1)} - h_1(z^{(0\sim m_1-1)}, \zeta, t) \\ x_{k+1} = l_k^{-1}(z^{(0\sim p_k-1)}, y_k, \zeta, t), \ y_k = z^{(p_k)} - h_k(z^{(0\sim p_k-1)}, \zeta, t), \\ \qquad k = 2, 3, \ldots, n-1, \end{cases} \tag{8.89}$$

and

$$\begin{cases} x_1^{(i)} = z^{(i)}, \quad i = 1,2,\ldots,m_1-1 \\ x_{k+1}^{(i)} = \frac{\mathrm{d}^i}{\mathrm{d}t^i} l_k^{-1}(z^{(0\sim p_k-1)}, y_k, \zeta, t), \quad y_k = z^{(p_k)} - h_k(z^{(0\sim p_k-1)}, \zeta, t), \\ \qquad\qquad i = 1,2,\ldots,m_{k+1}-1,\ k = 1,2,\ldots,n-1, \end{cases} \tag{8.90}$$

the generalized nonlinear SFS (8.87) can be equivalently transformed into the following FAS model:

$$z^{(p)} = h_n(z^{(0\sim p-1)}, \zeta, t) + l_n(z^{(0\sim p-1)}, u, \zeta, t), \tag{8.91}$$

where the vector functions $h_n(z^{(0\sim p-1)}, \zeta, t)$ and $l_n(z^{(0\sim p-1)}, \zeta, u, t)$ are recursively given by

$$\begin{aligned} & h_k(z^{(0\sim p_k-1)}, \zeta, t) \\ = \ & F_{k-1}^{(m_k-1)}(z^{(0\sim p_{k-1})}, x_k, \zeta, t) \\ & + \sum_{j=0}^{m_k-2} C(j, m_k-1) \left(\frac{\partial}{\partial x_k} l_{k-1}(z^{(0\sim p_{k-1}-1)}, x_k, \zeta, t) \right)^{(m_k-1-j)} x_k^{(j+1)} \\ & + \frac{\partial l_{k-1}(z^{(0\sim p_{k-1}-1)}, x_k, \zeta, t)}{\partial x_k} f_k(x_i^{(0\sim m_i)}|_{i=1\sim k}, \zeta, t), \\ & \qquad\qquad k = 2,3,\ldots,n, \end{aligned} \tag{8.92}$$

and

$$\begin{cases} l_k\left(z^{(0\sim p_k-1)}, x_k, \zeta, t\right) = \frac{\partial g_1(x_2,\zeta,t)}{\partial x_2} \frac{\partial g_2(x_3,\zeta,t)}{\partial x_3} \cdots \frac{\partial g_{k-1}(x_k,\zeta,t)}{\partial x_k} \\ \qquad\qquad \times g_k(x_{k+1}, \zeta, t), \quad k = 2,\ldots,n-1 \\ l_n(z^{(0\sim p_n-1)}, u, \zeta, t) = \frac{\partial l_{n-1}(z^{(0\sim p_n-1)}, x_n, \zeta, t)}{\partial x_n} g_n(u, \zeta, t), \end{cases} \tag{8.93}$$

with

$$\begin{aligned} F_{k-1}(z^{(0\sim p_{k-1})}, \zeta, t) = & \dot{h}_{k-1}(z^{(0\sim p_{k-1}-1)}, \zeta, t) + \frac{\partial l_{k-1}(z^{(0\sim p_{k-1}-1)}, x_k, \zeta, t)}{\partial t} \\ & + \frac{\partial l_{k-1}(z^{(0\sim p_{k-1}-1)}, x_k, u, \zeta, t)}{\partial z^{(0\sim p_{k-1}-1)}} z^{(1\sim p_{k-1})}, \\ & \qquad k = 2,3,\ldots,n, \end{aligned} \tag{8.94}$$

and the initial values

$$\begin{aligned} h_1(z^{(0\sim m_1-1)}, \zeta, t) &= f_1(x_1^{(0\sim m_1-1)}, \zeta, t), \\ l_1(z^{(0\sim m_1-1)}, x_2, \zeta, t) &= g_1(x_2, \zeta, t), \end{aligned}$$

and in (8.92)-(8.94) the states $x_k^{(0\sim m_k-1)}$, $k = 1,2,\ldots,n$, are all given by transformation (8.89)-(8.90).

Remark 8.10 Another way of treating the special SFS (8.87) is to convert it into an affine SFS. In fact, applying the transformation

$$\begin{cases} \hat{x}_1 = x_1 \\ \hat{x}_{k+1} = g_k(x_{k+1}, \zeta, t), \ \ k = 1, 2, \ldots, n-1 \\ \hat{u} = g_n(u, \zeta, t), \end{cases} \tag{8.95}$$

to SFS (8.87), immediately converts it into the following SFS:

$$\begin{cases} \hat{x}_1^{(m_1)} = \hat{f}_1(\hat{x}_1^{(0\sim p_1-1)}, \zeta, t) + \hat{x}_2 \\ \hat{x}_2^{(m_2)} = \hat{f}_2(\hat{x}_1^{(0\sim p_2-1)}, \zeta, t) + G_2(\hat{x}_2, \zeta, t)\hat{x}_3 \\ \qquad \vdots \\ \hat{x}_{n-1}^{(m_{n-1})} = \hat{f}_{n-1}(\hat{x}_1^{(0\sim p_{k-1}-1)}, \zeta, t) + G_{n-1}(\hat{x}_{n-1}, \zeta, t)\hat{x}_n \\ \hat{x}_n^{(m_n)} = \hat{f}_n(\hat{x}_1^{(0\sim p-1)}, \zeta, t) + G_n(\hat{x}_n, \zeta, t)\hat{u}, \end{cases} \tag{8.96}$$

where

$$\begin{aligned} \hat{f}_1(\hat{x}_1^{(0\sim p_1-1)}, \zeta, t) &= f_1(x_1^{(0\sim m_1-1)}, \zeta, t), \\ \hat{f}_k(\hat{x}_1^{(0\sim p_k-1)}, \zeta, t) &= \frac{\partial g_{k-1}(x_k)}{\partial x_k} f_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k}, \zeta, t) \\ &\quad + \left(\frac{\partial g_{k-1}(x_k, \zeta, t)}{\partial t}\right)^{(m_k-1)} \\ &\quad + \sum_{j=0}^{m_k-2} C(j, m_k-1) \\ &\quad \times \left(\frac{\partial g_{k-1}(x_k, \zeta, t)}{\partial x_k}\right)^{(m_k-1-j)} x_k^{(j+1)}, \\ &\quad k = 2, 3, \ldots, n, \end{aligned}$$

and

$$G_k(\hat{x}_k, \zeta, t) = \frac{\partial g_{k-1}(x_k, \zeta, t)}{\partial x_k}, k = 2, 3, \ldots, n,$$

which has been treated in Duan (2021b).

Case of $f_k(\cdot) = 0$, $k = 1, 2, \ldots, n$

In this subsection, let us consider the special case of

$$\begin{cases} x_1^{(m_1)} = g_1(x_1^{(0\sim m_1-1)}, x_2, \zeta, t) \\ x_2^{(m_2)} = g_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2}, x_3, \zeta, t) \\ \qquad \vdots \\ x_{n-1}^{(m_{n-1})} = g_{n-1}(x_i^{(0\sim m_i-1)}|_{i=1\sim n-1}, x_n, \zeta, t) \\ x_n^{(m_n)} = g_n(x_i^{(0\sim m_i-1)}|_{i=1\sim n}, u, \zeta, t). \end{cases} \tag{8.97}$$

Setting

$$f_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k},\zeta,t)=0,\ \ k=1,2,\ldots,n,$$

in Theorem 8.3, immediately gives the following corollary.

Corollary 8.6 *Let Assumption 8.5 be satisfied, then, under the following transformations:*

$$\left\{\begin{array}{l} x_1=z\\ x_2=l_1^{-1}(z^{(0\sim m_1-1)},y_1,\zeta,t),\ y_1=z^{(m_1)}-h_1(z^{(0\sim m_1-1)},\zeta,t)\\ x_{k+1}=l_k^{-1}(z^{(0\sim p_k-1)},y_k,\zeta,t),\ y_k=z^{(p_k)}-h_k(z^{(0\sim p_k-1)},\zeta,t),\\ \qquad k=2,3,\ldots,n-1, \end{array}\right. \tag{8.98}$$

and

$$\left\{\begin{array}{l} x_1^{(i)}=z^{(i)},\quad i=1,2,\ldots,m_1-1\\ x_{k+1}^{(i)}=\frac{\mathrm{d}^i}{\mathrm{d}t^i}l_k^{-1}(z^{(0\sim p_k-1)},y_k,\zeta,t),\quad y_k=z^{(p_k)}-h_k(z^{(0\sim p_k-1)},\zeta,t),\\ \qquad i=1,2,\ldots,m_{k+1}-1,\ k=1,2,\ldots,n-1, \end{array}\right. \tag{8.99}$$

the generalized nonlinear SFS (8.97) can be equivalently transformed into the following FAS model:

$$z^{(p)}=h_n(z^{(0\sim p-1)},\zeta,t)+l_n(z^{(0\sim p-1)},u,\zeta,t), \tag{8.100}$$

where the vector functions $h_n(z^{(0\sim p-1)},\zeta,t)$ and $l_n(z^{(0\sim p-1)},\zeta,u,t)$ are recursively given by

$$\begin{array}{rcl} & & h_k(z^{(0\sim p_k-1)},\zeta,t)\\ &=& F_{k-1}^{(m_k-1)}(z^{(0\sim p_{k-1})},x_k,t)\\ & & +\sum_{j=0}^{m_k-2}C(j,m_k-1)\left(\frac{\partial}{\partial x_k}l_{k-1}(z^{(0\sim p_{k-1}-1)},x_k,t)\right)^{(m_k-1-j)}x_k^{(j+1)}\\ & & \qquad k=2,3,\ldots,n, \end{array} \tag{8.101}$$

and

$$\left\{\begin{array}{l} l_k(z^{(0\sim p_k-1)},x_{k+1},\zeta,t)=\frac{\partial l_{k-1}(z^{(0\sim p_{k-1}-1)},x_k,\zeta,t)}{\partial x_k}\\ \qquad\qquad \times g_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k},x_{k+1},\zeta,t),\ \ k=2,\ldots,n-1\\ l_n(z^{(0\sim p-1)},u,\zeta,t)=\frac{\partial l_{n-1}(z^{(0\sim p_{k-1}-1)},x_n,\zeta,t)}{\partial x_n}g_n(u,\zeta,t), \end{array}\right. \tag{8.102}$$

with

$$\begin{array}{rcl} F_{k-1}(z^{(0\sim p_{k-1})},\zeta,t) &=& \dot{h}_{k-1}(z^{(0\sim p_{k-1}-1)},\zeta,t)+\frac{\partial l_{k-1}(z^{(0\sim p_{k-1}-1)},x_k,t)}{\partial t}\\ & & +\frac{\partial l_{k-1}(z^{(0\sim p_{k-1}-1)},x_k,t)}{\partial z^{(0\sim p_{k-1}-1)}}z^{(1\sim p_{k-1})},\\ & & \qquad k=2,3,\ldots,n, \end{array} \tag{8.103}$$

and the initial values

$$\begin{aligned} h_1(z^{(0\sim m_1-1)},\zeta,t) &= 0, \\ l_1(z^{(0\sim m_1-1)},x_2,\zeta,t) &= g_1(x_1^{(0\sim m_1-1)},x_2,\zeta,t), \end{aligned}$$

and in (8.101)-(8.103) the $x_k^{(0\sim m_k-1)}$, $k=1,2,\ldots,n$, *are all given by the transformations (8.98) and (8.99).*

Remark 8.11 Obviously, the nonaffine strict-feedback nonlinear system (8.97) is a more generalized form of the mixed-order pseudo pure-feedback nonlinear system treated in Zhang et al. (2025a). Thus, the result in Corollary 8.6 is a more generalized result of that in Zhang et al. (2025a).

8.4 Examples

8.4.1 First-order Example

Consider the following nonaffine SFS which is considered previously in Zhang and Ge (2008):

$$\begin{cases} \dot{x}_1 = x_1 + x_2 + \frac{x_2^3}{5} \\ \dot{x}_2 = x_3 + \frac{x_3^3}{2} \\ \dot{x}_3 = x_1x_2x_3 + (1+0.1\sin(0.5x_1x_2x_3))\,u + \frac{u^3}{7}. \end{cases} \tag{8.104}$$

By utilizing Theorem 8.1, the transformation for the first equation of (8.104) is given as

$$\begin{cases} z = x_1 \\ y_1 = \dot{x}_1 - x_1 \\ g_1 = x_2 + \frac{x_2^3}{5}. \end{cases} \tag{8.105}$$

Based on the third equation of (8.105), x_2 can be obtained:

$$\begin{aligned} x_2 &= g_1^{-1}\left(z^{(0\sim 1)}\right) \\ &= \sqrt[3]{\frac{5}{2}(\dot{z}-z)+\sqrt{\frac{25}{4}(\dot{z}-z)^2+\frac{125}{27}}} - \frac{5}{3\sqrt[3]{\frac{5}{2}(\dot{z}-z)+\sqrt{\frac{25}{4}(\dot{z}-z)^2+\frac{125}{27}}}} \\ &= \hat{y}^{\frac{1}{3}} - \frac{5}{3}\hat{y}^{-\frac{1}{3}}, \end{aligned} \tag{8.106}$$

where

$$\hat{y}\left(z^{(0\sim 1)}\right) = \frac{5}{2}(\dot{z}-z) + \left(\frac{25}{4}(\dot{z}-z)^2+\frac{125}{27}\right)^{\frac{1}{2}}.$$

First, take the derivative of both sides of the first equation in (8.104), which gives

$$\begin{aligned}\ddot{x}_1 &= \dot{x}_1 + \dot{x}_2 + \frac{3x_2^2}{5}\dot{x}_2 \\ &= x_1 + x_2 + \frac{x_2^3}{5} + \left(1 + \frac{3x_2^2}{5}\right)\left(x_3 + \frac{x_3^3}{2}\right).\end{aligned} \tag{8.107}$$

By utilizing transformation (8.105), the above equation can be reexpressed as

$$\ddot{z} = h_2\left(z^{(0\sim 1)}, t\right) + l_2\left(z^{(0\sim 1)}, x_3, t\right), \tag{8.108}$$

where

$$\begin{aligned}h_2\left(z^{(0\sim 1)}, t\right) &= z + \hat{y}^{\frac{1}{3}} - \frac{5}{3\hat{y}^{\frac{1}{3}}} + \frac{1}{5}\left(\hat{y}^{\frac{1}{3}} - \frac{5}{3\hat{y}^{\frac{1}{3}}}\right)^3, \\ l_2\left(z^{(0\sim 1)}, x_3, t\right) &= \left(1 + \frac{3}{5}\left(\hat{y}^{\frac{1}{3}} - \frac{5}{3}\hat{y}^{-\frac{1}{3}}\right)^2\right)\left(x_3 + \frac{x_3^3}{2}\right).\end{aligned}$$

With the help of (8.108), system (8.104) is rewritten as

$$\begin{cases} \ddot{z} = h_2\left(z^{(0\sim 1)}, t\right) + l_2\left(z^{(0\sim 1)}, x_3, t\right) \\ \dot{x}_3 = x_1 x_2 x_3 + (1 + 0.1\sin(0.5 x_1 x_2 x_3))\, u + \frac{u^3}{7}. \end{cases} \tag{8.109}$$

By introducing a function y_2 and substituting it into the function l_2 in the first equation of (8.109), it follows that

$$x_3 = l_2^{-1}\left(z^{(0\sim 1)}, y_2, t\right),\ y_2 = \ddot{z} - h_2\left(z^{(0\sim 1)}, t\right), \tag{8.110}$$

and x_3 can be further given as

$$\begin{aligned}x_3 &= l_2^{-1}\left(z^{(0\sim 1)}, y_2, t\right) \\ &= \left(\ddot{z} - h_2 + \left((\ddot{z} - h_2)^2 + \frac{8}{27}\right)^{\frac{1}{2}}\right)^{\frac{1}{3}} - \frac{2}{3}\left(\ddot{z} - h_2 + \left((\ddot{z} - h_2)^2 + \frac{8}{27}\right)^{\frac{1}{2}}\right)^{-\frac{1}{3}} \\ &= \tilde{y}^{\frac{1}{3}} - \frac{2}{3}\tilde{y}^{-\frac{1}{3}},\end{aligned}$$

where

$$\tilde{y} = \ddot{z} - h_2 + \left((\ddot{z} - h_2)^2 + \frac{8}{27}\right)^{\frac{1}{2}}.$$

Then, taking the derivative of the first sub-equation of (8.109), we have

$$\begin{aligned}
\dddot{z} &= \dot{h}_2\left(z^{(0\sim1)},t\right)+\frac{\partial l_2\left(z^{(0\sim1)},x_3,t\right)}{\partial z^{(0\sim1)}}\dot{z}^{(0\sim1)} \\
&\quad +\frac{\partial l_2\left(z^{(0\sim1)},x_3,t\right)}{\partial x_3}\dot{x}_3+\frac{\partial l_2\left(z^{(0\sim1)},x_3,t\right)}{\partial t} \\
&= \dot{h}_2\left(z^{(0\sim1)},t\right)+\frac{\partial l_2\left(z^{(0\sim1)},x_3,t\right)}{\partial z^{(0\sim1)}}\dot{z}^{(0\sim1)} \\
&\quad +\left(1+\frac{3}{5}\left(\hat{y}^{\frac{1}{3}}-\frac{5}{3}\hat{y}^{-\frac{1}{3}}\right)^2\right)\left(1+\frac{3x_3^2}{2}\right)\dot{x}_3 \\
&= \dot{h}_2\left(z^{(0\sim1)},t\right)+\frac{\partial l_2\left(z^{(0\sim1)},x_3,t\right)}{\partial z^{(0\sim1)}}\dot{z}^{(0\sim1)} \\
&\quad +\left(1+\frac{3}{5}\left(\hat{y}^{\frac{1}{3}}-\frac{5}{3}\hat{y}^{-\frac{1}{3}}\right)^2\right)\left(1+\frac{3x_3^2}{2}\right) \\
&\quad \times\left(x_1x_2x_3+(1+0.1\sin(0.5x_1x_2x_3))\,u+\frac{u^3}{7}\right). \qquad (8.111)
\end{aligned}$$

To clear the following statement, we define

$$\begin{aligned}
z_x &= x_1x_2x_3 \\
&= z\left(\hat{y}^{\frac{1}{3}}-\frac{5}{3}\hat{y}^{-\frac{1}{3}}\right)\left(\tilde{y}^{\frac{1}{3}}-\frac{2}{3}\tilde{y}^{-\frac{1}{3}}\right). \qquad (8.112)
\end{aligned}$$

Finally, with the help of (8.112), equation (8.111) can be further reexpressed as:

$$\dddot{z}=h_3\left(z^{(0\sim2)},t\right)+l_3\left(z^{(0\sim2)},u,t\right),$$

where

$$\begin{aligned}
h_3\left(z^{(0\sim2)},t\right) &= \dot{h}_2\left(z^{(0\sim1)},t\right)+\frac{\partial l_2\left(z^{(0\sim1)},x_3,t\right)}{\partial z^{(0\sim1)}}\dot{z}^{(0\sim1)} \\
&\quad +\left(1+\frac{3}{5}\left(\hat{y}^{\frac{1}{3}}-\frac{5}{3}\hat{y}^{-\frac{1}{3}}\right)^2\right)\left(1+\frac{3x_3^2}{2}\right)z_x, \\
l_3\left(z^{(0\sim2)},u,t\right) &= \left(1+\frac{3}{5}\left(\hat{y}^{\frac{1}{3}}-\frac{5}{3}\hat{y}^{-\frac{1}{3}}\right)^2\right)\left(1+\frac{3}{2}\left(\tilde{y}^{\frac{1}{3}}-\frac{2}{3}\tilde{y}^{-\frac{1}{3}}\right)^2\right)\hat{u},
\end{aligned}$$

with

$$\hat{u}=(1+0.1\sin(0.5z_x))\,u+\frac{u^3}{7}.$$

The control law is designed as

$$\hat{u}\left(z^{0\sim2}\right)=-\tilde{l}_3^{-1}\left(h_3\left(z^{(0\sim2)},t\right)+A_{0\sim2}\begin{bmatrix} z \\ \dot{z} \\ \ddot{z}\end{bmatrix}\right),$$

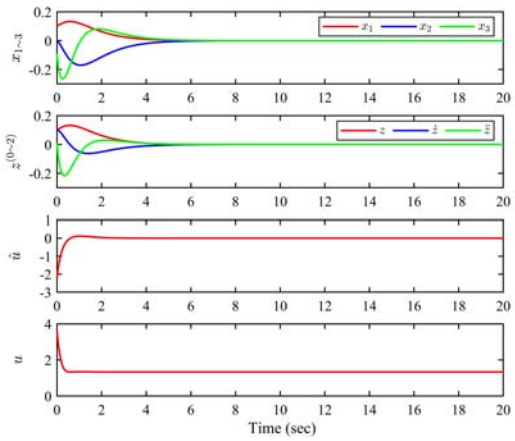


Figure 8.1: State responses and control signal

where

$$\tilde{l}_3 = \left(1+\frac{3}{5}\left(\hat{y}^{\frac{1}{3}}-\frac{5}{3}\hat{y}^{-\frac{1}{3}}\right)^2\right)\left(1+\frac{3}{2}\left(\tilde{y}^{\frac{1}{3}}-\frac{2}{3}\tilde{y}^{-\frac{1}{3}}\right)^2\right).$$

It worth noting that u can be obtained by the inverse function of $\hat{u}$ and it gives that

$$u\left(z^{0\sim 2}\right) = \hat{m}^{\frac{1}{3}} + \hat{m}^{-\frac{1}{3}}\left(0.233\,33\sin\left(0.5z_x\right)+2.333\right),$$

where

$$\hat{m} = 12.25\hat{u}^2 + 1.2704\times 10^{-2}\sin^3\left(0.5z_x\right) + 0.381\,11\sin^2\left(0.5z_x\right) + 3.811\,1\sin\left(0.5z_x\right) + 12.704.$$

The simulation parameters are set to be $A_{0\sim 2} = \begin{bmatrix} 6 & 11 & 6 \end{bmatrix}$ and the related eigenvalues are -1, -2 and -3. The initial conditions for the simulation are given as

$$z(0) = \dot{z}(0) = 0.1, \ddot{z}(0) = 0.01.$$

The simulation results are shown in Fig. 8.1, and the results show that all the transformed states $z^{(0\sim 2)}$ and original system states $x_{1\sim 3}$ converge to the origin with the design controller. ◄

8.4.2 Second-order Example

Consider the following second-order nonaffine SFS:

$$\begin{cases} \ddot{x}_1 = -\frac{1}{3}\dot{x}_1 - x_1^2 + \sinh(x_2) \\ \ddot{x}_2 = \dot{x}_2 + x_1x_2 + (1+x_1^2)x_3 \\ \ddot{x}_3 = x_1^2 + \dot{x}_2x_3 + u + u^3, \end{cases} \tag{8.113}$$

where x_i, $i = 1, 2, 3$, are the system states, u denotes the system input variable. Let

$$\begin{aligned} f_1(x_1^{(0\sim 1)}) =& -\frac{1}{3}\dot{x}_1 - x_1^2, \\ f_2(x_{1\sim 2}^{(0\sim 1)}) =& \dot{x}_2 + x_1 x_2, \\ f_3(x_{1\sim 3}^{(0\sim 1)}) =& x_1^2 + \dot{x}_2 x_3, \\ g_1(x_2) =& \sinh(x_2), \\ g_2(x_1, x_3) =& (1 + x_1^2)x_3, \\ g_3(u) =& u + u^3, \end{aligned}$$

then system (8.113) is clearly a nonaffine SFS in the form of (8.29), which corresponds to the case of $n = 3$. Therefore, Theorem 8.2 can be readily applied to solve the problem.

Note that

$$\frac{\partial g_1(x_2)}{\partial x_2} = [\sinh(x_2)]' = \cosh(x_2), \quad \frac{\partial^2 g_1(x_2)}{\partial x_2^2} = [\sinh(x_2)]'' = \sinh(x_2).$$

Taking the second-order derivatives of both sides of the first equation in (8.113), gives

$$\begin{aligned} x_1^{(4)} =& \ddot{f}_1(x_1^{(0\sim 1)}) + \frac{\partial^2 g_1(x_2)}{\partial x_2^2}(\dot{x}_2)^2 + \frac{\partial g_1(x_2)}{\partial x_2}\ddot{x}_2 \\ =& \ddot{f}_1(x_1^{(0\sim 1)}) + \sinh(x_2)(\dot{x}_2)^2 + \cosh(x_2)\ddot{x}_2. \end{aligned} \tag{8.114}$$

Introduce the transformation $z = x_1$. According to the first equation in (8.113), the state x_2 can be solved by

$$x_2 = g_1^{-1}(z^{(0\sim 2)}, y_1) = \ln\left(y_1 + \sqrt{y_1^2 + 1}\right), \; y_1 = \ddot{z} - f_1(z^{(0\sim 1)}), \tag{8.115}$$

which further gives

$$\dot{x}_2 = \frac{\dot{y}_1}{\sqrt{1 + y_1^2}} \triangleq d_1^{x_2}(z^{(0\sim 3)}), \tag{8.116}$$

with

$$\dot{y}_1 = z^{(3)} - \dot{f}_1(z^{0\sim 1}).$$

Substituting (8.115) and (8.116) into the second equation in (8.113), gives the following form of $\ddot{x}_2$ expressed only by x_1 and its derivatives:

$$\ddot{x}_2 = \frac{\dot{y}_1}{\sqrt{1 + y_1^2}} + z\ln\left(y_1 + \sqrt{y_1^2 + 1}\right) + (1 + z^2)x_3. \tag{8.117}$$

Further, applying $\sinh(x_2) = y_1$, and substituting the above equations (8.116) and (8.117) into (8.114), yield the following fourth-order FAS:

$$\begin{aligned} z^{(4)} &= \ddot{f}_1(z^{(0\sim1)}) + \frac{y_1\dot{y}_1^2}{1+y_1^2} \\ &\quad + \cosh\left(g_1^{-1}(z^{(0\sim2)}, y_1)\right)\left(\frac{\dot{y}_1}{\sqrt{1+y_1^2}} + z\ln\left(y_1+\sqrt{y_1^2+1}\right) + (1+z^2)x_3\right) \\ &= h_2(z^{(0\sim3)}) + l_2(z^{(0\sim3)})x_3, \end{aligned} \tag{8.118}$$

where

$$\begin{aligned} h_2(z^{(0\sim3)}) &= \ddot{f}_1(z^{(0\sim1)}) + \frac{y_1\dot{y}_1^2}{1+y_1^2} \\ &\quad + \cosh\left(g_1^{-1}(z^{(0\sim2)}, y_1)\right)\left(\frac{\dot{y}_1}{\sqrt{1+y_1^2}} + z\ln\left(y_1+\sqrt{y_1^2+1}\right)\right), \\ l_2(z^{(0\sim3)}) &= \cosh\left(g_1^{-1}(z^{(0\sim2)}, y_1)\right)(1+z^2). \end{aligned}$$

Hence, according to the above transformation process, system (8.113) can be expressed as

$$\begin{cases} z^{(4)} = h_2(z^{(0\sim3)}) + l_2(z^{(0\sim3)})x_3 \\ \ddot{x}_3 = z^2 + \dot{x}_2x_3 + u + u^3. \end{cases} \tag{8.119}$$

Next, it follows from the first equation in (8.119) that the state x_3 can be solved by

$$x_3 = \frac{1}{l_2(z^{(0\sim3)})}y_2, \quad y_2 = z^{(4)} - h_2(z^{(0\sim3)}), \tag{8.120}$$

which further gives

$$\dot{x}_3 = \frac{\dot{y}_2l_2(z^{(0\sim3)}) - \dot{l}_2(z^{(0\sim3)})y_2}{l_2^2(z^{(0\sim3)})} \triangleq d_1^{x_3}(z^{(0\sim5)}, t), \tag{8.121}$$

with

$$\dot{y}_2 = z^{(5)} - \dot{h}_2(z^{(0\sim3)}).$$

Taking the second-order derivatives of both sides of the first equation in (8.121), gives

$$z^{(6)} = \ddot{h}_2(z^{(0\sim3)}) + \ddot{l}_2(z^{(0\sim3)})x_3 + 2\dot{l}_2(z^{(0\sim3)})\dot{x}_3 + l_2(z^{(0\sim3)})\ddot{x}_3. \tag{8.122}$$

Then, by combining equations (8.116), (8.121) and the second equation in (8.119), we can obtain the following sixth-order fully-actuated system:

$$\begin{aligned} z^{(6)} = {} & \ddot{h}_2(z^{(0\sim3)}) + \frac{\ddot{l}_2(z^{(0\sim3)})}{l_2(z^{(0\sim3)})}y_2 + 2\dot{l}_2(z^{(0\sim3)})\frac{\dot{y}_2l_2(z^{(0\sim3)}) - \dot{l}_2(z^{(0\sim3)})y_2}{l_2^2(z^{(0\sim3)})} \\ & + l_2(z^{(0\sim3)})\left(z^2 + \frac{\dot{y}_1}{\sqrt{1+y_1^2}}\frac{y_2}{l_2(z^{(0\sim3)})} + u + u^3\right). \end{aligned} \tag{8.123}$$

Based on the above derivation, equation (8.123) is clearly a high-order FAS model that depends solely on z and its derivatives. Let

$$\begin{aligned} h_3(z^{(0\sim5)}) &= \ddot{h}_2(z^{(0\sim3)}) + \frac{\ddot{l}_2(z^{(0\sim3)})}{l_2(z^{(0\sim3)})} y_2 \\ &\quad + 2\dot{l}_2(z^{(0\sim3)}) \frac{\dot{y}_2 l_2(z^{(0\sim3)}) - \dot{l}_2(z^{(0\sim3)}) y_2}{l_2^2(z^{(0\sim3)})} \\ &\quad + l_2(z^{(0\sim3)}) \left(z^2 + \frac{\dot{y}_1}{\sqrt{1+y_1^2}} \frac{y_2}{l_2(z^{(0\sim3)})} \right). \end{aligned}$$

Then the sixth-order fully-actuated system (8.123) can be rewritten as the following simplified form:

$$z^{(6)} = h_3(z^{(0\sim5)}) + l_2(z^{(0\sim3)})(u + u^3). \tag{8.124}$$

Next, let

$$\hat{u} = u + u^3,$$

and design

$$\hat{u} = \frac{1}{l_2(z^{(0\sim3)})} \left(-h_3(z^{(0\sim5)}) - \sum_{i=0}^{5} a_i z^{(i)} + \upsilon \right), \tag{8.125}$$

where a_i, $i = 0 \sim 5$ are a set of parameters, υ is the reference input. Then, substituting the above equation (8.125) into system (8.124), the following linear closed-loop system can be obtained:

$$z^{(6)} + \sum_{i=0}^{5} a_i z^{(i)} = \upsilon. \tag{8.126}$$

According to the Cardano equation, the control input u can be solved by

$$u = \left(\frac{\hat{u}}{2} + \sqrt{\frac{\hat{u}^2}{4} + \frac{1}{27}} \right)^{\frac{1}{3}} + \left(\frac{\hat{u}}{2} - \sqrt{\frac{\hat{u}^2}{4} + \frac{1}{27}} \right)^{\frac{1}{3}}. \tag{8.127}$$

Thus, the state feedback control law can be designed by (8.127) to make system (8.124) a linear system.

In the simulation, we design the desired poles of the constant linear system (8.126) as

$$[s_1 \ s_2 \ s_3 \ s_4 \ s_5 \ s_6] = -[1 \ 2 \ 3 \ 1 \ 3 \ 3],$$

which yields the parameter vector $A_{0\sim5}$ as

$$A_{0\sim5} = [54 \ 189 \ 261 \ 182 \ 68 \ 13].$$

Using control law (8.127) and selecting the initial value $z(0) = z^{(3)}(0) = z^{(5)}(0) = 0.1$, $\dot{z}(0) = \ddot{z}(0) = z^{(4)}(0) = 0.2$, we carried out the simulation results which are shown in Fig. 8.2. Clearly, Fig. 8.2 demonstrates that all the transformed system states $z^{(0\sim5)}$ and original system states $x_{1\sim3}$ converge to the origin, which indicates the control algorithm proposed in this section is effective.

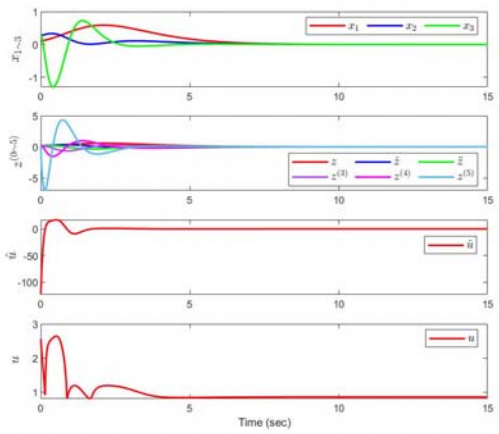


Figure 8.2: State responses and control signals

8.4.3 Mixed-order Example

Consider the following mixed-order nonaffine nonlinear SFS:

$$\begin{cases} \dddot{x}_1 = -\frac{1}{3}\ddot{x}_1 + \dot{x}_1 + \left(1+\zeta x_1^2\right)x_2 \\ \ddot{x}_2 = \sin(x_1) + x_2^2 + \sinh(x_3) + \mathrm{e}^{-t} \\ \dot{x}_3 = x_1 x_2 x_3 + (1 + 0.1\sin(0.5 x_1 x_2 x_3))\,u, \end{cases} \tag{8.128}$$

where $x_i \in \mathbb{R}$, $i = 1, 2, 3$, are system states, $u \in \mathbb{R}$ is the control input.

To obtain the high-order FAS of system (8.128), we employ the following state transformation:

$$\begin{cases} x_1 = z, \\ x_2 = \frac{y_1}{1+\zeta z^2},\ y_1 = \dddot{z} + \frac{1}{3}\ddot{z} - \dot{z}, \\ x_3 = \ln\left(y_2 + \sqrt{y_2^2+1}\right), y_2 = \ddot{x}_2(z) - \sin(z) - x_2^2(z) - \mathrm{e}^{-t}. \end{cases} \tag{8.129}$$

Taking the first-order and second-order derivatives of the first equation in (8.128), yields

$$\begin{cases} x_1^{(4)} = -\frac{1}{3}\dddot{x}_1 + \ddot{x}_1 + 2\zeta x_1 \dot{x}_1 x_2 + \left(1+\zeta x_1^2\right)\dot{x}_2 \\ x_1^{(5)} = -\frac{1}{3}x_1^{(4)} + \dddot{x}_1 + 2\zeta\left(\dot{x}_1^2 + x_1\ddot{x}_1\right)x_2 + 4\zeta x_1 \dot{x}_1 \dot{x}_2 + \left(1+\zeta x_1^2\right)\ddot{x}_2. \end{cases} \tag{8.130}$$

Substituting the second equation of (8.128) into the result, and using $x_1 = z$, give

$$z^{(5)} = H_2\left(z^{(0\sim4)}, \zeta, t\right) + L_2(z, \zeta)\sinh(x_3), \tag{8.131}$$

where

$$\begin{aligned} H_2\left(z^{(0\sim4)},\zeta,t\right) &= -\frac{1}{3}z^{(4)}+\dddot{z}+2\zeta\left(z^2+z\ddot{z}\right)x_2+4\zeta z\dot{z}\dot{x}_2 \\ &\quad +\left(1+\zeta z^2\right)\left(\sin(z)+x_2^2+\mathrm{e}^{-t}\right), \qquad (8.132) \\ L_2(z,\zeta) &= \left(1+\zeta z^2\right). \qquad (8.133) \end{aligned}$$

Taking the first-order derivative of (8.131), yields

$$z^{(6)}=\dot{H}_2\left(z^{(0\sim4)},\zeta,t\right)+\dot{L}_2(z,\zeta)\sinh(x_3)+L_2(z,\zeta)\cosh(x_3)\dot{x}_3. \qquad (8.134)$$

Substituting the third equation of (8.128) into system (8.134), produces

$$z^{(6)}=H_3\left(z^{(0\sim5)},\zeta,t\right)+L_3\left(z^{(0\sim5)},\zeta\right)(1+0.1\sin(0.5x_1x_2x_3))\,u, \qquad (8.135)$$

where

$$\begin{aligned} H_3\left(z^{(0\sim5)},\zeta,t\right) &= -\frac{1}{3}z^{(5)}+z^{(4)}+2\zeta\left(3\dot{z}\ddot{z}+z\dddot{z}\right)x_2 \\ &\quad +2\zeta\left(\dot{z}^2+z\ddot{z}\right)\dot{x}_2+4\zeta\left(\dot{z}^2\dot{x}_2+z\ddot{z}\dot{x}_2+z\dot{z}\ddot{x}_2\right) \\ &\quad +2\zeta x_1\dot{z}\left(\sin(z)+x_2^2+\sinh(x_3)+\mathrm{e}^{-t}\right) \\ &\quad +\left(1+\zeta z^2\right)\left(\cos(z)\dot{z}+2x_2\dot{x}_2-\mathrm{e}^{-t}\right) \\ &\quad +\left(1+\zeta z^2\right)\cosh(x_3)zx_2x_3, \\ L_3\left(z^{(0\sim5)},\zeta\right) &= \left(1+\zeta z^2\right)\cosh(x_3), \end{aligned}$$

in which x_2, x_3 and their derivatives are all given or determined by transformation (8.129).

Once the high-order FAS model (8.135) is obtained, the controller of the system can then be simply designed as

$$\begin{aligned} u &= -\frac{1}{L_3\left(z^{(0\sim5)},\zeta\right)(1+0.1\sin(0.5zx_2x_3))} \\ &\quad \times\left(A_{0\sim5}z^{(0\sim5)}+H_3\left(z^{(0\sim5)},\zeta,t\right)-\upsilon\right) \qquad (8.136) \end{aligned}$$

which gives the following constant linear closed-loop system:

$$z^{(6)}+A_{0\sim5}z^{(0\sim5)}=\upsilon, \qquad (8.137)$$

where υ is some external signal, while $A_{0\sim5}$ is an arbitrary parameter vector which

makes $\Phi(A_{0\sim5})$ stable, $z^{(i)}$, $i = 0 \sim 5$, can be obtained from transformation (8.129) as

$$\begin{cases} z = x_1 \\ \dot{z} = \dot{x}_1 \\ \ddot{z} = \ddot{x}_1 \\ \dddot{z} = -\frac{1}{3}\ddot{x}_1 + \dot{x}_1 + \left(1 + \zeta x_1^2\right) x_2 \\ z^{(4)} = -\frac{1}{3}\dddot{x}_1 + \ddot{x}_1 + 2\zeta x_1 \dot{x}_1 x_2 + \left(1 + \zeta x_1^2\right) \dot{x}_2 \\ z^{(5)} = H_2\left(x_1^{(0\sim4)}, x_2^{(0\sim1)}, \zeta, t\right) + L_2\left(x_1, \zeta\right) \sinh\left(x_3\right), \end{cases} \tag{8.138}$$

in which $H_2\left(x_1^{(0\sim4)}, x_2^{(0\sim1)}, \zeta, t\right)$ and $L_2\left(x_1, \zeta\right)$ are given by (8.132) and (8.133), respectively.

In the simulation test, we set $\zeta = 0.5$, and design the desired poles of the constant linear system (8.137) as

$$[s_1 \ s_2 \ s_3 \ s_4 \ s_5 \ s_6] = -[1.2 \ \ 1.5 \ \ 2.0 \ \ 0.8 \ \ 0.5 \ \ 1.0],$$

which yields the parameter vector $A_{0\sim5}$ as

$$A_{0\sim5} = [1.44 \ \ 9 \ \ 22.4 \ \ 28.55 \ \ 19.71 \ \ 7].$$

Choosing the initial state values of the original system (8.128) as

$$[x_1(0) \ \ \dot{x}_1(0) \ \ \ddot{x}_1(0) \ \ x_2(0) \ \ \dot{x}_2(0) \ \ x_3(0)] = [1.0 \ \ 0 \ \ 0 \ \ 0.5 \ \ 0 \ \ 0.8],$$

the state responses and the control input under the explicit control law in (8.136) are shown in Figure 8.3. It shows that all the system states x_i, $i = 1, 2, 3$, $z^{(i)}$, $i = 0 \sim 5$, and the control input u converge to zero, which demonstrates the efficacy of the proposed approach.

8.5 Notes and References

8.5.1 First-order Nonaffine SFSs

The first section of this chapter has proposed a novel scheme for stabilization control of a class of nonaffine SFSs. The proposed scheme introduces a recursive solution that defines an intermediate function, determines its inverse, and subsequently transforms the nonaffine SFS into an equivalent FAS model. Moreover, two special cases are considered. First, when the i-th control vector function depends solely on the $(i+1)$-th state, the nonaffine SFS can either be directly transformed into a FAS model using the recursive solution or first converted into an affine SFS, followed by applying existing results to obtain an equivalent FAS model. Second, when the nonlinear terms vanish, the nonaffine SFS reduces to the well-known pseudo pure-feedback system, for which a corresponding transformation into a FAS model is also provided. The FAS approach enables direct control law design for the transformed system, offering improved effectiveness over the conventional backstepping method while mitigating the well-known "explosion of complexity" issue in backstepping. Future research will extend this framework to more complex scenarios, such as nonaffine SFSs with dynamic or parametric uncertainties.

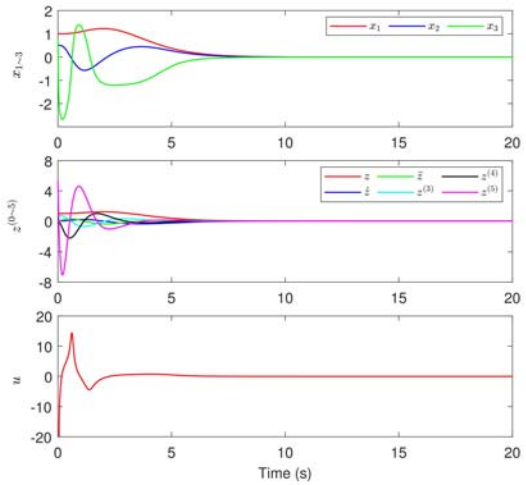


Figure 8.3: State responses and control signal

8.5.2 Second-order Nonaffine SFSs

The synthesis issue of generalized nonaffine second-order SFSs has been investigated in the second section of this chapter under the framework of FAS method. To avoid the complexity of controller design caused by the state space method, a set of appropriate differential homeomorphism transformations has been provided, and based on this, a recursive design scheme has been presented to convert equivalently generalized nonaffine SFSs into HOFASs. By designing a controller for the transformed HOFAS system, a linear steady-state system with the expected characteristic structure has also been obtained. This scheme avoids the differentiation explosion problem commonly caused by classical backstepping.

8.5.3 Mixed-order Nonaffine SFSs

In the third section of this chapter, the problem of synthesis of a class of mixed-order nonaffine SFSs has been studied. It is shown that the nonaffine SFS can be further generalized to mixed orders and the derivatives of the state vectors, and a recursive solution exists to convert the generalized version of SFS equivalently into a high-order FAS model. By using the FAS approach, the nonlinear characteristics in the original system can be eliminated, and an LTI system with the desired characteristic structure can be attained. A simulation example has been presented to demonstrate the effectiveness of the proposed approach. The concept and approach proposed in this series of papers can be naturally generalized to discrete-time systems and time-delay systems. Particularly, the substabilization of the nonaffine nonlinear SFSs will be addressed in separate papers. The proposed FAS approach will definitely provide a powerful tool to deal with the stabilization and control of more complicated dynamical systems.

Chapter 9

High-order Backstepping for Robust Control

The idea and the technology of backstepping design for nonlinear strict feedback systems (SFSs) are indeed very exquisite and ingenious. The trick in tying the virtual controllers of all layers together to guarantee the stability of the designed system is extremely wonderful and magnificent. Fortunately, the idea and technology are extendable to the second-order SFS (6.18) and the mixed-order SFS (8.69). Specifically, we have proposed in Duan (2021c) the robust high-order backstepping designs for both the second-order and the mixed-order SFSs with dynamical uncertainties, and also in Duan (2021d) the adaptive high-order backstepping designs for both the second-order and the mixed-order SFSs with an unknown parameter vector.

The proposed high-order backstepping clearly possesses the following advantages:

1. Since many systems are originally high-order SFSs, high-order backstepping designs are directly applicable, removing the procedure of converting the systems into first-order SFSs.

2. It is well-known that the typical first-order backstepping design has the problem of "differential explosion", especially for SFSs with more layers (a large number of subsystems). When a high-order SFS is converted into a first-order one, the number of layers increases several-fold. Applying high-order backstepping designs would greatly reduce the problem of "differential explosion".

Obviously, the invention of high-order backstepping designs is clearly due to the high-order FAS approach. Like the main results in Chapters 6, the techniques of high-order backstepping produced in this chapter and the next would not be produced within the state-space approach framework.

9.1 Uncertain High-order SFSs

9.1.1 Uncertain Second-order SFSs

Physical laws such as Lagrangian Equations and Theorem of Linear and Angular Momentum are often used to model physical systems. In such applications, the obtained subsystem models are generally of second-order (Duan (2020a, 2021a,b)). Due to this fact, we introduce the following second-order strict-feedback nonlinear system with uncertainties (refer also to Duan (2021b)):

$$\begin{cases} \ddot{x}_1 = f_1(x_1,\dot{x}_1) + \Delta f_1(x_1,\dot{x}_1) + G_1(x_1,\dot{x}_1)x_2 \\ \ddot{x}_2 = f_2(x_{1\sim 2}^{(0\sim 1)}) + \Delta f_2(x_{1\sim 2}^{(0\sim 1)}) + G_2(x_{1\sim 2}^{(0\sim 1)})x_3 \\ \qquad\vdots \\ \ddot{x}_{n-1} = f_{n-1}(x_{1\sim n-1}^{(0\sim 1)}) + \Delta f_{n-1}(x_{1\sim n-1}^{(0\sim 1)}) + G_{n-1}(x_{1\sim n-1}^{(0\sim 1)})x_n \\ \ddot{x}_n = f_n(x_{1\sim n}^{(0\sim 1)}) + \Delta f_n(x_{1\sim n}^{(0\sim 1)}) + G_n(x_{1\sim n}^{(0\sim 1)})u, \end{cases} \tag{9.1}$$

where $x_i \in \mathbb{R}^r$, $i = 1,2,\ldots,n$, are the state vectors, $u \in \mathbb{R}^r$ is the control input vector, $f_i(x_{1\sim i}^{(0\sim 1)}) \in \mathbb{R}^r$, $i = 1,2,\ldots,n$, are a set of sufficiently smooth vector functions, while $\Delta f_i(x_{1\sim i}^{(0\sim 1)}) \in \mathbb{R}^r$, $i = 1,2,\ldots,n$, are a set of sufficiently smooth unknown vector functions. The coefficient matrices $G_i(x_{1\sim i}^{(0\sim 1)}) \in \mathbb{R}^{r\times r}$, $i = 1,2,\ldots,n$, are a set of sufficiently smooth matrix functions satisfying the following full-actuation assumption:

Assumption 9.1 $\det G_i(x_{1\sim i}^{(0\sim 1)}) \neq 0,\ \forall x_i \in \mathbb{R}^r,\ i = 1,2,\ldots,n.$

In the case that the nonlinear uncertainties in the above second-order SFS do not appear, it has been shown in Duan (2021b) that the model can be equivalently converted into the form of (5.1). However, when nonlinear uncertainties do exist, in some cases, an equivalent single FAS model in the form of (5.2) may not be able to be derived, or in other words, the derived FAS model may be too conservative in the sense that too much certain information has to be included into the uncertain terms $\Delta f_i(x_{1\sim i}^{(0\sim 1)})$, $i = 1,2,\ldots,n$, since it is sometimes hard to separate the certain part from the uncertain terms. Therefore, in certain applications, it may be more favorable to treat directly the set of models in (9.1), or, only convert a few of the subsystems in (9.1) into several FAS models in the form of (5.2). After all, all the subsystems in (9.1), as they stand, are a set of cascade fully actuated subsystems, when the x_{i+1} in the i-th subsystem is looked upon as the control variable.

It is worth pointing out that the above second-order SFS (9.1) also has the following companion form:

$$\begin{cases} \ddot{x}_1 = f_1(x_1,\dot{x}_1) + \Delta f_1(x_1,\dot{x}_1) + G_1(x_1,\dot{x}_1)\dot{x}_2 \\ \ddot{x}_2 = f_2(x_{1\sim 2}^{(0\sim 1)}) + \Delta f_2(x_{1\sim 2}^{(0\sim 1)}) + G_2(x_{1\sim 2}^{(0\sim 1)})\dot{x}_3 \\ \qquad\vdots \\ \ddot{x}_{n-1} = f_{n-1}(x_{1\sim n-1}^{(0\sim 1)}) + \Delta f_{n-1}(x_{1\sim n-1}^{(0\sim 1)}) + G_{n-1}(x_{1\sim n-1}^{(0\sim 1)})\dot{x}_n \\ \ddot{x}_n = f_n(x_{1\sim n}^{(0\sim 1)}) + \Delta f_n(x_{1\sim n}^{(0\sim 1)}) + G_n(x_{1\sim n}^{(0\sim 1)})u. \end{cases} \tag{9.2}$$

As a matter of fact, the last term $G_i(x_{1\sim i}^{(0\sim 1)})\dot{x}_{i+1}$ in each subsystem can be further generalized into

$$G_{0i}(x_{1\sim i}^{(0\sim 1)})x_{i+1} + G_{1i}(x_{1\sim i}^{(0\sim 1)})\dot{x}_{i+1},$$

with $\left[G_{0i}(x_{1\sim i}^{(0\sim 1)}) \quad G_{1i}(x_{1\sim i}^{(0\sim 1)})\right]$ being of full rank.

9.1.2 Uncertain Mixed-order SFSs

Parallel to the uncertain second-order SFS (9.1), we introduce the following high-order (mixed-order) strict-feedback nonlinear systems with unknown nonlinearities:

$$\begin{cases} x_1^{(m_1)} = f_1(x_1^{(0\sim m_1-1)}) + \Delta f_1(x_1^{(0\sim m_1-1)}) + G_1(x_1^{(0\sim m_1-1)})x_2 \\ x_2^{(m_2)} = f_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2}) + \Delta f_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2}) + G_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2})x_3 \\ \quad\vdots \\ x_{n-1}^{(m_{n-1})} = f_{n-1}(x_i^{(0\sim m_i-1)}|_{i=1\sim n-1}) \\ \qquad\qquad + \Delta f_{n-1}(x_i^{(0\sim m_i-1)}|_{i=1\sim n-1}) + G_{n-1}(x_i^{(0\sim m_i-1)}|_{i=1\sim n-1})x_n \\ x_n^{(m_n)} = f_n(x_i^{(0\sim m_i-1)}|_{i=1\sim n}) + \Delta f_n(x_i^{(0\sim m_i-1)}|_{i=1\sim n}) + G_n(x_i^{(0\sim m_i-1)}|_{i=1\sim n})u, \end{cases} \tag{9.3}$$

where $m_i, i = 1, 2, \ldots, n$, are a set of positive integers, $f_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k})$, $\Delta f_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k}) \in \mathbb{R}^r, k = 1, 2, \ldots, n$, are two sets of sufficiently smooth vector functions, $G_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k}) \in \mathbb{R}^{r\times r}$, $k = 1, 2, \ldots, n$, are a set of sufficiently smooth matrix functions satisfying the following full-actuation assumption:

Assumption 9.2 $\det G_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k}) \neq 0,\ \forall x_i \in \mathbb{R}^r,\ k = 1, 2, \ldots, n.$

Regarding the derivation of the above uncertain mixed-order SFS (9.3), we make the following statements:

- In the modeling of physical systems, we often use physical laws such as Lagrangian Equations and Theorem of Linear and Angular Momentum. In such cases the original models of the obtained subsystems are of second-order. On the other side, there are also subsystems modelled by other physical laws, such as the well-known Hamiltonian Theorem, and the obtained original models are of first-order. Such operations give a mixed-order SFS (9.3), with m_i, $i = 1, 2, \ldots, n$, taking the values of 1 or 2.

- It has been pointed out above that the mixed-order SFS may be obtained from the second-order SFS by converting equivalently some of the subsystems in (9.1) into several FAS models in the form of (5.2). Meanwhile, some of the subsystems in (9.1) can also be converted into several first-order equations. Both operations give mixed-order SFSs.

Parallelly, the above mixed-order SFS (9.3) also has a companion form which appears as follows:

$$\left\{\begin{array}{l} x_1^{(m_1)} = f_1(x_1^{(0\sim m_1-1)}) + \Delta f_1(x_1^{(0\sim m_1-1)}) + G_1(x_1^{(0\sim m_1-1)})\dot{x}_2 \\ x_2^{(m_2)} = f_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2}) + \Delta f_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2}) + G_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2})\dot{x}_3 \\ \qquad\vdots \\ x_{n-1}^{(m_{n-1})} = f_{n-1}(x_i^{(0\sim m_i-1)}|_{i=1\sim n-1}) \\ \qquad + \Delta f_{n-1}(x_i^{(0\sim m_i-1)}|_{i=1\sim n-1}) + G_{n-1}(x_i^{(0\sim m_i-1)}|_{i=1\sim n-1})\dot{x}_n \\ x_n^{(m_n)} = f_n(x_i^{(0\sim m_i-1)}|_{i=1\sim n}) + \Delta f_n(x_i^{(0\sim m_i-1)}|_{i=1\sim n}) + G_n(x_i^{(0\sim m_i-1)}|_{i=1\sim n})u. \end{array}\right. \tag{9.4}$$

It is reminded that this form can be often converted into the form of (9.3) by lowering the orders of the subsystems. A by-product of doing this is that the number of equations in the system will be increased.

9.2 Backstepping for Second-order SFSs

Based on Theorem 5.1, in this section we give a direct backstepping design method of the second-order SFS (9.1) without turning the system into a conventional first-order SFS. To do this, we need to make some preparations.

Suppose $A_i^{0\sim 1} \in \mathbb{R}^{1\times 2}$, $i = 1, 2, \ldots, n$, are a set of matrices satisfying

$$\operatorname{Re}\lambda_k\left(\Phi\left(A_i^{0\sim 1}\right)\right) \le -\frac{\mu}{2}, \quad k = 1, 2, \tag{9.5}$$

where μ is an arbitrarily given positive number, and

$$P_i\left(A_i^{0\sim 1}\right) = \left[P_{iF}\left(A_i^{0\sim 1}\right) \quad P_{iL}\left(A_i^{0\sim 1}\right)\right] \in \mathbb{R}^{2\times 2}, \tag{9.6}$$

is a positive definite solution to the matrix inequality

$$\Phi^{\mathrm{T}}\left(A_i^{0\sim 1}\right)P_i\left(A_i^{0\sim 1}\right) + P_i\left(A_i^{0\sim 1}\right)\Phi\left(A_i^{0\sim 1}\right) \le -\gamma\, P_i\left(A_i^{0\sim 1}\right), \tag{9.7}$$

for $i = 1, 2, \ldots, n$, and some positive number γ. Further, define, for $i = 1, 2, \ldots, n$,

$$\tilde{P}_i\left(A_i^{0\sim 1}\right) = I_2^{\circ} P_i\left(A_i^{0\sim 1}\right),$$

and denote

$$\begin{aligned} \tilde{P}_i^{-1}\left(A_i^{0\sim 1}\right) &= \left[\begin{array}{c} Q_{i1}\left(A_i^{0\sim 1}\right) \\ Q_{i2}\left(A_i^{0\sim 1}\right) \end{array}\right] \\ &= \left[\begin{array}{cc} & Q_{i1}\left(A_i^{0\sim 1}\right) \\ Q_{iF}\left(A_i^{0\sim 1}\right) & Q_{iL}\left(A_i^{0\sim 1}\right) \end{array}\right], \end{aligned}$$

where $Q_{iF}\left(A_i^{0\sim 1}\right)$ and $Q_{iL}\left(A_i^{0\sim 1}\right)$ are two scalars, and it can be easily shown that

$$Q_{iL}\left(A_i^{0\sim 1}\right) \ne 0, \quad i = 1, 2, \ldots, n. \tag{9.8}$$

It is worth pointing out that all the variables defined above are independent of system (9.1), and can all be easily obtained beforehand.

9.2.1 The First Step

Let

$$z_1^{(0\sim 1)} = x_1^{(0\sim 1)}, \tag{9.9}$$

and

$$\tilde{P}_2\left(A_2^{0\sim 1}\right) z_2^{(0\sim 1)} = x_2^{(0\sim 1)} - \begin{bmatrix} \alpha_1 \\ 0 \end{bmatrix}. \tag{9.10}$$

In view of the symmetry property of $P_2\left(A_2^{0\sim 1}\right)$, the above equation (9.10) can be equivalently decomposed into

$$P_{2L}^{\mathrm{T}}\left(A_2^{0\sim 1}\right) z_2^{(0\sim 1)} = x_2 - \alpha_1, \tag{9.11}$$

and

$$P_{2F}^{\mathrm{T}}\left(A_2^{0\sim 1}\right) z_2^{(0\sim 1)} = \dot{x}_2. \tag{9.12}$$

Therefore, by using (9.9) and (9.11), the first equation in (9.1) is transformed into

$$\ddot{z}_1 = \hat{f}_1(z_1, \dot{z}_1) + \Delta\, \hat{f}_1(z_1, \dot{z}_1) + L_1(z, \dot{z}_1)\left(P_{2L}^{\mathrm{T}}\left(A_2^{0\sim 1}\right) z_2^{(0\sim 1)} + \alpha_1\right), \tag{9.13}$$

where

$$\begin{cases} \hat{f}_1(z_1, \dot{z}_1) = f_1(z_1, \dot{z}_1) \\ \Delta\, \hat{f}_1(z_1, \dot{z}_1) = \Delta f_1(z_1, \dot{z}_1) \\ L_1(z_1, \dot{z}_1) = G_1(z_1, \dot{z}_1). \end{cases} \tag{9.14}$$

If there exists a non-negative scalar continuous function $\rho_1(z_1, \dot{z}_1)$ such that

$$|\Delta f_1(z_1, \dot{z}_1)| \le \rho_1(z_1, \dot{z}_1),$$

then, by applying Theorem 5.1 to the above system (9.13), the first virtual control law can be designed as

$$\begin{cases} \alpha_1 = -L_1^{-1}(z_1, \dot{z}_1)\left(A_1^{0\sim 1} z_1^{(0\sim 1)} + \alpha_1^*\right) \\ \alpha_1^* = \hat{f}_1(z_1, \dot{z}_1) + \frac{1}{4\varepsilon}\rho_1^2(z_1, \dot{z}_1) P_{1L}^{\mathrm{T}}(A_1^{0\sim 1}) z_1^{(0\sim 1)}. \end{cases} \tag{9.15}$$

The closed-loop subsystem resulted in by this control law is

$$\ddot{z}_1 + A_1^{0\sim 1} z_1^{(0\sim 1)} = \phi_1 + w_1, \tag{9.16}$$

with

$$\phi_1 = -\frac{1}{4\varepsilon}\rho_1^2(z_1^{(0\sim 1)}) P_{1L}^{\mathrm{T}}(A_1^{0\sim 1}) z_1^{(0\sim 1)} + \Delta\, \hat{f}_1(z_1^{(0\sim 1)}), \tag{9.17}$$

and

$$w_1 = L_1(z_1, \dot{z}_1) P_{2L}^{\mathrm{T}}\left(A_2^{0\sim 1}\right) z_2^{(0\sim 1)}. \tag{9.18}$$

9.2.2 The Second Step

The virtual control law α_1 given by (9.15) is a function with respect to z_1 and $\dot{z}_1$, that is,

$$\alpha_1 = \alpha_1(z_1, \dot{z}_1).$$

Thus we have, using the first equation in (9.1) again,

$$\begin{aligned} \dot{\alpha}_1 &= \frac{\partial \alpha_1}{\partial z_1}\dot{z}_1 + \frac{\partial \alpha_1}{\partial \dot{z}_1}\ddot{z}_1 \\ &= h_1(z_{1\sim 2}^{(0\sim 1)}) + \Delta h_1(z_{1\sim 2}^{(0\sim 1)}), \end{aligned} \tag{9.19}$$

with

$$\begin{aligned} h_1 &= \frac{\partial \alpha_1}{\partial z_1}\dot{z}_1 + \frac{\partial \alpha_1}{\partial \dot{z}_1}(f_1(z_1, \dot{z}_1) + G_1 x_2), \\ \Delta h_1 &= \frac{\partial \alpha_1}{\partial \dot{z}_1}\Delta f_1(z_1, \dot{z}_1). \end{aligned}$$

On the other hand, we obtain, from (9.10),

$$z_2^{(0\sim 1)} = \tilde{P}_2^{-1}\left(A_2^{0\sim 1}\right)\left(x_2^{(0\sim 1)} - \begin{bmatrix} \alpha_1 \\ 0 \end{bmatrix}\right), \tag{9.20}$$

which gives

$$\begin{aligned} \dot{z}_2 &= Q_{22}\left(A_2^{0\sim 1}\right)\left(x_2^{(0\sim 1)} - \begin{bmatrix} \alpha_1 \\ 0 \end{bmatrix}\right) \\ &= Q_{2F}\left(A_2^{0\sim 1}\right)(x_2 - \alpha_1) + Q_{2L}\left(A_2^{0\sim 1}\right)\dot{x}_2. \end{aligned}$$

Further taking differentials of the above equation, and using (9.19) and the second equation in (9.1), yield

$$\begin{aligned} \ddot{z}_2 &= Q_{2F}\left(A_2^{0\sim 1}\right)(\dot{x}_2 - \dot{\alpha}_1) + Q_{2L}\left(A_2^{0\sim 1}\right)\ddot{x}_2 \\ &= Q_{2F}\left(A_2^{0\sim 1}\right)(\dot{x}_2 - h_1(z_1, \dot{z}_1, z_2) - \Delta h_1(z_1, \dot{z}_1)) \\ &\quad + Q_{2L}\left(A_2^{0\sim 1}\right)\left(f_2(x_{1\sim 2}^{(0\sim 1)}) + \Delta f_2(x_{1\sim 2}^{(0\sim 1)}) + G_2(x_{1\sim 2}^{(0\sim 1)})x_3\right) \\ &= \hat{f}_2(z_{1\sim 2}^{(0\sim 1)}) + \Delta \hat{f}_2(z_{1\sim 2}^{(0\sim 1)}) + L_2(x_{1\sim 2}^{(0\sim 1)})x_3, \end{aligned} \tag{9.21}$$

where

$$\begin{cases} \hat{f}_2(z_{1\sim 2}^{(0\sim 1)}) = Q_{2F}\left(A_2^{0\sim 1}\right)(\dot{x}_2 - h_1(z_1, \dot{z}_1, z_2)) + Q_{2L}\left(A_2^{0\sim 1}\right) f_2(x_{1\sim 2}^{(0\sim 1)}) \\ \Delta \hat{f}_2(z_{1\sim 2}^{(0\sim 1)}) = -Q_{2F}\left(A_2^{0\sim 1}\right)\Delta h_1(z_1, \dot{z}_1) + Q_{2L}\left(A_2^{0\sim 1}\right)\Delta f_2(x_{1\sim 2}^{(0\sim 1)}) \\ L_2(x_{1\sim 2}^{(0\sim 1)}) = Q_{2L}\left(A_2^{0\sim 1}\right)G_2(x_{1\sim 2}^{(0\sim 1)}). \end{cases} \tag{9.22}$$

Further let

$$\tilde{P}_3\left(A_3^{0\sim 1}\right)z_3^{(0\sim 1)} = x_3^{(0\sim 1)} - \begin{bmatrix} \alpha_2 \\ 0 \end{bmatrix}, \tag{9.23}$$

which can be equivalently decomposed, again in view of the symmetry property of $P_3\left(A_3^{0\sim1}\right)$, into

$$P_{3L}^{\mathrm{T}}\left(A_3^{0\sim1}\right)z_3^{(0\sim1)}=x_3-\alpha_2, \tag{9.24}$$

and

$$P_{3F}^{\mathrm{T}}\left(A_3^{0\sim1}\right)z_3^{(0\sim1)}=\dot{x}_3. \tag{9.25}$$

Substituting (9.24) into (9.21), gives

$$\ddot{z}_2=\hat{f}_2\left(z_{1\sim2}^{(0\sim1)}\right)+\Delta\hat{f}_2\left(z_{1\sim2}^{(0\sim1)}\right)+L_2(z_{1\sim2}^{(0\sim1)})\left(P_{3L}^{\mathrm{T}}\left(A_3^{0\sim1}\right)z_3^{(0\sim1)}+\alpha_2\right). \tag{9.26}$$

It follows from (9.8) that $L_2(z_{1\sim2}^{(0\sim1)})\neq0,\forall z_{1\sim2}^{(0\sim1)}\in\mathbb{R}^4$. If there exists a continuous function $\rho_2(z_{1\sim2}^{(0\sim1)})$ satisfying

$$\left|\Delta\hat{f}_2(z_{1\sim2}^{(0\sim1)})\right|\leqslant\rho_2(z_{1\sim2}^{(0\sim1)}), \tag{9.27}$$

then, applying Theorem 5.1 to system (9.26) gives the second virtual control law of the system as

$$\begin{cases}\alpha_2=-L_2^{-1}(z_{1\sim2}^{(0\sim1)})\left(A_2^{0\sim1}z_2^{(0\sim1)}+\alpha_2^*\right)\\ \alpha_2^*=\frac{1}{4\varepsilon}\rho_2^2(z_{1\sim2}^{(0\sim1)})P_{2L}^{\mathrm{T}}(A_2^{0\sim1})z_2^{(0\sim1)}+\hat{f}_2(z_{1\sim2}^{(0\sim1)})+v_1,\end{cases} \tag{9.28}$$

where

$$v_1=L_1(z_1,\dot{z}_1)P_{1L}^{\mathrm{T}}\left(A_1^{0\sim1}\right)z_1^{(0\sim1)}. \tag{9.29}$$

The closed-loop subsystem resulted in by this control law is

$$\ddot{z}_2+A_2^{0\sim1}z_2^{(0\sim1)}=\phi_2+w_2-v_1, \tag{9.30}$$

where

$$\phi_2=-\frac{1}{4\varepsilon}\rho_2^2(z_{1\sim2}^{(0\sim1)})P_{2L}^{\mathrm{T}}(A_2^{0\sim1})z_2^{(0\sim1)}+\Delta\hat{f}_2(z_{1\sim2}^{(0\sim1)}), \tag{9.31}$$

and

$$w_2=L_2(z_{1\sim2}^{(0\sim1)})P_{3L}^{\mathrm{T}}\left(A_3^{0\sim1}\right)z_3^{(0\sim1)}. \tag{9.32}$$

9.2.3 The Third Step

The function α_2 defined by (9.28) is the function of $z_{1\sim2}^{(0\sim1)}$, that is,

$$\alpha_2=\alpha_2\left(z_1,\dot{z}_1;z_2,\dot{z}_2\right). \tag{9.33}$$

Similar to the second step, taking derivative of the above equation, and further separating its certain part and uncertain part, yield

$$\begin{aligned}
\dot{\alpha}_2 &= \frac{\partial \alpha_2}{\partial z_{1\sim 2}}\dot{z}_{1\sim 2}+\frac{\partial \alpha_2}{\partial \dot{z}_1}\ddot{z}_1+\frac{\partial \alpha_2}{\partial \dot{z}_2}\ddot{z}_2 \\
&= \frac{\partial \alpha_2}{\partial z_{1\sim 2}}\dot{z}_{1\sim 2}+\frac{\partial \alpha_2}{\partial \dot{z}_1}(f_1(z_1,\dot{z}_1)+\Delta f_1(z_1,\dot{z}_1)+G_1x_2) \\
&\quad +\frac{\partial \alpha_2}{\partial \dot{z}_2}\left(\hat{f}_2\left(z_{1\sim 2}^{(0\sim 1)}\right)+\Delta \hat{f}_2\left(z_{1\sim 2}^{(0\sim 1)}\right)+L_2x_3\right) \\
&= \frac{\partial \alpha_2}{\partial z_{1\sim 2}}\dot{z}_{1\sim 2}+\frac{\partial \alpha_2}{\partial \dot{z}_1}(f_1(z_1,\dot{z}_1)+G_1x_2) \\
&\quad +\frac{\partial \alpha_2}{\partial \dot{z}_2}\left(\hat{f}_2\left(z_{1\sim 2}^{(0\sim 1)}\right)+L_2x_3\right)+\frac{\partial \alpha_2}{\partial \dot{z}_1}\Delta f_1(z_1,\dot{z}_1)+\frac{\partial \alpha_2}{\partial \dot{z}_2}\Delta \hat{f}_2\left(z_{1\sim 2}^{(0\sim 1)}\right) \\
&= h_2\left(z_{1\sim 3}^{(0\sim 1)}\right)+\Delta h_2\left(z_{1\sim 3}^{(0\sim 1)}\right), \qquad (9.34)
\end{aligned}$$

where

$$\begin{aligned}
h_2\left(z_{1\sim 3}^{(0\sim 1)}\right) &= \frac{\partial \alpha_2}{\partial z_{1\sim 2}}\dot{z}_{1\sim 2}+\frac{\partial \alpha_2}{\partial \dot{z}_1}(f_1(z_1,\dot{z}_1)+G_1x_2) \\
&\quad +\frac{\partial \alpha_2}{\partial \dot{z}_2}\left(\hat{f}_2\left(z_{1\sim 2}^{(0\sim 1)}\right)+L_2x_3\right), \\
\Delta h_2\left(z_{1\sim 3}^{(0\sim 1)}\right) &= \frac{\partial \alpha_2}{\partial \dot{z}_1}\Delta f_1(z_1,\dot{z}_1)+\frac{\partial \alpha_2}{\partial \dot{z}_2}\Delta \hat{f}_2\left(z_{1\sim 2}^{(0\sim 1)}\right).
\end{aligned}$$

Please note that $\Delta h_2(z_{1\sim 3}^{(0\sim 1)})$ is a function directly related to the unmodeled items Δf_1 and Δf_2.

On the other hand, we can obtain, from (9.23),

$$z_3^{(0\sim 1)}=\tilde{P}_3^{-1}\left(A_3^{0\sim 1}\right)\left(x_3^{(0\sim 1)}-\begin{bmatrix}\alpha_2\\0\end{bmatrix}\right), \qquad (9.35)$$

which gives

$$\dot{z}_3=Q_{3F}\left(A_3^{0\sim 1}\right)(x_3-\alpha_2)+Q_{3L}\left(A_3^{0\sim 1}\right)\dot{x}_3.$$

Taking differentials of the above equation, and using (9.34) and the third equation in (9.1), we obtain

$$\ddot{z}_3=\hat{f}_3(z_{1\sim 3}^{(0\sim 1)})+\Delta \hat{f}_3(z_{1\sim 3}^{(0\sim 1)})+L_3(z_{1\sim 3}^{(0\sim 1)})x_4, \qquad (9.36)$$

Further, introduce the transformation

$$\tilde{P}_4\left(A_4^{0\sim 1}\right)z_4^{(0\sim 1)}=x_4^{(0\sim 1)}-\begin{bmatrix}\alpha_3\\0\end{bmatrix},$$

which can be divided into

$$\begin{aligned}
P_{4L}^{\mathsf{T}}\left(A_4^{0\sim 1}\right)z_4^{(0\sim 1)} &= x_4-\alpha_3, \qquad (9.37)\\
P_{4F}^{\mathsf{T}}\left(A_4^{0\sim 1}\right)z_4^{(0\sim 1)} &= \dot{x}_4. \qquad (9.38)
\end{aligned}$$

Thus we have, from (9.37),

$$x_4 = P_{4L}^{\mathrm{T}}\left(A_4^{0\sim1}\right) z_4^{(0\sim1)} + \alpha_3. \tag{9.39}$$

Submitting this into (9.36), gives

$$\ddot{z}_3 = \hat{f}_3(z_{1\sim3}^{(0\sim1)}) + \Delta\,\hat{f}_3(z_{1\sim3}^{(0\sim1)}) + L_3(z_{1\sim3}^{(0\sim1)})\left(P_{4L}^{\mathrm{T}}\left(A_4^{0\sim1}\right) z_4^{(0\sim1)} + \alpha_3\right). \tag{9.40}$$

Assume that there exists a non-negative continuous function $\rho_3(z_{1\sim3}^{(0\sim1)})$ satisfying

$$\left|\Delta\,\hat{f}_3(z_{1\sim3}^{(0\sim1)})\right| \leqslant \rho_3(z_{1\sim3}^{(0\sim1)}), \tag{9.41}$$

then, by applying Theorem 5.1 to system (9.40), the third virtual control law of the system can be designed as

$$\begin{cases} \alpha_3 = -L_3^{-1}(x_{1\sim3}^{(0\sim1)})\left(A_3^{0\sim1} z_3^{(0\sim1)} + \alpha_3^*\right) \\ \alpha_3^* = \frac{1}{4\varepsilon}\rho_3^2(x_{1\sim3}^{(0\sim1)}) P_{3L}^{\mathrm{T}}(A_3^{0\sim1}) z_3^{(0\sim1)} + \hat{f}_3\left(x_{1\sim3}^{(0\sim1)}\right) - w_2 + v_2, \end{cases} \tag{9.42}$$

where

$$v_2 = L_2(x_{1\sim2}^{(0\sim1)}) P_{2L}^{\mathrm{T}}\left(A_2^{0\sim1}\right) z_2^{(0\sim1)}. \tag{9.43}$$

The closed-loop subsystem resulted in by this control law is

$$\ddot{z}_3 + A_3^{0\sim1} z_3^{(0\sim1)} = \phi_3 + w_3 - v_2, \tag{9.44}$$

where

$$\phi_3 = -\frac{1}{4\varepsilon}\rho_3^2(x_{1\sim3}^{(0\sim1)}) P_{3L}^{\mathrm{T}}(A_3^{0\sim1}) z_3^{(0\sim1)} + \Delta\,\hat{f}_3(x_{1\sim3}^{(0\sim1)}), \tag{9.45}$$

and

$$w_3 = L_3(x_{1\sim3}^{(0\sim1)}) P_{4L}^{\mathrm{T}}\left(A_4^{0\sim1}\right) z_4^{(0\sim1)}. \tag{9.46}$$

9.2.4 The n-th Step

Continue the above process, and suppose that the expressions of α_{n-1}, $\dot{\alpha}_{n-1}$, $h_{n-1}(z_{1\sim n}^{(0\sim1)})$ and $\Delta h_{n-1}(z_{1\sim n}^{(0\sim1)})$ have all been obtained. Then, similar to the previous treatment, we can obtain

$$\ddot{z}_n = \hat{f}_n(z_{1\sim n}^{(0\sim1)}) + \Delta\,\hat{f}_n(z_{1\sim n}^{(0\sim1)}) + L_n(z_{1\sim n-1}^{(0\sim1)})u. \tag{9.47}$$

Assume that there exist continuous functions $\rho_i(z_{1\sim i}^{(0\sim1)})$, $i = 1, 2, \ldots, n$, satisfying

$$\left|\Delta\,\hat{f}_i(z_{1\sim i}^{(0\sim1)})\right| \leqslant \rho_i(z_{1\sim i}^{(0\sim1)}),\ i = 1, 2, \ldots, n, \tag{9.48}$$

then, by applying Theorem 5.1 to system (9.47), the control law of system (9.1) can be finally designed as

$$\begin{cases} u = -L_n^{-1}(z_{1\sim n}^{(0\sim1)})\left(A_n^{0\sim1} z_n^{(0\sim1)} + u^*\right) \\ u^* = \frac{1}{4\varepsilon}\rho_n^2(z_{1\sim n}^{(0\sim1)}) P_{nL}^{\mathrm{T}}(A_n^{0\sim1}) z_n^{(0\sim1)} + \hat{f}_n\left(z_{1\sim n}^{(0\sim1)}\right) + v_{n-1}, \end{cases} \tag{9.49}$$

where

$$v_{n-1} = L_{n-1}(z_{1\sim n-1}^{(0\sim 1)})P_{n-1,L}^{\mathrm{T}}\left(A_{n-1}^{0\sim 1}\right)z_{n-1}^{(0\sim 1)}. \tag{9.50}$$

The last closed-loop subsystem resulted in by this control law is

$$\ddot{z}_n + A_n^{0\sim 1}z_n^{(0\sim 1)} = \phi_n - v_{n-1}, \tag{9.51}$$

where

$$\phi_n = -\frac{1}{4\varepsilon}\rho_n^2(z_{1\sim n}^{(0\sim 1)})P_{nL}^{\mathrm{T}}(A_n^{0\sim 1})z_n^{(0\sim 1)} + \Delta\,\hat{f}_n(z_{1\sim n}^{(0\sim 1)}). \tag{9.52}$$

9.2.5 The Conclusion

The above process defines a transformation between the $x_{1\sim n}^{(0\sim 1)}$ and $z_{1\sim n}^{(0\sim 1)}$, as follows:

$$\begin{cases} z_1^{(0\sim 1)} = x_1^{(0\sim 1)} \\ \tilde{P}_i\left(A_i^{0\sim 1}\right)z_i^{(0\sim 1)} = x_i^{(0\sim 1)} - \begin{bmatrix} \alpha_{i-1}(x_{1\sim i-1}) \\ 0 \end{bmatrix}, \\ \qquad\qquad i = 2,3,\ldots,n. \end{cases} \tag{9.53}$$

In order to prove that the above control law (9.49) stabilizes system (9.1), we define the following Lyapunov function:

$$V = \sum_{k=1}^{n} V_k, \tag{9.54}$$

with

$$V_i = \left(z_i^{(0\sim 1)}\right)^{\mathrm{T}}P_i(A_i^{0\sim 1})z_i^{(0\sim 1)},\ i = 1,2,\ldots,n. \tag{9.55}$$

Following the proof of Theorem 5.1, it is easy to obtain

$$\begin{cases} \dot{V}_1 \le -\mu V_1 + \varepsilon + 2\left(z_1^{(0\sim 1)}\right)^{\mathrm{T}}P_{1L}(A_1^{0\sim 1})w_1 \\ \dot{V}_2 \le -\mu V_2 + \varepsilon + 2\left(z_2^{(0\sim 1)}\right)^{\mathrm{T}}P_{2L}(A_2^{0\sim 1})(w_2 - v_1) \\ \quad\vdots \\ \dot{V}_{n-1} \le -\mu V_{n-1} + \varepsilon + 2\left(z_{n-1}^{(0\sim 1)}\right)^{\mathrm{T}}P_{n-1,L}(A_{n-1}^{0\sim 1})(w_{n-1} - v_{n-2}) \\ \dot{V}_n \le -\mu V_n + \varepsilon + 2\left(z_n^{(0\sim 1)}\right)^{\mathrm{T}}P_{nL}(A_{n-1}^{0\sim 1})(-v_{n-1}). \end{cases} \tag{9.56}$$

In view of

$$\begin{cases} v_i = L_i\left(x_i^{(0\sim 1)}\right)P_{iL}^{\mathrm{T}}\left(A_i^{0\sim 1}\right)z_i^{(0\sim 1)} \\ w_i = L_i\left(x_i^{(0\sim 1)}\right)P_{i+1,L}^{\mathrm{T}}\left(A_{i+1}^{0\sim 1}\right)z_{i+1}^{(0\sim 1)},\ i = 1,2,\ldots,n-1, \end{cases} \tag{9.57}$$

we have

$$\begin{aligned}
&\left(z_i^{(0\sim 1)}\right)^{\mathrm{T}} P_{iL}(A_i^{(0\sim 1)})w_i \\
= &\left(z_i^{(0\sim 1)}\right)^{\mathrm{T}} P_{iL}(A_i^{(0\sim 1)})L_i\left(x_i^{(0\sim 1)}\right)P_{i+1,L}^{\mathrm{T}}\left(A_{i+1}^{0\sim 1}\right)z_{i+1}^{(0\sim 1)} \\
= &\left(z_{i+1}^{(0\sim 1)}\right)^{\mathrm{T}} P_{i+1,L}\left(A_{i+1}^{0\sim 1}\right)L_i\left(x_i^{(0\sim 1)}\right)P_{iL}^{\mathrm{T}}(A_i^{(0\sim 1)})z_i^{(0\sim 1)} \\
= &\left(z_{i+1}^{(0\sim 1)}\right)^{\mathrm{T}} P_{i+1,L}(A_{i+1}^{(0\sim 1)})v_i, \qquad (9.58)\\
&\qquad i=1,2,\ldots,n-1.
\end{aligned}$$

Adding the equations in (9.56) side by side, and using the relations in (9.58), give

$$\dot{V}\leqslant -\mu V+n\varepsilon. \qquad (9.59)$$

The above process clearly proves the following result.

Theorem 9.1 *Suppose that Assumption 9.1 holds. Let μ and ε be two arbitrarily given positive numbers, and $A_i^{0\sim 1}$, $P_i(A_i^{(0\sim 1)})$ and $P_{iL}(A_i^{(0\sim 1)})$, $i=1,2,\ldots,n-1$, be three sets of matrices as described above. Assume that there exist a set of non-negative continuous scalar functions $\rho_i(x_{1\sim i}^{(0\sim 1)})$, $i=1,2,\ldots,n$, satisfying (9.48). Then, controller (9.49) for the uncertain second-order SFS (9.1) guarantees that the transformed state $z_{1\sim n}^{(0\sim 1)}$ converges into the following ellipsoid:*

$$\left\{z_{1\sim n}^{(0\sim 1)}\left|\sum_{i=1}^{n}\left[\left(z_{1\sim i}^{(0\sim 1)}\right)^{\mathrm{T}}P_i(A_i^{(0\sim 1)})z_{1\sim i}^{(0\sim 1)}\right]\leqslant\frac{n\varepsilon}{\mu}\right.\right\}.$$

It follows from the above theorem that the state $z_i^{(0\sim 1)}$, $i=1,2,\ldots,n$, of each subsystem eventually converges into the following ellipsoid centered at the origin:

$$\Theta_i^{\mu,\varepsilon}(0)=\left\{z_i^{(0\sim 1)}\left|\left(z_i^{(0\sim 1)}\right)^{\mathrm{T}}P_i(A_i^{(0\sim 1)})z_i^{(0\sim 1)}\leqslant\frac{n\varepsilon}{\mu}\right.\right\},$$

which implies that the state $x_1^{(0\sim 1)}$ of the original system also converges into the following ellipsoid centered at the origin:

$$\Theta_1^{\mu,\varepsilon}(0)=\left\{x_1^{(0\sim 1)}\left|\left(x_1^{(0\sim 1)}\right)^{\mathrm{T}}P_1(A_1^{(0\sim 1)})x_1^{(0\sim 1)}\leqslant\frac{n\varepsilon}{\mu}\right.\right\}.$$

Remark 9.1 The above proposed second-order backstepping method can be easily modified to suit the other companion SFS (9.2). As a matter of fact, for SFS (9.2), we can simply take $\tilde{P}_i\left(A_i^{0\sim 1}\right)=P_i\left(A_i^{0\sim 1}\right)$, $\quad i=1,2,\ldots,n$, and in this case we also have theoretically $Q_{iL}\left(A_i^{0\sim 1}\right)\neq 0$, $\quad i=1,2,\ldots,n$, which ensure $L_i\left(z_{1\sim i}^{(0\sim 1)}\right)\neq 0$, $\quad i=1,2,\ldots,n$.

9.3 Backstepping for Mixed-order SFSs

Similar to Section 9.2, in this section we also give, based on Theorem 5.1, a direct backstepping design method for the high-order SFS (9.3) without turning it into a conventional first-order SFS. Parallel to the second-order case, we also make the following preparations.

Suppose $A_i^{0\sim m_i-1} \in \mathbb{R}^{1\times m_i}$, $i = 1, 2, \ldots, n$, are a set of matrices satisfying

$$\operatorname{Re}\lambda_k\left(\Phi\left(A_i^{0\sim m_i-1}\right)\right) \le -\frac{\mu}{2}, \quad k = 1, 2...,m_i, \tag{9.60}$$

where μ is an arbitrarily given positive number, and $P_i\left(A_i^{0\sim m_i-1}\right)$ is a positive definite solution to the Lyapunov inequality

$$\Phi^{\mathrm{T}}\left(A_i^{0\sim m_i-1}\right)P_i\left(A_i^{0\sim m_i-1}\right) + P_i\left(A_i^{0\sim m_i-1}\right)\Phi\left(A_i^{0\sim m_i-1}\right) \le -\gamma\, P_i\left(A_i^{0\sim m_i-1}\right), \tag{9.61}$$

for $i = 1, 2, \ldots, n$, and some positive number γ. Let

$$P_i\left(A_i^{0\sim m_i-1}\right) = \left[P_{iF}\left(A_i^{0\sim m_i-1}\right) \quad \cdots \quad P_{iL}\left(A_i^{0\sim m_i-1}\right)\right] \in \mathbb{R}^{m_i\times m_i},$$

where $P_{iF}\left(A_i^{0\sim m_i-1}\right)$ and $P_{iL}\left(A_i^{0\sim m_i-1}\right)$ are respectively the first and the last column of $P_i\left(A_i^{0\sim m_i-1}\right)$.

Further, define

$$\tilde{P}_i\left(A_i^{0\sim m_i-1}\right) = I_{m_i}^{\circ} P_i\left(A_i^{0\sim m_i-1}\right),$$

and we denote

$$\begin{aligned}\tilde{P}_i^{-1}\left(A_i^{0\sim m_i-1}\right) &= \begin{bmatrix} Q_{i1}\left(A_i^{0\sim m_i-1}\right) \\ Q_{i2}\left(A_i^{0\sim m_i-1}\right)\end{bmatrix} \\ &= \begin{bmatrix} & & Q_{i1}\left(A_i^{0\sim m_i-1}\right) \\ Q_{iF}\left(A_i^{0\sim m_i-1}\right) & Q_{iM}\left(A_i^{0\sim m_i-1}\right) & Q_{iL}\left(A_i^{0\sim m_i-1}\right)\end{bmatrix},\end{aligned}$$

where $Q_{iF}\left(A_i^{0\sim m_i-1}\right)$ and $Q_{iL}\left(A_i^{0\sim m_i-1}\right)$ are two scalars. Note that $P_i\left(A_i^{0\sim m_i-1}\right)$ is not unique, proper matrix $P_i\left(A_i^{0\sim m_i-1}\right)$ can be found such that

$$Q_{iL}\left(A_i^{0\sim m_i-1}\right) \ne 0, \quad i = 1, 2, \ldots, n. \tag{9.62}$$

For simplicity, only the basic idea is given in this section.

9.3.1 The First Step

Let

$$z_1^{(0\sim m_1-1)} = x_1^{(0\sim m_1-1)}, \tag{9.63}$$

and

$$\tilde{P}_2 z_2^{(0\sim m_2-1)} = x_2^{(0\sim m_2-1)} - \begin{bmatrix} \alpha_1 \\ 0_{m_2-1} \end{bmatrix}, \tag{9.64}$$

which implies, in view of the symmetry property of $P_2\left(A_2^{0\sim m_2-1}\right)$,

$$P_{2L}^{\mathrm{T}}\left(A_2^{0\sim m_2-1}\right) z_2^{(0\sim m_2-1)} = x_2 - \alpha_1. \tag{9.65}$$

Thus the first equation in (9.3) can be transformed into

$$\begin{aligned} z_1^{(m_1)} &= \hat{f}_1(z_1^{(0\sim m_1-1)}) + \Delta\hat{f}_1(z_1^{(0\sim m_1-1)}) \\ &\quad + G_1(z_1^{(0\sim m_1-1)})\left(P_{2L}^{\mathrm{T}}\left(A_2^{0\sim m_1-1}\right) z_2^{(0\sim m_2-1)} + \alpha_1\right), \end{aligned} \tag{9.66}$$

where

$$\begin{cases} \hat{f}_1(z_1^{(0\sim m_1-1)}) = f_1(x_1^{(0\sim m_1-1)}) \\ \Delta\hat{f}_1(z_1^{(0\sim m_1-1)}) = \Delta f_1(x_1^{(0\sim m_1-1)}) \\ L_1(z_1^{(0\sim m_1-1)}) = G_1(x_1^{(0\sim m_1-1)}). \end{cases} \tag{9.67}$$

Assume that there exists a non-negative continuous scalar function $\rho_1(z_1^{(0\sim m_1-1)})$ satisfying

$$\left|\Delta\hat{f}_1(z_1^{(0\sim m_1-1)})\right| \le \rho_1(z_1^{(0\sim m_1-1)}),$$

applying Theorem 5.1 to system (9.66) gives the first virtual control law as

$$\begin{cases} \alpha_1 = -L_1^{-1}(z_1^{(0\sim m_1-1)})\left(A_1^{0\sim m_1-1} z_1^{(0\sim m_1-1)} + \alpha_1^*\right) \\ \alpha_1^* = \frac{1}{4\varepsilon}\rho_1^2(z_1^{(0\sim m_1-1)}) P_{1L}^{\mathrm{T}}(A_1^{0\sim m_1-1}) z_1^{(0\sim m_1-1)} + \hat{f}_1(z_1^{(0\sim m_1-1)}). \end{cases} \tag{9.68}$$

The closed-loop subsystem resulted in by this control law is

$$z_1^{(m_1)} + A_1^{0\sim m_1-1} z_1^{(0\sim m_1-1)} = \phi_1 + w_1, \tag{9.69}$$

where

$$\phi_1 = -\frac{1}{4\varepsilon}\rho_1^2(z_1^{(0\sim m_1-1)}) P_{1L}^{\mathrm{T}}(A_1^{0\sim m_1-1}) z_1^{(0\sim m_1-1)} + \Delta\hat{f}_1(z_1^{(0\sim m_1-1)}), \tag{9.70}$$

and

$$w_1 = L_1(z_1^{(0\sim m_1-1)}) P_{2L}^{\mathrm{T}}\left(A_2^{0\sim m_2-1}\right) z_2^{(0\sim m_2-1)}. \tag{9.71}$$

9.3.2 The Second Step

The virtual control law α_1 given by (9.68) is a function with respect to $z_1^{(0\sim m_1-1)}$, that is,

$$\alpha_1 = \alpha_1\left(z_1^{(0\sim m_1-1)}\right).$$

Taking differentials, we can obtain

$$\begin{aligned}
\dot{\alpha}_1 &= \frac{\partial \alpha_1}{\partial z_1^{(0\sim m_1-2)}} z_1^{(1\sim m_1-1)} + \frac{\partial \alpha_1}{\partial z_1^{(m_1-1)}} z_1^{(m_1)} \\
&= \frac{\partial \alpha_1}{\partial z_1^{(0\sim m_1-2)}} z_1^{(1\sim m_1-1)} + \frac{\partial \alpha_1}{\partial z_1^{(m_1-1)}} \left(\hat{f}_1(z_1^{(0\sim m_1-1)}) + \Delta \hat{f}_1 \left(z_1^{(0\sim m_1-1)} \right) \right) \\
&\quad + \frac{\partial \alpha_1}{\partial z_1^{(m_1-1)}} G_1 \left(z_1^{(0\sim m_1-1)} \right) \left(P_{2L}^{\mathsf{T}} \left(A_2^{0\sim m_1-1} \right) z_2^{(0\sim m_2-1)} + \alpha_1 \right) \\
&= h_1 \left(z_i^{(0\sim m_i-1)} |_{i=1\sim 2} \right) + \Delta h_1 \left(z_i^{(0\sim m_i-1)} |_{i=1\sim 2} \right), \qquad (9.72)
\end{aligned}$$

where

$$\begin{aligned}
h_1 \left(z_i^{(0\sim m_i-1)} |_{i=1\sim 2} \right) &= \frac{\partial \alpha_1}{\partial z_1^{(0\sim m_1-2)}} z_1^{(1\sim m_1-1)} + \frac{\partial \alpha_1}{\partial z_1^{(m_1-1)}} \hat{f}_1 \left(z_1^{(0\sim m_1-1)} \right) \\
&\quad + \frac{\partial \alpha_1}{\partial z_1^{(m_1-1)}} G_1 \left(z_1^{(0\sim m_1-1)} \right) \left(P_{2L}^{\mathsf{T}} \left(A_2^{0\sim m_1-1} \right) z_2^{(0\sim m_2-1)} + \alpha_1 \right), \\
\Delta h_1 \left(z_i^{(0\sim m_i-1)} |_{i=1\sim 2} \right) &= \frac{\partial \alpha_1}{\partial z_1^{(m_1-1)}} \Delta \hat{f}_1 \left(z_1^{(0\sim m_1-1)} \right).
\end{aligned}$$

Please note that $h_1(z_i^{(0\sim m_i-1)}|_{i=1\sim 2})$ is a known function, while $\Delta h_1(z_i^{(0\sim m_i-1)}|_{i=1\sim 2})$ is related to the system uncertainties.

On the other hand, we can obtain, from (9.64),

$$z_2^{(0\sim m_2-1)} = \tilde{P}_2^{-1} \left(A_2^{0\sim m_2-1} \right) \left(x_2^{(0\sim m_2-1)} - \begin{bmatrix} \alpha_1 \\ 0 \end{bmatrix} \right), \qquad (9.73)$$

which gives,

$$\begin{aligned}
z_2^{(m_2-1)} &= Q_{22} \left(A_2^{0\sim m_2-1} \right) \left(x_2^{(0\sim m_2-1)} - \begin{bmatrix} \alpha_1 \\ 0 \end{bmatrix} \right) \\
&= Q_{2F} \left(A_2^{0\sim m_2-1} \right) (x_2 - \alpha_1) \\
&\quad + Q_{2M} \left(A_2^{0\sim m_2-1} \right) x_2^{(1\sim m_2-2)} + Q_{2L} \left(A_2^{0\sim m_2-1} \right) x_2^{(m_2-1)}.
\end{aligned}$$

Taking differentials of the above equation, and using (9.72) and the second equation in (9.3), produce

$$\begin{aligned}
z_2^{(m_2)} &= Q_{2F} \left(A_2^{0\sim 1} \right) (\dot{x}_2 - \dot{\alpha}_1) + Q_{2M} \left(A_2^{0\sim 1} \right) \dot{x}_2^{(1\sim m_2-2)} + Q_{2L} \left(A_2^{0\sim 1} \right) x_2^{(m_2)} \\
&= \hat{f}_2(z_i^{(0\sim m_i-1)}|_{i=1\sim 2}) + \Delta \hat{f}_2(z_i^{(0\sim m_i-1)}|_{i=1\sim 2}) \\
&\quad + L_2(z_i^{(0\sim m_i-1)}|_{i=1\sim 2}) x_3. \qquad (9.74)
\end{aligned}$$

Further let

$$\tilde{P}_3 \left(A_3^{0\sim m_3-1} \right) z_3^{(0\sim m_3-1)} = x_3^{(0\sim m_3-1)} - \begin{bmatrix} \alpha_2 \\ 0 \end{bmatrix}, \qquad (9.75)$$

which implies, in view of the symmetry property of $P_3\left(A_3^{0\sim m_3-1}\right)$,

$$P_{3L}^{\mathrm{T}}\left(A_3^{0\sim m_3-1}\right)z_3^{(0\sim m_3-1)}=x_3-\alpha_2. \tag{9.76}$$

Substituting (9.76) into (9.74), yields

$$\begin{aligned} z_2^{(m_2)} &= \hat{f}_2\left(z_i^{(0\sim m_i-1)}|_{i=1\sim 2}\right)+\Delta\,\hat{f}_2\left(z_i^{(0\sim m_i-1)}|_{i=1\sim 2}\right)\\ &\quad +L_2(z_i^{(0\sim m_i-1)}|_{i=1\sim 2})\left(P_{3L}^{\mathrm{T}}\left(A_3^{0\sim m_3-1}\right)z_3^{(0\sim m_3-1)}+\alpha_2\right). \end{aligned} \tag{9.77}$$

Assume that there exists a continuous function $\rho_2(z_i^{(0\sim m_i-1)}|_{i=1\sim 2})$ satisfying

$$\left|\Delta\,\hat{f}_2(z_i^{(0\sim m_i-1)}|_{i=1\sim 2})\right|\leqslant\rho_2(z_i^{(0\sim m_i-1)}|_{i=1\sim 2}), \tag{9.78}$$

applying Theorem 5.1 to system (9.77) produces the second virtual control law of the system as

$$\begin{cases} \alpha_2=-L_2^{-1}(z_i^{(0\sim m_i-1)}|_{i=1\sim 2})\left(A_2^{0\sim 1}z_2^{(0\sim m_2-1)}+\alpha_2^*\right)\\ \alpha_2^*=\frac{1}{4\varepsilon}\rho_2^2(z_i^{(0\sim m_i-1)}|_{i=1\sim 2})P_{2L}^{\mathrm{T}}(A_2^{0\sim 1})z_2^{(0\sim 1)}+\hat{f}_2(z_i^{(0\sim m_i-1)}|_{i=1\sim 2})+v_1, \end{cases} \tag{9.79}$$

where

$$v_1=L_1(z_1^{(0\sim m_1-1)})P_{1L}^{\mathrm{T}}\left(A_1^{0\sim m_1-1}\right)z_1^{(0\sim m_1-1)}. \tag{9.80}$$

The closed-loop subsystem resulted in by this control law is

$$z_2^{(m_2)}+A_2^{0\sim m_2-1}z_2^{(0\sim m_2-1)}=\phi_2+w_2-v_1, \tag{9.81}$$

where

$$\phi_2=-\frac{1}{4\varepsilon}\rho_2^2(z_i^{(0\sim m_i-1)}|_{i=1\sim 2})P_{2L}^{\mathrm{T}}(A_2^{0\sim m_2-1})z_2^{(0\sim m_2-1)}+\Delta\,\hat{f}_2(z_i^{(0\sim m_i-1)}|_{i=1\sim 2}), \tag{9.82}$$

and

$$w_2=L_2(z_i^{(0\sim m_i-1)}|_{i=1\sim 2})P_{3L}^{\mathrm{T}}\left(A_3^{0\sim m_3-1}\right)z_3^{(0\sim m_3-1)}. \tag{9.83}$$

9.3.3 The n-th Step

Suppose that the expressions of α_{n-1}, $\dot{\alpha}_{n-1}$, $h_{n-1}(z_i^{(0\sim m_i-1)}|_{i=1\sim n})$ and $\triangle h_{n-1}(z_i^{(0\sim m_i-1)}|_{i=1\sim n})$ have all been obtained in the $(n-1)$-th step. Similar to the previous deduction, we can obtain

$$z_n^{(m_n)}=\hat{f}_n(z_i^{(0\sim m_i-1)}|_{i=1\sim n})+\Delta\,\hat{f}_n(z_i^{(0\sim m_i-1)}|_{i=1\sim n})+L_n(z_i^{(0\sim m_i-1)}|_{i=1\sim n})u. \tag{9.84}$$

Assume that there exist a set of non-negative continuous scalar functions $\rho_k(z_i^{(0\sim m_i-1)}|_{i=1\sim k})$, $k=1,2,\ldots,n$, satisfying

$$\left|\Delta\,\hat{f}_k(z_i^{(0\sim m_i-1)}|_{i=1\sim k})\right|\leq\rho_k(z_i^{(0\sim m_i-1)}|_{i=1\sim k}),\ k=1,2,\ldots,n. \tag{9.85}$$

Then, by applying Theorem 5.1 to system (9.84), the control law of system (9.3) can be finally designed as

$$\begin{cases} u = -L_n^{-1}(z_i^{(0\sim m_i-1)}|_{i=1\sim n})\left(A_n^{0\sim m_n-1}z_n^{(0\sim m_n-1)} + u^*\right) \\ u^* = \frac{1}{4\varepsilon}\rho_n^2(z_i^{(0\sim m_i-1)}|_{i=1\sim n})P_{nL}^{\mathrm{T}}(A_n^{0\sim m_n-1})z_n^{(0\sim m_n-1)} \\ \qquad\quad + \hat{f}_n\left(z_i^{(0\sim m_i-1)}|_{i=1\sim n}\right) + v_{n-1}, \end{cases} \tag{9.86}$$

where

$$v_{n-1} = L_{n-1}(z_i^{(0\sim m_i-1)}|_{i=1\sim n-1})P_{n-1,L}^{\mathrm{T}}\left(A_{n-1}^{0\sim m_{n-1}-1}\right)z_{n-1}^{(0\sim m_{n-1}-1)}. \tag{9.87}$$

The last closed-loop subsystem resulted in by this control law is

$$z_n^{(m_n)} + A_n^{0\sim m_n-1}z_n^{(0\sim m_n-1)} = \phi_n - v_{n-1}, \tag{9.88}$$

where

$$\phi_n = -\frac{1}{4\varepsilon}\rho_n^2(z_i^{(0\sim m_i-1)}|_{i=1\sim n})P_{nL}^{\mathrm{T}}(A_n^{0\sim m_n-1})z_n^{(0\sim m_n-1)} + \Delta\,\hat{f}_n(x_i^{(0\sim m_i-1)}|_{i=1\sim n}). \tag{9.89}$$

9.3.4 The Conclusion

It should be noted that the above procedure also defines a state transformation between $z_i^{(0\sim m_i-1)}$ and $x_i^{(0\sim m_i-1)}$, $i = 1, 2, \ldots, n$, in the form of

$$\begin{cases} z_1^{(0\sim m_1-1)} = x_1^{(0\sim m_1-1)}, \\ \tilde{P}_i\left(A_i^{0\sim m_i-1}\right)z_i^{(0\sim m_i-1)} = x_i^{(0\sim m_i-1)} - \begin{bmatrix} \alpha_{i-1} \\ 0 \end{bmatrix}, \\ \qquad\qquad\qquad\qquad i = 2, 3, \ldots, n, \end{cases} \tag{9.90}$$

where $\alpha_{i-1} = \alpha_{i-1}(x_k^{(0\sim m_k-1)}|_{k=1\sim i-1})$. Similar to the proof of Theorem 9.1, we can also prove the following conclusion.

Theorem 9.2 *Suppose that Assumption 9.2 holds. Let μ and ε be two arbitrarily given positive numbers, and $A_i^{0\sim m_i-1} \in \mathbb{R}^{1\times m_i}$, $i = 1, 2, \ldots, n$, be a set of matrices satisfying (9.60). Assume that there exist a set of non-negative continuous scalar functions $\rho_k(z_i^{(0\sim m_i-1)}|_{i=1\sim k})$, $k = 1, 2, \ldots, n$, satisfying (9.85). Then, controller (9.86) for the uncertain high-order SFS (9.3) guarantees that the transformed states $z_i^{(0\sim m_i-1)}$, $i = 1, 2, \ldots, n$, finally converge into the following ellipsoid:*

$$\Theta^{\mu,\varepsilon}(0) = \left\{ z_i^{(0\sim m_i-1)}|_{i=1\sim n} \left| \sum_{i=1}^{n}\left(z_i^{(0\sim m_i-1)}\right)^{\mathrm{T}}P_i z_i^{(0\sim m_i-1)} \le \frac{n\varepsilon}{\mu} \right. \right\},$$

where $P_i = P_i(A_i^{0\sim m_i-1})$.

It is known from the above theorem that the states $z_i^{(0\sim m_i-1)}$, $i=1,2,\ldots,n$, of each subsystem eventually converge into the following ellipsoid centered at the origin:

$$\Theta_i^{\mu,\varepsilon}(0)=\left\{z_i^{(0\sim m_i-1)}\left|\left(z_i^{(0\sim m_i-1)}\right)^{\mathrm{T}}P_i z_i^{(0\sim m_i-1)}\leq\frac{n\varepsilon}{\mu}\right.\right\}.$$

Since $z_1=x_1$, the state $x_1^{(0\sim m_1-1)}$ converges into the following ellipsoid centered at the origin:

$$\Theta_1^{\mu,\varepsilon}(0)=\left\{x_1^{(0\sim m_1-1)}\left|\left(x_1^{(0\sim m_1-1)}\right)^{\mathrm{T}}P_1 x_1^{(0\sim m_1-1)}\leq\frac{n\varepsilon}{\mu}\right.\right\}.$$

To end this section, we make a few remarks.

Remark 9.2 The above proposed mixed-order backstepping method can also be easily modified to suit the other companion SFS (9.4). As a matter of fact, for SFS (9.4), we can simply take

$$\tilde{P}_i\left(A_i^{0\sim m_i-1}\right)=P_i\left(A_i^{0\sim m_i-1}\right),\quad i=1,2,\ldots,n,$$

and in this case we also have theoretically $Q_{iL}\left(A_i^{0\sim m_i-1}\right)\neq 0$, $i=1,2,\ldots,n$, since $Q_{iL}\left(A_i^{0\sim m_i-1}\right)$ can be easily shown to be a diagonal element of a positive definite matrix.

Remark 9.3 The second- and high-order (mixed-order) backstepping methods discussed in the previous and the current sections can both be properly modified to cope with the case of signal tracking. Specifically, in order to let x_1 track a given sufficiently smooth signal $x_1^*(t)$, we need only to adjust the first step of these second- and high-order methods of backstepping by substituting (9.10) or (9.64) by

$$z_1^{(0\sim 1)}=x_1^{(0\sim 1)}-\left(x_1^*\right)^{(0\sim 1)},$$

or

$$z_1^{(0\sim m_1-1)}=x_1^{(0\sim m_1-1)}-\left(x_1^*\right)^{(0\sim m_1-1)},$$

and then in the sequential steps treat the terms related to the reference signal $x_1^*(t)$ properly.

Remark 9.4 The above high-order backstepping method for high-order systems appears to be more complicated in description. It should be noted that, in many specific engineering applications, m_i, $i=1,2,\ldots,n$, often take the values of 1 and 2. So the fact is, in such applications, the mixed-order backstepping method is even simpler than that of the second-order systems.

Remark 9.5 For a specific practical system of high-order, direct application of the high-order backstepping generally requires fewer steps than application of the normal

first-order backstepping method, since there are more subsystems in the converted first-order SFS. What is more, application of high-order backstepping also saves the process of converting the high-order subsystems originally obtained from physical modelling into first-order ones. Therefore, to solve a specific practical problem, the method of high-order (or mixed-order) backstepping is generally simpler than the normal first-order method of backstepping.

9.4 Example

Again, let us consider Example 5.1.

9.4.1 Designs of Virtual Controllers

The results for each step are given below.

Step 1

Corresponding to the form of system (9.13), we have

$$\begin{cases} \hat{f}_1(z_1, \dot{z}_1) = z_1^2 \\ \Delta \hat{f}_1(z_1, \dot{z}_1) = \Delta f_1(z_1) \\ L_1(z_1, \dot{z}_1) = 1. \end{cases} \tag{9.91}$$

In view of (5.19), we can take

$$\rho_1(z_1, \dot{z}_1) = \sigma_1 z_1^2,$$

it then follows from Theorem 5.1 that the virtual control law can be designed as

$$\alpha_1 = -A_1^{0\sim 1} z_1^{(0\sim 1)} - z_1^2 - \frac{1}{4\varepsilon}\sigma_1^2 z_1^4 P_{1L}^{\mathrm{T}}(A_1^{0\sim 1}) z_1^{(0\sim 1)}. \tag{9.92}$$

Step 2

Partition $P_{1L}(A_1^{0\sim 1})$ into

$$P_{1L}(A_1^{0\sim 1}) = \begin{bmatrix} P_{1L}^{\mathrm{I}}(A_1^{0\sim 1}) \\ P_{1L}^{\mathrm{II}}(A_1^{0\sim 1}) \end{bmatrix},$$

then, we have, from (9.92),

$$\dot{\alpha}_1 = h(z_1, \dot{z}_1, x_2) + \Delta h(z_1), \tag{9.93}$$

with

$$\begin{aligned} h(z_1, \dot{z}_1, x_2) \;=\; & -\left(A_1^0 \dot{z}_1 + 2 z_1 \dot{z}_1 + A_1^1 \left(z_1^2 + x_2\right)\right) \\ & -\frac{1}{\varepsilon}\sigma_1^2 z_1^3 \dot{z}_1 P_{1L}^{\mathrm{T}}(A_1^{0\sim 1}) z_1^{(0\sim 1)} - \frac{1}{4\varepsilon}\sigma_1^2 P_{1L}^{\mathrm{I}}(A_1^{0\sim 1}) z_1^4 \dot{z}_1 \\ & -\frac{1}{4\varepsilon}\sigma_1^2 P_{1L}^{\mathrm{II}}(A_1^{0\sim 1}) \left(z_1^2 + x_2\right) z_1^4, \end{aligned} \tag{9.94}$$

and

$$\Delta h(z_1) = -\left(A_1^1 + \frac{1}{4\varepsilon}\sigma_1^2 z_1^4 P_{1L}^{\mathrm{II}}(A_1^{0\sim1})\right)\Delta f_1(z_1). \tag{9.95}$$

By formula (9.22), we can obtain

$$\ddot{z}_2 = \hat{f}_2(z_{1\sim2}^{(0\sim1)}) + \Delta\hat{f}_2(z_{1\sim2}^{(0\sim1)}) + Q_{2L}\left(A_2^{0\sim1}\right)u, \tag{9.96}$$

with

$$\begin{cases} \hat{f}_2(z_{1\sim2}^{(0\sim1)}) = Q_{2F}\left(A_2^{0\sim1}\right)(\dot{x}_2 - h_1(z_1, \dot{z}_1, x_2)) \\ \Delta\hat{f}_2(z_{1\sim2}^{(0\sim1)}) = Q_{2F}\left(A_2^{0\sim1}\right)\Delta h_1(z_1) + Q_{2L}\left(A_2^{0\sim1}\right)\Delta f_2(x_1, \dot{x}_2). \end{cases} \tag{9.97}$$

Noting (9.95), we have

$$\begin{aligned} |\Delta h_1(z_1)| &\leq \left|A_1^1 + \frac{1}{4\varepsilon}\sigma_1^2 z_1^4 P_{1L}^{\mathrm{II}}(A_1^{0\sim1})\right| |\Delta f_1(z_1)| \\ &\leq \sigma_1\left(\left|A_1^1\right| + \frac{1}{4\varepsilon}\sigma_1^2 z_1^4 \left|P_{1L}^{\mathrm{II}}(A_1^{0\sim1})\right|\right)x_1^2 \\ &\triangleq \delta(z_1). \end{aligned} \tag{9.98}$$

Further using (9.97) and (5.19), we have

$$\begin{aligned} \left|\Delta\hat{f}_2(z_{1\sim2}^{(0\sim1)})\right| &\leq \left|Q_{2F}\left(A_2^{0\sim1}\right)\right| |\Delta h_1(z_1)| + \left|Q_{2L}\left(A_2^{0\sim1}\right)\right| |\Delta f_2(x_1, \dot{x}_2)| \\ &\leq \left|Q_{2F}\left(A_2^{0\sim1}\right)\right| \delta(z_1) + \sigma_2\left|Q_{2L}\left(A_2^{0\sim1}\right)\right| |x_1\dot{x}_2|. \end{aligned}$$

Thus we can choose

$$\rho_2\left(z_{1\sim2}^{(0\sim1)}\right) = \left|Q_{2F}\left(A_2^{0\sim1}\right)\right| \delta(z_1) + \sigma_2\left|Q_{2L}\left(A_2^{0\sim1}\right)\right| |x_1\dot{x}_2|.$$

Applying Theorem 5.1 to system (9.96) gives the robust stabilizing control law of the system as

$$\begin{cases} u = -Q_{2L}^{-1}\left(A_2^{0\sim1}\right)\left(A_2^{0\sim1}z_2^{(0\sim1)} + \hat{f}_2(z_{1\sim2}^{(0\sim1)}) + \alpha_2^* + v_1\right) \\ \alpha_2^* = \frac{1}{4\varepsilon}\rho_2^2\left(z_{1\sim2}^{(0\sim1)}\right)P_{2L}^{\mathrm{T}}(A_2^{0\sim1})z_2^{(0\sim1)} \\ v_1 = P_{1L}^{\mathrm{T}}\left(A_1^{0\sim1}\right)z_1^{(0\sim1)}. \end{cases} \tag{9.99}$$

where $A_i^{0\sim1} = [A_i^0\ A_i^1] \in \mathbb{R}^{1\times2}$, $i = 0, 1$, are taken to satisfy

$$\mathrm{Re}\,\lambda_i\left(\Phi\left(A_k^{0\sim1}\right)\right) < -\frac{\mu}{2},\ i, k = 1, 2, \tag{9.100}$$

and $P_{iL}(A_i^{0\sim1})$, $i = 1, 2$, are determined by (9.6)-(9.7).

9.4.2 The Closed-loop System

It follows from the steps of the second-order method of backstepping described in Section 9.2 that the closed-loop system resulted in by this robust control law is

$$\begin{cases} \ddot{z}_1 + A_1^{0\sim 1} z_1^{(0\sim 1)} = \phi_1 + w_1 \\ \ddot{z}_2 + A_2^{0\sim 1} z_2^{(0\sim 1)} = \phi_2 - v_1, \end{cases} \tag{9.101}$$

with

$$\begin{cases} \phi_1 = -\frac{1}{4\varepsilon}\sigma_1^2 z_1^2 P_{1L}^{\mathrm{T}}(A_1^{0\sim 1}) z_1^{(0\sim 1)} + \Delta f_1(z_1) \\ \phi_2 = -\frac{1}{4\varepsilon}\rho_2^2(z_{1\sim 2}^{(0\sim 1)}) P_{2L}^{\mathrm{T}}(A_2^{0\sim 1}) z_2^{(0\sim 1)} + \Delta \hat{f}_2(z_{1\sim 2}^{(0\sim 1)}), \end{cases} \tag{9.102}$$

and

$$w_1 = P_{2L}^{\mathrm{T}}\left(A_2^{0\sim 1}\right) z_2^{(0\sim 1)}. \tag{9.103}$$

It follows from Theorem 9.1 that the transformed state $z_{1\sim 2}^{(0\sim 1)}$ converges into the following ellipsoid:

$$\Theta^{\mu,\varepsilon}(0) = \left\{ z_{1\sim 2}^{(0\sim 1)} \left| \sum_{i=1}^{2} \left[\left(z_{1\sim i}^{(0\sim 1)}\right)^{\mathrm{T}} P_i (A_i^{0\sim 1}) z_{1\sim i}^{(0\sim 1)} \right] \leqslant \frac{2\varepsilon}{\mu} \right. \right\}.$$

Particularly, the state $x_1^{(0\sim 1)}$ of the original system converges into the following ellipsoid centered at the origin:

$$\Theta_1^{\mu,\varepsilon}(0) = \left\{ x_1^{(0\sim 1)} \left| \left(x_1^{(0\sim 1)}\right)^{\mathrm{T}} P_1 (A_1^{(0\sim 1)}) x_1^{(0\sim 1)} \leqslant \frac{2\varepsilon}{\mu} \right. \right\}.$$

Remark 9.6 If system (5.18) is converted into the first-order state-space model, then the converted model is composed of 4 subsystems, and eventually four steps of backstepping are needed to solve the problem. Experience with normal first-order backstepping designs tells us that the design process for the converted first-order systems with four steps of backstepping is definitely much more complicated than the above process.

9.5 Notes and References

The reference closely related to this chapter is Duan (2021c), which first proposes high-order SFSs and the corresponding high-order backstepping methods.

For the newly proposed second- and high-order uncertain SFSs, effective designs of robust controllers for nonlinear systems are proposed, which turn out to be the generalized second- and high-order methods of backstepping for second- and high-order SFSs.

It is commonly known that the method of backstepping can hardly be applicable to an SFS with many subsystems due to the well-known "differential explosion" problem. Such a problem of course also applies to the newly proposed second- and high-order methods of backstepping. Nevertheless, for a specific system physically modelled by, e.g., Lagrangian Equation, a second-order method of backstepping needs only half

of the steps of the first-order method of backstepping, and also saves the process of converting the second-order system into a first-order one, and is thus much simpler to apply than the normal first-order method of backstepping.

The second-order and mixed-order SFSs with dynamical uncertainties discussed in this chapter can also be extended in several ways. Firstly, these can be generalized to second-order and mixed-order SFSs with increasing dimensions. Secondly, they can be also generalized into second-order and mixed-order nonaffine SFSs. Correspondingly, robust high-order backstepping schemes for all these extended cases can also be derived.

Chapter 10

High-order Backstepping for Adaptive Control

In this Chapter, two types of high-order strict feedback system (SFS) models with parametric uncertainties are introduced, namely, the second- and high-order strict-feedback SFS models, which possess a common full-actuation structure. Using the direct design approaches of adaptive stabilizing controllers and adaptive tracking controllers of FAS models proposed in Chapter 5, direct second- and high-order backstepping methods for the designs of adaptive stabilizing controllers of the second- and high-order SFSs with parametric uncertainties are also proposed. These approaches do not need to convert the high-order systems into first-order ones, and are thus more direct and effective. Furthermore, for a specific second- or higher-order SFS, application of the proposed high-order backstepping methods needs fewer steps than the usual first-order backstepping method, thereby saving much design and computation complexity.

10.1 High-order FASs with Parametric Uncertainties

In this section, three types of high-order systems with parametric uncertainties are investigated.

10.1.1 Second-order SFSs

Lagrangian Equations, Theorem of Linear and Angular Momentum, etc., are the common tools in modelling physical systems. In modelling using such physical laws, a set of second-order differential equations are obtained. Due to such a phenomenon, we introduce the following second-order strict-feedback nonlinear system with unknown

parameters:

$$\begin{cases} \dot{x}_1 = f_1(x_1, \dot{x}_1)\vartheta_1 + g_1(x_1, \dot{x}_1)x_2 \\ \dot{x}_2 = f_2(x_{1\sim 2}^{(0\sim 1)})\vartheta_2 + g_2(x_{1\sim 2}^{(0\sim 1)})x_3 \\ \qquad\vdots \\ \dot{x}_{n-1} = f_{n-1}(x_{1\sim n-1}^{(0\sim 1)})\vartheta_{n-1} + g_{n-1}(x_{1\sim n-1}^{(0\sim 1)})x_n \\ \dot{x}_n = f_n(x_{1\sim n}^{(0\sim 1)})\vartheta_n + g_n(x_{1\sim n}^{(0\sim 1)})u, \end{cases} \tag{10.1}$$

where $x_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, are the state variables, $u \in \mathbb{R}$ is the control input, $\vartheta_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, are a set of unknown constant scalar parameters, $f_i(x_{1\sim i}^{(0\sim 1)})$, $g_i(x_{1\sim i}^{(0\sim 1)}) \in \mathbb{R}$, $i = 1, 2, \ldots, n$, are two sets of sufficiently smooth scalar functions, and $g_i(x_{1\sim i}^{(0\sim 1)}) \in \mathbb{R}$, $i = 1, 2, \ldots, n$, satisfy the following full-actuation condition:

Assumption 10.1 $g_i(x_{1\sim i}^{(0\sim 1)}) \neq 0,\ \forall x_i, \dot{x}_i \in \mathbb{R},\ i = 1, 2, \ldots, n.$

Besides physical modelling, many conventional first-order SFSs can also be converted into the above second-order form (10.1).

Clearly, a companion form of the above SFS (10.1) is the following:

$$\begin{cases} \ddot{x}_1 = f_1(x_1, \dot{x}_1)\vartheta_1 + g_1(x_1, \dot{x}_1)\dot{x}_2 \\ \ddot{x}_2 = f_2(x_{1\sim 2}^{(0\sim 1)})\vartheta_2 + g_2(x_{1\sim 2}^{(0\sim 1)})\dot{x}_3 \\ \qquad\vdots \\ \ddot{x}_{n-1} = f_{n-1}(x_{1\sim n-1}^{(0\sim 1)})\vartheta_{n-1} + g_{n-1}(x_{1\sim n-1}^{(0\sim 1)})\dot{x}_n \\ \ddot{x}_n = f_n(x_{1\sim n}^{(0\sim 1)})\vartheta_n + g_n(x_{1\sim n}^{(0\sim 1)})u. \end{cases} \tag{10.2}$$

10.1.2 High-order SFSs

A straight forward generalization of the above second-order SFS (10.1) is the following high-order (mixed-order) strict-feedback nonlinear system with unknown parameters:

$$\begin{cases} x_1^{(m_1)} = f_1(x_1^{(0\sim m_1-1)})\vartheta_1 + g_1(x_1^{(0\sim m_1-1)})x_2 \\ x_2^{(m_2)} = f_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2})\vartheta_2 + g_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2})x_3 \\ \qquad\vdots \\ x_{n-1}^{(m_{n-1})} = f_{n-1}(x_i^{(0\sim m_i-1)}|_{i=1\sim n-1})\vartheta_{n-1} + g_{n-1}(x_i^{(0\sim m_i-1)}|_{i=1\sim n-1})x_n \\ x_n^{(m_n)} = f_n(x_i^{(0\sim m_i-1)}|_{i=1\sim n})\vartheta_n + g_n(x_i^{(0\sim m_i-1)}|_{i=1\sim n})u, \end{cases} \tag{10.3}$$

where m_i, $i = 1, 2, \ldots, n$, are a set of positive integers, and $f_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k})$, $g_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k}) \in \mathbb{R}$, $k = 1, 2, \ldots, n$, are two sets of sufficiently smooth scalar functions satisfying the following full-actuation condition:

Assumption 10.2 $g_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k}) \neq 0,\ \forall x_i^{(0\sim m_i-1)} \in \mathbb{R}^{m_i},\ k = 1, 2, \ldots, n.$

Parallel to (10.2), a companion form of the above SFS (10.3) is the following:

$$\begin{cases} x_1^{(m_1)} = f_1(x_1^{(0\sim m_1-1)})\vartheta_1 + g_1(x_1^{(0\sim m_1-1)})\dot{x}_2 \\ x_2^{(m_2)} = f_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2})\vartheta_2 + g_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2})\dot{x}_3 \\ \quad\vdots \\ x_{n-1}^{(m_{n-1})} = f_{n-1}(x_i^{(0\sim m_i-1)}|_{i=1\sim n-1})\vartheta_{n-1} + g_{n-1}(x_i^{(0\sim m_i-1)}|_{i=1\sim n-1})\dot{x}_n \\ x_n^{(m_n)} = f_n(x_i^{(0\sim m_i-1)}|_{i=1\sim n})\vartheta_n + g_n(x_i^{(0\sim m_i-1)}|_{i=1\sim n})u. \end{cases} \tag{10.4}$$

In Duan (2021c), the relations among three types of high-order models with uncertainties are discussed, while the above three types of high-order models with unknown parameters have similar relations.

10.2 Backstepping for Second-order SFSs

Based on Theorem 5.3, we can give a direct backstepping design method for the second-order SFS (10.1) without turning the system into a conventional first-order SFS. To do this we need some preparations.

Suppose $A_i^{0\sim 1} \in \mathbb{R}^{1\times 2},\ i = 1, 2, \ldots, n$, are a set of matrices which make $\Phi\left(A_i^{0\sim 1}\right) \in \mathbb{R}^{2\times 2},\ i = 1, 2, \ldots, n$, stable, and

$$P_i\left(A_i^{0\sim 1}\right) = \left[P_{iF}\left(A_i^{0\sim 1}\right) \quad P_{iL}\left(A_i^{0\sim 1}\right)\right] \in \mathbb{R}^{2\times 2},$$

is the unique positive definite solution to the Lyapunov equation

$$\Phi^{\mathrm{T}}\left(A_i^{0\sim 1}\right) P_i\left(A_i^{0\sim 1}\right) + P_i\left(A_i^{0\sim 1}\right)\Phi\left(A_i^{0\sim 1}\right) = -I_2, \tag{10.5}$$

for $i = 1, 2, \ldots, n$. Further, define, for $i = 1, 2, \ldots, n$,

$$\tilde{P}_i\left(A_i^{0\sim 1}\right) = I_2^{\circ} P_i\left(A_i^{0\sim 1}\right),$$

and denote

$$\begin{aligned} \tilde{P}_i^{-1}\left(A_i^{0\sim 1}\right) &= \begin{bmatrix} Q_{i1}\left(A_i^{0\sim 1}\right) \\ Q_{i2}\left(A_i^{0\sim 1}\right) \end{bmatrix} \\ &= \begin{bmatrix} Q_{i1}\left(A_i^{0\sim 1}\right) & \\ Q_{iF}\left(A_i^{0\sim 1}\right) & Q_{iL}\left(A_i^{0\sim 1}\right) \end{bmatrix}, \end{aligned}$$

where $Q_{iF}\left(A_i^{0\sim 1}\right)$ and $Q_{iL}\left(A_i^{0\sim 1}\right)$ are two scalars, then it can be easily shown that

$$Q_{iL}\left(A_i^{0\sim 1}\right) \neq 0, \quad i = 1, 2, \ldots, n. \tag{10.6}$$

It is worth pointing out that all the variables introduced above are independent of system (10.1) and can all be easily obtained beforehand.

10.2.1 The First Step

Let

$$z_1^{(0\sim 1)} = x_1^{(0\sim 1)}, \tag{10.7}$$

and

$$\tilde{P}_2\left(A_2^{0\sim 1}\right) z_2^{(0\sim 1)} = x_2^{(0\sim 1)} - \begin{bmatrix} \alpha_1 \\ 0 \end{bmatrix}. \tag{10.8}$$

Then, in view of the symmetry property of $P_2\left(A_2^{0\sim 1}\right)$, the above equation can be equivalently decomposed into

$$P_{2L}^{\mathrm{T}}\left(A_2^{0\sim 1}\right) z_2^{(0\sim 1)} = x_2 - \alpha_1, \tag{10.9}$$

and

$$P_{2F}^{\mathrm{T}}\left(A_2^{0\sim 1}\right) z_2^{(0\sim 1)} = \dot{x}_2. \tag{10.10}$$

Thus, by using (10.7) and (10.9), the first equation in (10.1) is transformed into

$$\ddot{z}_1 = H_1^{\mathrm{T}}(z_1, \dot{z}_1)\theta_1 + L_1(z_1, \dot{z}_1)\left(P_{2L}^{\mathrm{T}}\left(A_2^{0\sim 1}\right) z_2^{(0\sim 1)} + \alpha_1\right), \tag{10.11}$$

where

$$\begin{cases} \theta_1 = \vartheta_1 \\ H_1^{\mathrm{T}}(z_1, \dot{z}_1) = f_1(z_1, \dot{z}_1) \\ L_1(z_1, \dot{z}_1) = g_1(z_1, \dot{z}_1). \end{cases} \tag{10.12}$$

Applying Theorem 5.3 to the above system (10.11) gives the first virtual control law as

$$\begin{cases} \dot{\hat{\theta}}_1 = H_1(z_1, \dot{z}_1) P_{1L}^{\mathrm{T}}\left(A_1^{0\sim 1}\right) z_1^{(0\sim 1)} \\ \alpha_1 = -L_1^{-1}(z_1, \dot{z}_1)\left(A_1^{0\sim 1} z_1^{(0\sim 1)} + H_1^{\mathrm{T}}(z_1, \dot{z}_1)\hat{\theta}_1\right), \end{cases} \tag{10.13}$$

and a corresponding closed-loop subsystem resulted in by this control law is

$$\ddot{z}_1 + A_1^{0\sim 1} z_1^{(0\sim 1)} = H_1^{\mathrm{T}}(z_1, \dot{z}_1)\tilde{\theta}_1 + w_1, \tag{10.14}$$

where

$$w_1 = L_1(z_1, \dot{z}_1) P_{2L}^{\mathrm{T}}\left(A_2^{0\sim 1}\right) z_2^{(0\sim 1)}. \tag{10.15}$$

10.2.2 The Second Step

The virtual control α_1 is a function with respect to $z_1, \dot{z}_1$ and $\hat{\theta}_1$, that is,

$$\alpha_1 = \alpha_1(z_1, \dot{z}_1, \hat{\theta}_1).$$

In view of the expression of $\ddot{x}_1$ given in (10.1) and that of $\dot{\hat{\theta}}_1$ given in (10.13), we can obtain

$$\begin{aligned}\dot{\alpha}_1 &= \frac{\partial \alpha_1}{\partial z_1}\dot{z}_1+\frac{\partial \alpha_1}{\partial \dot{z}_1}\ddot{z}_1+\frac{\partial \alpha_1}{\partial \hat{\theta}_1}\dot{\hat{\theta}}_1\\ &= \frac{\partial \alpha_1}{\partial z_1}\dot{z}_1+\frac{\partial \alpha_1}{\partial \dot{z}_1}f_1(z_1,\dot{z}_1)\vartheta_1+\frac{\partial \alpha_1}{\partial \dot{z}_1}g_1(z_1,\dot{z}_1)x_2\\ &\quad +\frac{\partial \alpha_1}{\partial \hat{\theta}_1}H_1(z_1,\dot{z}_1)P_{1L}^{\mathrm{T}}\left(A_1^{0\sim 1}\right)z_1^{(0\sim 1)}\\ &= h_1^{\mathrm{T}}(z_1^{(0\sim 1)})\vartheta_1+\eta_1(z_{1\sim 2}^{(0\sim 1)}),\end{aligned} \tag{10.16}$$

where

$$\begin{cases} h_1^{\mathrm{T}}(z_1^{(0\sim 1)})=\frac{\partial \alpha_1}{\partial \dot{z}_1}f_1(z_1,\dot{z}_1)\\ \eta_1(z_{1\sim 2}^{(0\sim 1)})=\frac{\partial \alpha_1}{\partial z_1}\dot{z}_1+\frac{\partial \alpha_1}{\partial \dot{z}_1}g_1(z_1,\dot{z}_1)x_2+\frac{\partial \alpha_1}{\partial \hat{\theta}_1}H_1(z_1,\dot{z}_1)P_{1L}^{\mathrm{T}}\left(A_1^{0\sim 1}\right)z_1^{(0\sim 1)}. \end{cases}$$

On the other hand, we can obtain, from (10.8),

$$z_2^{(0\sim 1)}=\tilde{P}_2^{-1}\left(A_2^{0\sim 1}\right)\left(x_2^{(0\sim 1)}-\begin{bmatrix}\alpha_1\\ 0\end{bmatrix}\right), \tag{10.17}$$

which gives

$$\begin{aligned}\dot{z}_2 &= Q_{22}\left(A_2^{0\sim 1}\right)\left(x_2^{(0\sim 1)}-\begin{bmatrix}\alpha_1\\ 0\end{bmatrix}\right)\\ &= Q_{2F}\left(A_2^{0\sim 1}\right)(x_2-\alpha_1)+Q_{2L}\left(A_2^{0\sim 1}\right)\dot{x}_2.\end{aligned}$$

Taking differentials of the above equation, and using (10.16) and the second equation in (10.1), produce

$$\begin{aligned}\ddot{z}_2 &= Q_{2F}\left(A_2^{0\sim 1}\right)(\dot{x}_2-\dot{\alpha}_1)+Q_{2L}\left(A_2^{0\sim 1}\right)\ddot{x}_2\\ &= Q_{2F}\left(A_2^{0\sim 1}\right)\left(\dot{x}_2-h_1^{\mathrm{T}}(z_1^{(0\sim 1)})\vartheta_1-\eta_1(z_{1\sim 2}^{(0\sim 1)})\right)\\ &\quad +Q_{2L}\left(A_2^{0\sim 1}\right)\left(f_2(x_{1\sim 2}^{(0\sim 1)})\vartheta_2+g_2(x_{1\sim 2}^{(0\sim 1)})x_3\right),\end{aligned}$$

which gives

$$\ddot{z}_2=H_2^{\mathrm{T}}\left(z_{1\sim 2}^{(0\sim 1)}\right)\theta_2+q_2(z_{1\sim 2}^{(0\sim 1)})+L_2(z_{1\sim 2}^{(0\sim 1)})x_3, \tag{10.18}$$

where

$$\begin{cases} \theta_2=\begin{bmatrix}\vartheta_1 & \vartheta_2\end{bmatrix}^{\mathrm{T}}\\ H_2(z_{1\sim 2}^{(0\sim 1)})=\begin{bmatrix}-Q_{2F}\left(A_2^{0\sim 1}\right)h_1(z_1^{(0\sim 1)})\\ Q_{2L}\left(A_2^{0\sim 1}\right)f_2(x_{1\sim 2}^{(0\sim 1)})\end{bmatrix}\\ L_2(z_{1\sim 2}^{(0\sim 1)})=Q_{2L}\left(A_2^{0\sim 1}\right)g_2(x_{1\sim 2}^{(0\sim 1)})\\ q_2(z_{1\sim 2}^{(0\sim 1)})=Q_{2F}\left(A_2^{0\sim 1}\right)\left[\dot{x}_2-\eta_1(z_{1\sim 2}^{(0\sim 1)})\right]. \end{cases} \tag{10.19}$$

Further let

$$\tilde{P}_3\left(A_3^{0\sim1}\right)z_3^{(0\sim1)} = x_3^{(0\sim1)} - \begin{bmatrix} \alpha_2 \\ 0 \end{bmatrix}, \tag{10.20}$$

which can be equivalently decomposed, in view of the symmetry property of $P_3\left(A_3^{0\sim1}\right)$, into

$$P_{3L}^{\mathrm{T}}\left(A_3^{0\sim1}\right)z_3^{(0\sim1)} = x_3 - \alpha_2, \tag{10.21}$$

and

$$P_{3F}^{\mathrm{T}}\left(A_3^{0\sim1}\right)z_3^{(0\sim1)} = \dot{x}_3. \tag{10.22}$$

Substituting (10.21) into (10.18), yields

$$\ddot{z}_2 = H_2^{\mathrm{T}}\left(z_{1\sim2}^{(0\sim1)}\right)\theta_2 + q_2(z_{1\sim2}^{(0\sim1)}) + L_2(z_{1\sim2}^{(0\sim1)})\left(P_{3L}^{\mathrm{T}}\left(A_3^{0\sim1}\right)z_3^{(0\sim1)} + \alpha_2\right). \tag{10.23}$$

Applying Theorem 5.3 to the above system, gives the second virtual control law for the system as

$$\begin{cases} \dot{\hat{\theta}}_2 = H_2(z_{1\sim2}^{(0\sim1)})P_{2L}^{\mathrm{T}}(A_2^{0\sim1})z_2^{(0\sim1)} \\ \alpha_2 = -L_2^{-1}(z_{1\sim2}^{(0\sim1)})\left[A_2^{0\sim1}z_2^{(0\sim1)} + \alpha_2^*\right] \\ \alpha_2^* = H_2^{\mathrm{T}}(z_{1\sim2}^{(0\sim1)})\hat{\theta}_2 + q_2(z_{1\sim2}^{(0\sim1)}) + v_1, \end{cases} \tag{10.24}$$

where

$$v_1 = L_1(z_1, \dot{z}_1)P_{1L}^{\mathrm{T}}\left(A_1^{0\sim1}\right)z_1^{(0\sim1)}. \tag{10.25}$$

A corresponding closed-loop subsystem resulted in by this control law is

$$\ddot{z}_2 + A_2^{0\sim1}z_2^{(0\sim1)} = H_2^{\mathrm{T}}(z_{1\sim2}^{(0\sim1)})\tilde{\theta}_2 + w_2 - v_1, \tag{10.26}$$

where

$$w_2 = L_2(z_{1\sim2}^{(0\sim1)})P_{3L}^{\mathrm{T}}\left(A_3^{0\sim1}\right)z_3^{(0\sim1)}. \tag{10.27}$$

10.2.3 The Third Step

The variable α_2 given in (10.24) is a function with respect to $z_{1\sim2}^{(0\sim1)}$ and $\hat{\theta}_2$, that is,

$$\alpha_2 = \alpha_2\left(z_{1\sim2}^{(0\sim1)}, \hat{\theta}_2\right). \tag{10.28}$$

Similar to the derivation of $\dot{\alpha}_1$ in the second step, by taking the differentials to the above equation, we can obtain

$$\begin{aligned}
\dot{\alpha}_2 &= \frac{\partial \alpha_2}{\partial z_{1\sim 2}}\dot{z}_{1\sim 2}+\frac{\partial \alpha_2}{\partial \dot{z}_1}\ddot{z}_1+\frac{\partial \alpha_2}{\partial \dot{z}_2}\ddot{z}_2+\frac{\partial \alpha_2}{\partial \hat{\theta}_2}\dot{\hat{\theta}}_2 \\
&= \frac{\partial \alpha_2}{\partial z_{1\sim 2}}\dot{z}_{1\sim 2}+\frac{\partial \alpha_2}{\partial \dot{z}_1}\left(H_1^{\mathrm{T}}(z_1,\dot{z}_1)\theta_1+L_1(z_1,\dot{z}_1)\left(P_{2L}^{\mathrm{T}}\left(A_2^{0\sim 1}\right)z_2^{(0\sim 1)}+\alpha_1\right)\right) \\
&\quad +\frac{\partial \alpha_2}{\partial \dot{z}_2}\left(H_2^{\mathrm{T}}\left(z_{1\sim 2}^{(0\sim 1)}\right)\theta_2+q_2\left(z_{1\sim 2}^{(0\sim 1)}\right)+L_2\left(z_{1\sim 2}^{(0\sim 1)}\right)x_3\right) \\
&\quad +\frac{\partial \alpha_2}{\partial \hat{\theta}_2}H_2\left(z_{1\sim 2}^{(0\sim 1)}\right)P_{2L}^{\mathrm{T}}\left(A_2^{0\sim 1}\right)z_2^{(0\sim 1)} \\
&= \frac{\partial \alpha_2}{\partial \dot{z}_2}H_2^{\mathrm{T}}\left(z_{1\sim 2}^{(0\sim 1)}\right)\theta_2+\begin{bmatrix} \frac{\partial \alpha_2}{\partial \dot{z}_1}H_1^{\mathrm{T}}(z_1,\dot{z}_1) & 0 \end{bmatrix}\theta_2+\frac{\partial \alpha_2}{\partial z_{1\sim 2}}\dot{z}_{1\sim 2} \\
&\quad +\frac{\partial \alpha_2}{\partial \dot{z}_1}L_1(z_1,\dot{z}_1)\left(P_{2L}^{\mathrm{T}}\left(A_2^{0\sim 1}\right)z_2^{(0\sim 1)}+\alpha_1\right) \\
&\quad +\frac{\partial \alpha_2}{\partial \dot{z}_2}\left(q_2\left(z_{1\sim 2}^{(0\sim 1)}\right)+L_2\left(z_{1\sim 2}^{(0\sim 1)}\right)x_3\right)+\frac{\partial \alpha_2}{\partial \hat{\theta}_2}H_2\left(z_{1\sim 2}^{(0\sim 1)}\right)P_{2L}^{\mathrm{T}}\left(A_2^{0\sim 1}\right)z_2^{(0\sim 1)} \\
&= h_2^{\mathrm{T}}\left(z_{1\sim 3}^{(0\sim 1)}\right)\theta_2+\eta_2\left(z_{1\sim 3}^{(0\sim 1)}\right), \qquad (10.29)
\end{aligned}$$

where

$$\begin{cases}
h_2^{\mathrm{T}}\left(z_{1\sim 3}^{(0\sim 1)}\right)=\frac{\partial \alpha_2}{\partial \dot{z}_2}H_2^{\mathrm{T}}\left(z_{1\sim 2}^{(0\sim 1)}\right)+\begin{bmatrix} \frac{\partial \alpha_2}{\partial \dot{z}_1}H_1^{\mathrm{T}}(z_1,\dot{z}_1) & 0 \end{bmatrix} \\
\eta_2\left(z_{1\sim 3}^{(0\sim 1)}\right)=\frac{\partial \alpha_2}{\partial z_{1\sim 2}}\dot{z}_{1\sim 2}+\frac{\partial \alpha_2}{\partial \dot{z}_1}L_1(z_1,\dot{z}_1)\left(P_{2L}^{\mathrm{T}}\left(A_2^{0\sim 1}\right)z_2^{(0\sim 1)}+\alpha_1\right) \\
\qquad\qquad +\frac{\partial \alpha_2}{\partial \dot{z}_2}\left(q_2\left(z_{1\sim 2}^{(0\sim 1)}\right)+L_2\left(z_{1\sim 2}^{(0\sim 1)}\right)x_3\right) \\
\qquad\qquad +\frac{\partial \alpha_2}{\partial \hat{\theta}_2}H_2\left(z_{1\sim 2}^{(0\sim 1)}\right)P_{2L}^{\mathrm{T}}\left(A_2^{0\sim 1}\right)z_2^{(0\sim 1)}.
\end{cases}$$

Clearly, $h_2(z_{1\sim 3}^{(0\sim 1)})$ and $\eta_2(z_{1\sim 3}^{(0\sim 1)})$ are a two-dimensional vector function and a scalar function, respectively.

On the other hand, we can obtain, from (10.20),

$$z_3^{(0\sim 1)}=\tilde{P}_3^{-1}\left(A_3^{0\sim 1}\right)\left(x_3^{(0\sim 1)}-\begin{bmatrix} \alpha_2 \\ 0 \end{bmatrix}\right), \qquad (10.30)$$

which gives

$$\dot{z}_3=Q_{3F}\left(A_3^{0\sim 1}\right)(x_3-\alpha_2)+Q_{3L}\left(A_3^{0\sim 1}\right)\dot{x}_3.$$

Taking differentials of the above equation, and using (10.29) and the third equation in (10.1), we obtain

$$\ddot{z}_3=H_3^{\mathrm{T}}(z_{1\sim 3}^{(0\sim 1)})\theta_3+q_3(z_{1\sim 3}^{(0\sim 1)})+L_3(z_{1\sim 3}^{(0\sim 1)})x_4, \qquad (10.31)$$

where

$$\begin{cases} \theta_3 = \left[\vartheta_1\ \vartheta_2\ \vartheta_3\right]^{\mathrm{T}} \\ H_3(z_{1\sim3}^{(0\sim1)}) = \begin{bmatrix} -Q_{3F}\left(A_3^{0\sim1}\right) h_2(z_{1\sim3}^{(0\sim1)}) \\ Q_{3L}\left(A_3^{0\sim1}\right) f_3(x_{1\sim3}^{(0\sim1)}) \end{bmatrix} \\ L_3(z_{1\sim3}^{(0\sim1)}) = Q_{3L}\left(A_3^{0\sim1}\right) g_3(x_{1\sim3}^{(0\sim1)}) \\ q_3(z_{1\sim3}^{(0\sim1)}) = Q_{3F}\left(A_3^{0\sim1}\right)\left[\dot{x}_3 - \eta_2(z_{1\sim3}^{(0\sim1)})\right]. \end{cases} \tag{10.32}$$

Further, with the help of the transformation

$$\tilde{P}_4\left(A_4^{0\sim1}\right) z_4^{(0\sim1)} = x_4^{(0\sim1)} - \begin{bmatrix} \alpha_3 \\ 0 \end{bmatrix},$$

which can be partitioned as

$$\begin{aligned} P_{4L}^{\mathsf{T}}\left(A_4^{0\sim1}\right) z_4^{(0\sim1)} &= x_4 - \alpha_3, \\ P_{4F}^{\mathsf{T}}\left(A_4^{0\sim1}\right) z_4^{(0\sim1)} &= \dot{x}_4, \end{aligned}$$

we can obtain the expression of x_4. Substituting that into system (10.31), further gives

$$\ddot{z}_3 = H_3^{\mathrm{T}}(z_{1\sim3}^{(0\sim1)})\theta_3 + q_3(z_{1\sim3}^{(0\sim1)}) + L_3(z_{1\sim3}^{(0\sim1)})\left(P_{4L}^{\mathrm{T}}\left(A_4^{0\sim1}\right) z_4^{(0\sim1)} + \alpha_3\right). \tag{10.33}$$

Applying Theorem 5.3 to the above system, produces the third virtual control law for the system as

$$\begin{cases} \dot{\hat{\theta}}_3 = H_3(z_{1\sim3}^{(0\sim1)}) P_{3L}^{\mathrm{T}}\left(A_3^{0\sim1}\right) z_3^{(0\sim1)} \\ \alpha_3 = -L_3^{-1}(z_{1\sim3}^{(0\sim1)})\left[A_3^{0\sim1} z_3^{(0\sim1)} + \alpha_3^*\right] \\ \alpha_3^* = H_3^{\mathrm{T}}\left(z_{1\sim3}^{(0\sim1)}\right)\hat{\theta}_3 + q_3\left(z_{1\sim3}^{(0\sim1)}\right) + v_2, \end{cases} \tag{10.34}$$

where

$$v_2 = L_2(x_{1\sim2}^{(0\sim1)}) P_{2L}^{\mathrm{T}}\left(A_2^{0\sim1}\right) z_2^{(0\sim1)}. \tag{10.35}$$

A closed-loop subsystem resulted in by this control law is

$$\ddot{z}_3 + A_3^{0\sim1} z_3^{(0\sim1)} = H_3^{\mathrm{T}}(z_{1\sim3}^{(0\sim1)})\tilde{\theta}_3 + w_3 - v_2, \tag{10.36}$$

where

$$w_3 = L_3(z_{1\sim3}^{(0\sim1)}) P_{4L}^{\mathrm{T}}\left(A_4^{0\sim1}\right) z_4^{(0\sim1)}. \tag{10.37}$$

10.2.4 The n-th Step

Suppose that the expressions of α_{n-1}, $\dot{\alpha}_{n-1}$, $h_{n-1}(z_{1\sim n}^{(0\sim1)})$ and $\eta_{n-1}(z_{1\sim n}^{(0\sim1)})$ have all been obtained. Similar to the previous treatment, we can finally obtain

$$\ddot{z}_n = H_n^{\mathrm{T}}(z_{1\sim n}^{(0\sim1)})\theta_n + q_n(z_{1\sim n}^{(0\sim1)}) + L_n(z_{1\sim n}^{(0\sim1)})u, \tag{10.38}$$

where

$$\begin{cases} \theta_n = \left[\begin{array}{cccc} \vartheta_1 & \vartheta_2 & \cdots & \vartheta_n \end{array} \right]^{\mathrm{T}} \\ H_n(z_{1\sim n}^{(0\sim 1)}) = \left[\begin{array}{c} -Q_{nF}\left(A_n^{0\sim 1}\right) h_{n-1}(z_{1\sim n}^{(0\sim 1)}) \\ Q_{nL}\left(A_n^{0\sim 1}\right) f_n(x_{1\sim n}^{(0\sim 1)}) \end{array} \right] \\ L_n(z_{1\sim n}^{(0\sim 1)}) = Q_{nL}\left(A_n^{0\sim 1}\right) g_n(x_{1\sim n}^{(0\sim 1)}) \\ q_n(z_{1\sim n}^{(0\sim 1)}) = Q_{nF}\left(A_n^{0\sim 1}\right)\left[\dot{x}_n - \eta_{n-1}(z_{1\sim n}^{(0\sim 1)})\right]. \end{cases} \tag{10.39}$$

Therefore, by applying Theorem 5.3 to the above system (10.38), the adaptive control law for system (10.1) can be finally designed as

$$\begin{cases} \dot{\hat{\theta}}_n = H_n(z_{1\sim n}^{(0\sim 1)}) P_{nL}^{\mathrm{T}}\left(A_n^{0\sim 1}\right) z_n^{(0\sim 1)} \\ u = -L_n^{-1}(z_{1\sim n}^{(0\sim 1)})\left(A_n^{0\sim 1} z_n^{(0\sim 1)} + u^*\right) \\ u^* = H_n^{\mathrm{T}}\left(z_{1\sim n}^{(0\sim 1)}\right)\hat{\theta}_n + q_n\left(z_{1\sim n}^{(0\sim 1)}\right) + \upsilon_{n-1}, \end{cases} \tag{10.40}$$

where

$$\upsilon_{n-1} = L_{n-1}(z_{1\sim n-1}^{(0\sim 1)}) P_{n-1,L}^{\mathrm{T}}\left(A_{n-1}^{0\sim 1}\right) z_{n-1}^{(0\sim 1)}. \tag{10.41}$$

A closed-loop subsystem resulted in by this control law is

$$\ddot{z}_n + A_n^{0\sim 1} z_n^{(0\sim 1)} = H_n^{\mathrm{T}}\tilde{\theta}_n - \upsilon_{n-1}. \tag{10.42}$$

10.2.5 The Conclusion

Clearly, the above design process defines a transformation between $x_k^{(0\sim 1)}$ and $z_k^{(0\sim 1)}$, $i = 1, 2, 3, \ldots, n$, as

$$\begin{cases} z_1^{(0\sim 1)} = x_1^{(0\sim 1)}, \\ \tilde{P}_i\left(A_i^{0\sim 1}\right) z_i^{(0\sim 1)} = x_i^{(0\sim 1)} - \left[\begin{array}{c} \alpha_{i-1}(x_{1\sim i-1}) \\ 0 \end{array} \right], \\ \qquad\qquad i = 2, 3, \ldots, n, \end{cases} \tag{10.43}$$

In order to prove the stability of the system designed by the above process, we define the Lyapunov function

$$V = \sum_{i=1}^{n} V_i, \tag{10.44}$$

with

$$V_i = \left(z_i^{(0\sim 1)}\right)^{\mathrm{T}} P(A_i^{(0\sim 1)}) z_i^{(0\sim 1)} + \tilde{\theta}_i^{\mathrm{T}}\tilde{\theta}_i. \tag{10.45}$$

Following the proof of Theorem 5.3, it is easy to obtain

$$\begin{cases} \dot{V}_1 \le -\left\| z_1^{(0\sim 1)} \right\|^2 + 2\left(z_1^{(0\sim 1)}\right)^{\mathrm{T}} P_{1L}(A_1^{(0\sim 1)})w_1 \\ \dot{V}_2 \le -\left\| z_2^{(0\sim 1)} \right\|^2 + 2\left(z_2^{(0\sim 1)}\right)^{\mathrm{T}} P_{2L}(A_2^{(0\sim 1)})(w_2 - v_1) \\ \dot{V}_3 \le -\left\| z_3^{(0\sim 1)} \right\|^2 + 2\left(z_3^{(0\sim 1)}\right)^{\mathrm{T}} P_{3L}(A_3^{(0\sim 1)})(w_3 - v_2) \\ \quad \vdots \\ \dot{V}_{n-1} \le -\left\| z_{n-1}^{(0\sim 1)} \right\|^2 + 2\left(z_{n-1}^{(0\sim 1)}\right)^{\mathrm{T}} P_{n-1,L}(A_{n-1}^{(0\sim 1)})(w_{n-1} - v_{n-2}) \\ \dot{V}_n \le -\left\| z_n^{(0\sim 1)} \right\|^2 + 2\left(z_n^{(0\sim 1)}\right)^{\mathrm{T}} P_{nL}(A_n^{(0\sim 1)})(-v_{n-1}). \end{cases} \tag{10.46}$$

In view of

$$\begin{cases} v_i = L_i(x_{1\sim i}^{(0\sim 1)}) P_{iL}^{\mathrm{T}}\left(A_i^{0\sim 1}\right) z_i^{(0\sim 1)} \\ w_i = L_i(x_{1\sim i}^{(0\sim 1)}) P_{i+1,L}^{\mathrm{T}}\left(A_{i+1}^{0\sim 1}\right) z_{i+1}^{(0\sim 1)}, \end{cases} \tag{10.47}$$

it can be easily verified that

$$\begin{aligned} \left(z_i^{(0\sim 1)}\right)^{\mathrm{T}} P_{iL}(A_i^{(0\sim 1)})w_i &= \left(z_{i+1}^{(0\sim 1)}\right)^{\mathrm{T}} P_{i+1,L}(A_{i+1}^{(0\sim 1)})v_i, \\ i &= 1, 2, \ldots, n-1. \end{aligned} \tag{10.48}$$

Adding the equations in (10.46) side by side, and using the relations in (10.48), yield

$$\dot{V} \le -\sum_{k=1}^{n} \left\| z_k^{(0\sim 1)} \right\|^2. \tag{10.49}$$

The above process proves the following conclusion.

Theorem 10.1 *Let Assumption 10.1 hold, and $A_i^{0\sim 1} \in \mathbb{R}^{1\times 2}$, $i = 1, 2, \ldots, n$, be a set of matrices such that $\Phi\left(A_i^{0\sim 1}\right) \in \mathbb{R}^{2\times 2}$, $i = 1, 2, \ldots, n$, are stable. Then, for any set of unknown constant parameters $\vartheta_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, the adaptive control law (10.40) for the second-order SFS (10.1) guarantees that*

1. $z_i^{(0\sim 1)} \to 0$, $t \to \infty$, $\quad i = 1, 2, \ldots, n$;
2. $\left\| \theta_i - \hat{\theta}_i \right\|$, $i = 1, 2, \ldots, n$ *are bounded.*

It follows from the above theorem that $x_1 \to 0$, $t \to \infty$. However, if the state transformation (10.43) is a diffeomorphism from $x_{1\sim n}$ to $z_{1\sim n}$, and $x_{1\sim n}(0) = z_{1\sim n}(0)$, or equivalently, $\alpha_{i-1}(x_{1\sim i}(0)) = 0$, $i = 2, 3, \ldots, n$, then we also have

$$x_i^{(0\sim 1)} \to 0, \; t \to \infty, \quad i = 1, 2, \ldots, n.$$

Remark 10.1 The above proposed second-order backstepping method can be easily modified to suit the other companion SFS (10.2). As a matter of fact, for SFS (10.2), we can simply take $\tilde{P}_i\left(A_i^{0\sim 1}\right) = P_i\left(A_i^{0\sim 1}\right)$, $\quad i = 1, 2, \ldots, n$, and in this case we also have theoretically $Q_{iL}\left(A_i^{0\sim 1}\right) \neq 0$, $\quad i = 1, 2, \ldots, n$, which ensure $L_i\left(z_{1\sim i}^{(0\sim 1)}\right) \neq 0$, $i = 1, 2, \ldots, n$.

10.3 Backstepping for High-order SFSs

Similar to the above section, we can also give, based on Theorem 5.3, a direct mixed-order backstepping method for SFS (10.3) without turning it into a conventional first-order SFS. To do this we also need some preparations.

Suppose that $A_i^{0\sim m_i-1}\in\mathbb{R}^{1\times m_i},\ i=1,2,\ldots,n$, are a set of matrices which make $\Phi\left(A_i^{0\sim m_i-1}\right)\in\mathbb{R}^{m_i\times m_i},\ i=1,2,\ldots,n$, stable, and $P_i\left(A_i^{0\sim m_i-1}\right)\in\mathbb{R}^{m_i\times m_i}$ is a positive definite solution to the Lyapunov inequality

$$\Phi^{\mathrm{T}}\left(A_i^{0\sim m_i-1}\right)P_i\left(A_i^{0\sim m_i-1}\right)+P_i\left(A_i^{0\sim m_i-1}\right)\Phi\left(A_i^{0\sim m_i-1}\right)\leq -I_{m_i},\qquad(10.50)$$

for $i=1,2,\ldots,n$. Denote

$$P_i\left(A_i^{0\sim m_i-1}\right)=\left[P_{iF}\left(A_i^{0\sim m_i-1}\right)\quad\cdots\quad P_{iL}\left(A_i^{0\sim m_i-1}\right)\right]\in\mathbb{R}^{m_i\times m_i},$$

where $P_{iF}\left(A_i^{0\sim m_i-1}\right)$ and $P_{iL}\left(A_i^{0\sim m_i-1}\right)$ are the first and last column, respectively. Further, define, for $i=1,2,\ldots,n$,

$$\tilde{P}_i\left(A_i^{0\sim m_i-1}\right)=I_{m_i}^{\circ}P_i\left(A_i^{0\sim m_i-1}\right),$$

and denote

$$\begin{aligned}\tilde{P}_i^{-1}\left(A_i^{0\sim m_i-1}\right)&=\begin{bmatrix}Q_{i1}\left(A_i^{0\sim m_i-1}\right)\\ Q_{i2}\left(A_i^{0\sim m_i-1}\right)\end{bmatrix}\\&=\begin{bmatrix}&&Q_{i1}\left(A_i^{0\sim m_i-1}\right)\\ Q_{iF}\left(A_i^{0\sim m_i-1}\right)&Q_{iM}\left(A_i^{0\sim m_i-1}\right)&Q_{iL}\left(A_i^{0\sim m_i-1}\right)\end{bmatrix},\end{aligned}$$

where $Q_{iF}\left(A_i^{0\sim m_i-1}\right)$ and $Q_{iL}\left(A_i^{0\sim m_i-1}\right)$ are two scalars. Since $P_i\left(A_i^{0\sim m_i-1}\right)$ given by (10.50) is not unique, we can properly choose a $P_i\left(A_i^{0\sim m_i-1}\right)$ such that

$$Q_{iL}\left(A_i^{0\sim m_i-1}\right)\neq 0,\quad i=1,2,\ldots,n.\qquad(10.51)$$

For simplicity, only the basic ideas and steps are given here.

10.3.1 The First Step

Let

$$z_1^{(0\sim m_1-1)}=x_1^{(0\sim m_1-1)},\qquad(10.52)$$

and

$$\tilde{P}_2\left(A_2^{0\sim m_2-1}\right)z_2^{(0\sim m_2-1)}=x_2^{(0\sim m_2-1)}-\begin{bmatrix}\alpha_1\\0\end{bmatrix},\qquad(10.53)$$

which produces, in view of the symmetry property of $P_2\left(A_2^{0\sim m_2-1}\right)$,

$$P_{2L}^{\mathrm{T}}\left(A_2^{0\sim m_2-1}\right)z_2^{(0\sim m_2-1)}=x_2-\alpha_1. \tag{10.54}$$

Thus by using (10.52) and (10.54), the first equation in (10.3) is turned into

$$z_1^{(m_1)}=H_1^{\mathrm{T}}(z_1^{(0\sim m_1-1)})\theta_1+L_1(z_1^{(0\sim m_1-1)})\left(P_{2L}^{\mathrm{T}}\left(A_2^{0\sim m_2-1}\right)z_2^{(0\sim m_2-1)}+\alpha_1\right), \tag{10.55}$$

where

$$\begin{cases}\theta_1=\vartheta_1\\ H_1^{\mathrm{T}}(z_1^{(0\sim m_1-1)})=f_1(z_1^{(0\sim m_1-1)})\\ L_1(z_1^{(0\sim m_1-1)})=g_1(z_1^{(0\sim m_1-1)}),\end{cases} \tag{10.56}$$

then, by applying Theorem 5.3 to system (10.55), the first virtual control law can be designed as

$$\begin{cases}\dot{\hat{\theta}}_1=H_1(z_1^{(0\sim m_1-1)})P_{1L}^{\mathrm{T}}\left(A_1^{0\sim m_1-1}\right)z_1^{(0\sim m_1-1)}\\ \alpha_1=-L_1^{-1}(z_1^{(0\sim m_1-1)})\left(A_1^{0\sim m_1-1}z_1^{(0\sim m_1-1)}+H_1^{\mathrm{T}}(z_1^{(0\sim m_1-1)})\hat{\theta}_1\right),\end{cases} \tag{10.57}$$

and a closed-loop subsystem resulted in by this control law is

$$z_1^{(m_1)}+A_1^{0\sim m_1-1}z_1^{(0\sim m_1-1)}=H_1^{\mathrm{T}}(z_1^{(0\sim m_1-1)})\tilde{\theta}_1+w_1, \tag{10.58}$$

where

$$w_1=L_1(z_1^{(0\sim m_1-1)})P_{2L}^{\mathrm{T}}\left(A_2^{0\sim m_2-1}\right)z_2^{(0\sim m_2-1)}. \tag{10.59}$$

10.3.2 The Second Step

The virtual control α_1 is a function with respect to $z_1^{(0\sim m_1-1)}$ and $\hat{\theta}_1$, that is,

$$\alpha_1=\alpha_1(z_1^{(0\sim m_1-1)},\hat{\theta}_1).$$

Using the expression of $x_1^{(m_1)}$ given in (10.3) and that of $\dot{\hat{\theta}}_1$ given in (10.57), we obtain

$$\begin{aligned}\dot{\alpha}_1 &= \sum_{i=0}^{m_1-2}\frac{\partial\alpha_1}{\partial z_1^{(i)}}z_1^{(i+1)}+\frac{\partial\alpha_1}{\partial z_1^{(m_1-1)}}z_1^{(m_1)}+\frac{\partial\alpha_1}{\partial\hat{\theta}_1}\dot{\hat{\theta}}_1\\ &= \sum_{i=0}^{m_1-2}\frac{\partial\alpha_1}{\partial z_1^{(i)}}z_1^{(i+1)}+\frac{\partial\alpha_1}{\partial z_1^{(m_1-1)}}f_1(z_1^{(0\sim m_1-1)})\vartheta_1+\frac{\partial\alpha_1}{\partial z_1^{(m_1-1)}}g_1(z_1^{(0\sim m_1-1)})x_2\\ &\quad+\frac{\partial\alpha_1}{\partial\hat{\theta}_1}H_1(z_1^{(0\sim m_1-1)})P_{1L}^{\mathrm{T}}\left(A_1^{0\sim m_1-1}\right)z_1^{(0\sim m_1-1)}\\ &= h_1^{\mathrm{T}}(z_1^{(0\sim m_1-1)})\vartheta_1+\eta_1(z_i^{(0\sim m_i-1)}|_{\,i=1\sim 2}),\end{aligned} \tag{10.60}$$

where

$$\begin{cases} h_1^{\mathrm{T}}(z_1^{(0\sim m_1-1)}) = \frac{\partial \alpha_1}{\partial z_1^{(m_1-1)}} f_1(z_1^{(0\sim m_1-1)}) \\ \eta_1(z_i^{(0\sim m_i-1)}|_{i=1\sim 2}) = \sum_{i=0}^{m_1-2} \frac{\partial \alpha_1}{\partial z_1^{(i)}} z_1^{(i+1)} + \frac{\partial \alpha_1}{\partial z_1^{(m_1-1)}} g_1(z_1^{(0\sim m_1-1)}) x_2 \\ \qquad\qquad + \frac{\partial \alpha_1}{\partial \hat{\theta}_1} H_1(z_1^{(0\sim m_1-1)}) P_{1L}^{\mathrm{T}} \left(A_1^{0\sim m_1-1}\right) z_1^{(0\sim m_1-1)}. \end{cases}$$

On the other hand, we can obtain, from (10.8),

$$z_2^{(0\sim m_2-1)} = \tilde{P}_2^{-1} \left(A_2^{0\sim m_2-1}\right) \left(x_2^{(0\sim m_2-1)} - \begin{bmatrix} \alpha_1 \\ 0 \end{bmatrix}\right), \tag{10.61}$$

which gives

$$\begin{aligned} z_2^{(m_2-1)} &= Q_{22} \left(A_2^{0\sim m_2-1}\right) \left(x_2^{(0\sim m_2-1)} - \begin{bmatrix} \alpha_1 \\ 0 \end{bmatrix}\right) \\ &= Q_{2F} \left(A_2^{0\sim m_2-1}\right) (x_2 - \alpha_1) \\ &\quad + Q_{2M} \left(A_2^{0\sim m_2-1}\right) x_2^{(1\sim m_2-2)} + Q_{2L} \left(A_2^{0\sim m_2-1}\right) x_2^{(m_2-1)}. \end{aligned}$$

Taking differentials of the above equation, and using (10.60) and the second equation in (10.3), yield

$$\begin{aligned} z_2^{(m_2)} &= Q_{2F} \left(A_2^{0\sim m_2-1}\right) (\dot{x}_2 - \dot{\alpha}_1) \\ &\quad + Q_{2M} \left(A_2^{0\sim m_2-1}\right) \dot{x}_2^{(1\sim m_2-2)} + Q_{2L} \left(A_2^{0\sim m_2-1}\right) x_2^{(m_2)} \\ &= -Q_{2F} \left(A_2^{0\sim m_2-1}\right) h_1^{\mathrm{T}}(z_1^{(0\sim m_1-1)}) \vartheta_1 \\ &\quad + Q_{2L} \left(A_2^{0\sim m_2-1}\right) f_2(x_i^{(0\sim m_i-1)}|_{i=1,2}) \vartheta_2 \\ &\quad + Q_{2F} \left(A_2^{0\sim m_2-1}\right) \left(\dot{x}_2 - \eta_1(z_i^{(0\sim m_i-1)}|_{i=1\sim 2})\right) \\ &\quad + Q_{2M} \left(A_2^{0\sim m_2-1}\right) \dot{x}_2^{(1\sim m_2-2)} + Q_{2L} \left(A_2^{0\sim m_2-1}\right) g_2(x_i^{(0\sim m_i-1)}|_{i=1,2}) x_3 \\ &= H_2^{\mathrm{T}} \left(z_i^{(0\sim m_i-1)}|_{i=1,2}\right) \theta_2 + q_2(z_i^{(0\sim m_i-1)}|_{i=1,2}) \\ &\quad + L_2(z_i^{(0\sim m_i-1)}|_{i=1,2}) x_3, \end{aligned} \tag{10.62}$$

where

$$\begin{cases} \theta_2 = \begin{bmatrix} \vartheta_1 & \vartheta_2 \end{bmatrix}^{\mathrm{T}} \\ H_2(z_i^{(0\sim m_i-1)}|_{i=1\sim 2}) = \begin{bmatrix} -Q_{2F} \left(A_2^{0\sim m_2-1}\right) h_1(z_1^{(0\sim m_1-1)}) \\ Q_{2L} \left(A_2^{0\sim m_2-1}\right) f_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2}) \end{bmatrix} \\ L_2(z_i^{(0\sim m_i-1)}|_{i=1\sim 2}) = Q_{2L} \left(A_2^{0\sim m_2-1}\right) g_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2}) \\ q_2(z_i^{(0\sim m_i-1)}|_{i=1\sim 2}) = Q_{2F} \left(A_2^{0\sim m_2-1}\right) \left(\dot{x}_2 - \eta_1(z_i^{(0\sim m_i-1)}|_{i=1\sim 2})\right) \\ \qquad\qquad + Q_{2M} \left(A_2^{0\sim m_2-1}\right) \dot{x}_2^{(1\sim m_2-2)}. \end{cases} \tag{10.63}$$

Further let

$$\tilde{P}_3\left(A_3^{0\sim m_3-1}\right)z_3^{(0\sim m_3-1)}=x_3^{(0\sim m_3-1)}-\left[\begin{array}{c}\alpha_2\\0\end{array}\right], \tag{10.64}$$

then we have, in view of the symmetry property of $P_3\left(A_3^{0\sim m_3-1}\right)$,

$$P_{3L}^{\mathrm{T}}\left(A_3^{0\sim m_3-1}\right)z_3^{(0\sim m_3-1)}=x_3-\alpha_2. \tag{10.65}$$

Substituting the above (10.65) into (10.62), yields

$$\begin{aligned}z_2^{(m_2)} &= H_2^{\mathrm{T}}\left(z_i^{(0\sim m_i-1)}|_{i=1\sim 2}\right)\theta_2+q_2(z_i^{(0\sim m_i-1)}|_{i=1\sim 2})\\ &\quad +L_2(z_i^{(0\sim m_i-1)}|_{i=1\sim 2})\left(P_{3L}^{\mathrm{T}}\left(A_3^{0\sim m_3-1}\right)z_3^{(0\sim m_3-1)}+\alpha_2\right),\end{aligned} \tag{10.66}$$

By applying Theorem 5.3 to the above system, the second virtual control law for the system can be designed as

$$\left\{\begin{array}{l}\dot{\hat{\theta}}_2=H_2(z_i^{(0\sim m_i-1)}|_{i=1\sim 2})P_{2L}^{\mathrm{T}}(A_2^{0\sim m_2-1})z_2^{(0\sim m_2-1)}\\ \alpha_2=-L_2^{-1}(z_i^{(0\sim m_i-1)}|_{i=1\sim 2})\left[A_2^{0\sim m_2-1}z_2^{(0\sim m_2-1)}+\alpha_2^*\right]\\ \alpha_2^*=H_2^{\mathrm{T}}(z_i^{(0\sim m_i-1)}|_{i=1\sim 2})\hat{\theta}_2+q_2(z_i^{(0\sim m_i-1)}|_{i=1\sim 2})+v_1,\end{array}\right. \tag{10.67}$$

where

$$v_1=L_1(z_1^{(0\sim m_1-1)})P_{1L}^{\mathrm{T}}\left(A_1^{0\sim m_1-1}\right)z_1^{(0\sim m_1-1)}. \tag{10.68}$$

A corresponding closed-loop subsystem resulted in by this control law is

$$z_2^{(m_2)}+A_2^{0\sim m_2-1}z_2^{(0\sim m_2-1)}=H_2^{\mathrm{T}}(z_i^{(0\sim m_i-1)}|_{i=1\sim 2})\tilde{\theta}_2+w_2-v_1, \tag{10.69}$$

where

$$w_2=L_2(z_i^{(0\sim m_i-1)}|_{i=1\sim 2})P_{3L}^{\mathrm{T}}\left(A_3^{0\sim m_3-1}\right)z_3^{(0\sim m_3-1)}. \tag{10.70}$$

10.3.3 The n-th Step

Suppose that the expressions of α_{n-1}, $\dot{\alpha}_{n-1}$, $h_{n-1}(x_i^{(0\sim m_i-1)}|_{\,i=1\sim n})$ and $\eta_{n-1}(x_i^{(0\sim m_i-1)}|_{\,i=1\sim n})$ have all been obtained. Similar to the previous operation, we can obtain

$$z_n^{(m_n)}=H_n^{\mathrm{T}}(z_i^{(0\sim m_i-1)}|_{\,i=1\sim n})\theta_n+q_n(z_i^{(0\sim m_i-1)}|_{\,i=1\sim n})+L_n(z_i^{(0\sim m_i-1)}|_{\,i=1\sim n})u, \tag{10.71}$$

where

$$\left\{\begin{array}{l}\theta_n=\left[\begin{array}{cccc}\vartheta_1 & \vartheta_2 & \cdots & \vartheta_n\end{array}\right]^{\mathrm{T}}\\ H_n(z_i^{(0\sim m_i-1)}|_{\,i=1\sim n})=\left[\begin{array}{c}-Q_{nF}\left(A_n^{0\sim m_n-1}\right)h_{n-1}(z_i^{(0\sim m_i-1)}|_{\,i=1\sim n})\\ Q_{nL}\left(A_n^{0\sim m_n-1}\right)f_n(x_i^{(0\sim m_i-1)}|_{\,i=1\sim n})\end{array}\right]\\ L_n(z_i^{(0\sim m_i-1)}|_{\,i=1\sim n})=Q_{nL}\left(A_n^{0\sim m_n-1}\right)g_n(x_i^{(0\sim m_i-1)}|_{\,i=1\sim n})\\ q_n(z_i^{(0\sim m_i-1)}|_{\,i=1\sim n})=Q_{nF}\left(A_n^{0\sim m_n-1}\right)\left(\dot{x}_n-\eta_{n-1}(z_i^{(0\sim m_i-1)}|_{\,i=1\sim n})\right)\\ \qquad\qquad +Q_{nM}\left(A_n^{0\sim m_n-1}\right)\dot{x}_n^{(1\sim m_n-2)}.\end{array}\right. \tag{10.72}$$

Therefore, by applying Theorem 5.3 to system (10.71), the control law for the system can be finally designed as

$$\begin{cases} \dot{\hat{\theta}}_n = H_n P_{nL}^{\mathrm{T}} \left(A_n^{0\sim m_n-1}\right) z_n^{(0\sim m_n-1)} \\ u = -L_n^{-1}\left(A_n^{0\sim m_n-1} z_n^{(0\sim m_n-1)} + u^*\right) \\ u^* = H_n^{\mathrm{T}}\hat{\theta}_n + q_n + \upsilon_{n-1}, \end{cases} \tag{10.73}$$

where

$$\upsilon_{n-1} = L_{n-1} P_{n-1,L}^{\mathrm{T}} \left(A_{n-1}^{0\sim m_{n-1}-1}\right) z_{n-1}^{(0\sim m_{n-1}-1)}. \tag{10.74}$$

A closed-loop subsystem resulted in by this control law is

$$z_n^{(m_n)} + A_n^{0\sim m_n-1} z_n^{(0\sim m_n-1)} = H_n^{\mathrm{T}}\tilde{\theta}_n - \upsilon_{n-1}. \tag{10.75}$$

10.3.4 The Conclusion

It should be noted that the above procedure also defines a state transformation between $z_i^{(0\sim m_i-1)}$ and $x_i^{(0\sim m_i-1)}$ in the form of

$$\begin{cases} z_1^{(0\sim m_1-1)} = x_1^{(0\sim m_1-1)} \\ \tilde{P}_i \left(A_i^{0\sim m_i-1}\right) z_i^{(0\sim m_i-1)} = x_i^{(0\sim m_i-1)} - \begin{bmatrix} \alpha_{i-1} \\ 0 \end{bmatrix}, \\ \qquad\qquad\qquad\qquad i = 2, 3, \ldots, n, \end{cases} \tag{10.76}$$

where $\alpha_{i-1} = \alpha_{i-1}(x_k^{(0\sim m_k-1)}|_{k=1\sim i-1})$. Similar to the proof of Theorem 10.1, we can also obtain the following conclusion which gives a property of the transformed states $z_i^{(0\sim m_i-1)}$, $i = 1, 2, \ldots, n$.

Theorem 10.2 *Let Assumption 10.2 hold, and $A_i^{0\sim m_i-1} \in \mathbb{R}^{1\times m_i}$, $i = 1, 2, \ldots, n$, be a set of matrices which make $\Phi\left(A_i^{0\sim m_i-1}\right)$, $i = 1, 2, \ldots, n$, stable. Then, for any set of unknown constant parameters $\vartheta_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, the adaptive control law (10.73) for system (10.3) guarantees that*

1. $z_i^{(0\sim m_i-1)} \to 0, t \to \infty, \quad i = 1, 2, \ldots, n;$
2. $\left\|\theta_i - \hat{\theta}_i\right\|, i = 1, 2, \ldots, n,$ *are bounded.*

Since $z_1 = x_1$, we have

$$x_1^{(0\sim m_1-1)} \to 0,\ t \to \infty.$$

In order to let $x_1(t)$ track asymptotically some sufficiently smooth reference signal $x_1^*(t)$, it suffices to substitute, in the second- or high-order methods of backstepping, equation (10.7) or (10.52) by

$$z_1^{(0\sim 1)} = x_1^{(0\sim 1)} - \left(x_1^*\right)^{(0\sim 1)},$$

or

$$z_1^{(0\sim m_1-1)} = x_1^{(0\sim m_1-1)} - \left(x_1^*\right)^{(0\sim m_1-1)},$$

and then deal with the terms related to the reference signal $x_1^*(t)$ properly in the sequential processes.

In description, the above backstepping method of high-order systems appears to be much more complicated. Yet it should be noted that in many specific engineering applications, m_i, $i = 1, 2, \ldots, n$, often take the values of 1 and 2, and in such applications, the mixed-order backstepping method is in fact simpler than the second-order backstepping method.

Remark 10.2 The above proposed mixed-order backstepping method can also be easily modified to suit the other companion SFS (10.4). As a matter of fact, for SFS (10.4), we can simply take

$$\tilde{P}_i\left(A_i^{0\sim m_i-1}\right) = P_i\left(A_i^{0\sim m_i-1}\right), \quad i = 1, 2, \ldots, n,$$

and in this case we also have theoretically $Q_{iL}\left(A_i^{0\sim m_i-1}\right) \neq 0, \quad i = 1, 2, \ldots, n$, since $Q_{iL}\left(A_i^{0\sim m_i-1}\right)$ can now be shown to be a diagonal element of a positive definite matrix.

Remark 10.3 It is commonly known that the method of backstepping can hardly be applicable to an SFS with many subsystems due to the well-known "differential explosion" problem. Such a problem of course applies to the newly proposed second- and high-order methods of backstepping. Nevertheless, for a specific system physically modelled by, e.g., Lagrangian Equation, a second-order method of backstepping needs only half of the steps of the first-order method of backstepping, and also saves the process of converting the second-order system into a first-order one, and is thus simpler to apply than the normal first-order method of backstepping. Therefore, the method of high-order (or mixed-order) backstepping is generally simpler than the normal first-order method of backstepping.

10.4 Example

10.5 Notes and References

The chapter is closely related to Duan (2021d). It proposes a FAS approach for adaptive control of nonlinear systems with unknown parameters, and has demonstrated the following:

- Many nonlinear systems with an unknown parameter vector can be represented by second- and/or high-order SFSs with unknown parameters. Like the well-known first-order SFSs with unknown parameters, the newly proposed second- and high-order uncertain SFSs with unknown parameters are also system models closely related to uncertain FAS models for nonlinear systems since each of their subsystem obey a full-actuation structure.

- For second- and high-order SFSs with unknown parameters, the designs turn out to be the generalized second- and high-order methods of backstepping for second- and high-order SFSs, correspondingly.
- The proposed second- and high-order methods of backstepping for second- and high-order SFSs can be further generalized to the case that both x_i and ϑ_i, $i = 1, 2, \ldots, n$, are vectors.

The second-order and mixed-order SFSs with unknown parameters discussed in this chapter can also be extended in several ways. Firstly, these can be generalized to second-order and mixed-order SFSs with increasing dimensions. Secondly, they can be also generalized into second-order and mixed-order nonaffine SFSs. Correspondingly, robust high-order backstepping schemes for all these extended cases can also be derived.

Part IV

Output Feedback and Type I Systems

Chapter 11

Observer-based Control

This chapter addresses observer-based output feedback control of fully actuated systems (FASs). Both the full-order observer case and the reduced-order observer case are considered.

For full-order observer-based control of a class of FASs subject to Lipschitz assumption, a full-order observer-based controller is designed, which assures the uniformly exponential stability of the augmented closed-loop system, and the effect of the proposed method is demonstrated by the simulation of the thermoacoustic instability system. The parameters to be designed for the controller are determined using a parametric approach which provides a rich degree of design freedom.

For reduced-order observer-based output feedback stabilization of a class of FASs subject to a partial Lipschitz assumption, a dynamical output controller is provided based on a reduced-order observer. Conditions for global exponential stability of the corresponding closed-loop system are provided based on linear matrix inequality (LMI) technique and theory of input-to-state stability. The discussion is divided into several cases, and an illustrative example shows the effect of the proposed method.

11.1 Introduction

11.1.1 Observers and Observer-based Control

The observer-based control design for general nonlinear systems is quite difficult, and there is currently no universal and effective method. Different from the linear cases, the separation principle is not usually guaranteed due to the complex structure of nonlinear systems. Most of the existing results are obtained in a local sense for systems with special forms under some certain assumptions (Sun et al. (2022c,b); Katayama (2019)).

Since nonlinearities always exist in practice and the state variables of the system are not always measurable, observer-based control for nonlinear dynamic systems has attracted the attention of many researchers in recent decades and a large number of results have been produced (see Golubev et al. (2005) and the references therein). Among these studies, in order to make full use of the output information and reduce the dimen-

sionality of the state observer, many studies have been carried out on control based on reduced-order observers in the state-space framework (see, e.g., Bartolini and Punta (2010); Liu and Zhang (2004); Davari et al. (2012); Meng et al. (2024)). For a general nonlinear system in state-space representation, the reduced-order observer-based control is quite difficult, and many existing results are obtained for single-input systems (Sun et al. (2023); Oh and Choi (2021); Zhang et al. (2010); Hua et al. (2016); Ma and Ma (2020)). In Sun et al. (2023), a class of systems whose nonlinearity satisfies certain growth conditions is considered, for which an adaptive event-triggered output feedback control method is proposed to ensure that the tracking error converges to an arbitrarily small residual set. To achieve global exponential stabilization, in Oh and Choi (2021), a reduced-order observer-based output feedback controller with a gain-scaling factor is designed for a type of nonlinear systems with the extended structure of nonlinearity. In addition, some studies further consider the observer-based control for certain classes of systems with time delays. Specifically, the effects of time-varying time delays and disturbances are considered in Zhang et al. (2010), and the output feedback prescribed performance control of time-delay systems with unmodeled dynamics is considered in Hua et al. (2016). Different from the requirements that the control input coefficients of the systems are 1 in Sun et al. (2023); Oh and Choi (2021); Zhang et al. (2010); Hua et al. (2016), the control input coefficient of the fractional-order system considered in Ma and Ma (2020) is allowed to be a known positive constant.

Some results have also been obtained on observer-based control problems for multi-input nonlinear systems, most of which require the nonlinearities to have special structures or to satisfy certain assumptions. For example, Ornelas-Tellez et al. (2019) requires the nonlinearity to be in the so-called state-dependent coefficient factorized form and designs a reduced-order observer for such a class of systems. Furthermore, although there are no structural restrictions on the nonlinearity in the interconnected system considered in Kalsi et al. (2009), certain quadratic constraints need to be satisfied. In addition, one-sided Lipschitz conditions are frequently imposed on nonlinearities in some studies (see, e.g., Dong et al. (2021); Shi et al. (2024)). With the help of the quasi-one-sided Lipschitz condition, the reduced-order observer and feedback controller can be designed separately for the system studied in Dong et al. (2021). In some development results, in order to achieve better control effects, it is sometimes necessary to additionally satisfy other assumptions for the nonlinear part, such as the quadratically inner-bounded conditions (see Shi et al. (2024)).

11.1.2 The FAS Approach for Observer-based Control

Recently, as a unified control method for general multi-variable nonlinear systems, a so-called FAS approach has been proposed in Duan (2020a,b,c, 2021a,b,c). Due to the generality, the FAS method has been used to deal with many control problems (see Duan (2024e) and the references therein), among which a typical class of problems is observer-based control. In the framework of the FAS approach, rich results have been produced on disturbance observer-based control due to the ubiquity of disturbances in practical systems. For example, in Zhang and Liu (2023a), a disturbance observer-based predictive control method is developed for networked high-order FASs, in which the disturbance observer is used to handle external disturbances. Subsequently, some

researchers further developed disturbance observers with better performance for practical systems, such as fixed-time disturbance observers (Li et al. (2024f)) and prescribed-time disturbance observers (Lu et al. (2024b)). Besides, benefiting from the simple structure of the FAS model, fruitful results regarding the disturbance observer-based control can also be found in the references Liu et al. (2022a); Chen et al. (2023); Adıgüzel and Yalçın (2022); Cui and Duan (2023). Furthermore, another type of observers that have the ability to estimate the lumped disturbance and the states is the extended state observers (ESOs), and many results of extended state observer-based control for FASs are produced. In Zhao and Duan (2022a), an ESO-based control method is developed to achieve tracking of the motion of a 6-degree-of-freedom spacecraft. Different from the above traditional ESO, Cai et al. (2023b) presented a fully-actuated fault-tolerant control technique in conjunction with an ESO involving the sign function, but the chattering phenomenon became an inevitable problem. In the same year, the similar structure of ESO was utilized to estimate unknown state variables of a proton exchange membrane fuel cell air feed system (Li et al. (2023a)). Besides, many other results about ESO-based control can be found in Xiao and Chen (2024); Jiang et al. (2024b,a).

In addition to the disturbance observers and the ESOs mentioned above, state observers, as a very typical and common type of observers, have also received great attention, and the state observer-based control of FASs has been widely studied. In Zhao et al. (2023a), two design methods for exponentially stable full-order observers are developed, with which a parametric control method based on such an exponentially stable observer is proposed for FASs to ensure the exponential stability of the closed-loop system. Importantly, the separation principle is also proved to hold for the nonlinear time-varying FASs. Under the same Lipschitz condition, a generalized proportional-integral observer for single-order FASs is presented by using the LMI technique in Jiang and Duan (2023), and the output tracking problem is considered. Subsequently, in Chen et al. (2024b), an observer-based control problem for a class of FASs with time-varying delays is considered, and the uniformly exponential stability of the closed-loop system is achieved. The above studies all focus on the control problem based on the full-order state observer, while the related research on the reduced-order observer in the FAS framework is still very limited. Jiang and Duan (2024) designs a nonlinear reduced-order observer for FASs, based on which an output tracking control scheme is proposed and the condition for the exponential convergence of the tracking error is given.

The next section of this chapter proposes an observer-based control method for a class of FASs. The model of FASs is proposed, which represents a complex and general control model. An observer-based controller is given for the FAS and the augmented system is also derived. The uniformly exponential stability of this augmented system is strictly proved by applying Lyapunov stability analysis. Finally, the proposed method is applied to the control of a thermoacoustic instability system and the simulation results show the effectiveness.

In the third section of this chapter, the reduced-order observer-based output feedback stabilization of a class of FASs is discussed in detail. The contributions are of the following three aspects:

1. For the case that the nonlinear term is independent on its second argument, the solutions of the control gains and the observer gain to achieve stabilization control are given. It is shown that the solution of the control gains is turned into an eigenstructure assignment (ESA) problem, and three necessary and sufficient conditions for the existence of a solution to the observer gain are also given.

2. Based on the obtained results of the solutions of control and observer gains, for the complicated case where the nonlinear term is dependent on its second argument, linear matrix inequality (LMI) conditions that ensure the global exponential stability of the closed-loop system are further given.

3. Corresponding results are given in parallel for the case where the control gains are specifically taken to be zero, and a more convenient result is further provided for the case where the nonlinear term depends on its second argument.

11.2 Full-order Observer-based Control

11.2.1 Problem Formulation

Consider the system

$$\begin{cases} x^{(n)} = f\left(x^{(0\sim n-1)}, t\right) + B\left(y,t\right)u \\ y = C_{0\sim n-1}x^{(0\sim n-1)}, \end{cases} \tag{11.1}$$

where $x \in \mathbb{R}^r$ and $u \in \mathbb{R}^r$ are the system elementary state vector and the system input vector, respectively, $y \in \mathbb{R}^m$ is the output vector, and $C_{0\sim n-1} \in \mathbb{R}^{m\times nr}$ is a known constant matrix. Furthermore, $f\left(x^{(0\sim n-1)}, t\right)$ is a nonlinear function, and $B\left(y,t\right)$ is a matrix function satisfying the following assumptions.

Assumption 11.1 $\det B\left(y,t\right) \neq 0$ *or* ∞*, for all* $y \in \mathbb{R}^m$*, and* $t \geq 0$.

Assumption 11.2 *There exist a scalar* $\rho > 0$ *and a convex subset* $\Omega \subseteq \mathbb{R}^{nr}$ *containing the origin such that:*

$$\left\| f\left(x_1^{(0\sim n-1)}, t\right) - f\left(x_2^{(0\sim n-1)}, t\right)\right\| \leqslant \rho \left\| x_1^{(0\sim n-1)} - x_2^{(0\sim n-1)}\right\|, \tag{11.2}$$

holds for all $x_1^{(0\sim n-1)}, x_2^{(0\sim n-1)} \in \Omega$.

The observer-based dynamical output controller for system (11.1) can be designed as

$$\begin{cases} u = -B^{-1}\left(\cdot\right)\left(f\left(\hat{x}^{(0\sim n-1)}, t\right) + A_{0\sim n-1}\hat{x}^{(0\sim n-1)}\right) \\ \dot{\hat{x}}^{(0\sim n-1)} = \Phi\left(A_{0\sim n-1}\right)\hat{x}^{(0\sim n-1)} - L\left(y - C_{0\sim n-1}\hat{x}^{(0\sim n-1)}\right), \end{cases} \tag{11.3}$$

where $\hat{x}^{(0\sim n-1)}$ is the estimation of $x^{(0\sim n-1)}$, and $A_{0\sim n-1} \in \mathbb{R}^{r\times nr}$ and $L \in \mathbb{R}^{nr\times m}$ are matrices to be designed.

With the above controller, system (11.1) becomes

$$x^{(n)} = -A_{0\sim n-1}\hat{x}^{(0\sim n-1)} + \Delta f(\cdot), \tag{11.4}$$

where

$$\begin{aligned} \Delta f(\cdot) &\triangleq \Delta f\left(x^{(0\sim n-1)}, \hat{x}^{(0\sim n-1)}, t\right) \\ &= f\left(x^{(0\sim n-1)}, t\right) - f\left(\hat{x}^{(0\sim n-1)}, t\right). \end{aligned} \tag{11.5}$$

Note that the state-space form of (11.4) is

$$\dot{x}^{(0\sim n-1)} = \Phi(0_{0\sim n-1})\, x^{(0\sim n-1)} + B_c\left(\Delta f(\cdot) - A_{0\sim n-1}\hat{x}^{(0\sim n-1)}\right). \tag{11.6}$$

The closed-loop system is obviously composed of

$$\dot{\hat{x}}^{(0\sim n-1)} = \Phi(A_{0\sim n-1})\,\hat{x}^{(0\sim n-1)} - LC_{0\sim n-1}\left(x^{(0\sim n-1)} - \hat{x}^{(0\sim n-1)}\right), \tag{11.7}$$

and

$$\dot{x}^{(0\sim n-1)} = \Phi(0_{0\sim n-1})\, x^{(0\sim n-1)} + B_c\left(\Delta f(\cdot) - A_{0\sim n-1}\hat{x}^{(0\sim n-1)}\right). \tag{11.8}$$

Then, we impose the following assumption, which plays a role in the subsequent stability analysis.

Assumption 11.3 *The matrix pair* $(\Phi(0_{0\sim n-1}), C_{0\sim n-1})$ *is observable.*

Based on the above preparation, we can now state the problem to be solved in this section as follows.

Problem 11.1 *Under Assumptions 11.1∼11.3, find a dynamical output controller in the form of (11.3) such that the closed-loop system (11.7)-(11.8) is uniformly exponentially stable.*

11.2.2 Solution to Problem

This section treats the observer-based control design. Firstly, we convert the closed-loop system (11.7)-(11.8) to obtain an augmented system. Then we discuss the stability of the augmented system.

To realize the observer-based control of FAS (11.1), let us quote the following preliminary results in Subsection 5.1.1 (see also Duan (2021c)).

Lemma 11.1 *Let* $A \in \mathbb{R}^{n\times n}$ *satisfy*

$$\text{Re}\,\lambda_i(A) \le -\frac{\gamma}{2}, \quad i = 1, 2, \ldots, n, \tag{11.9}$$

where $\gamma > 0$*, then there exists a positive definite matrix* $P \in \mathbb{R}^{n\times n}$ *satisfying*

$$A^{\mathrm{T}}P + PA \le -\gamma P. \tag{11.10}$$

Lemma 11.2 *For any $\mu_c > 0$, there exist a set of matrices $A_i \in \mathbb{R}^{r\times r}$, $i = 0, 1, \ldots, n-1$, satisfying*

$$\text{Re}\ \lambda_i\left(\Phi\left(A_{0\sim n-1}\right)\right) \leq -\frac{\mu_c}{2},\ i = 1, 2, \ldots, nr. \tag{11.11}$$

It is known from Lemma 11.1 that, when the condition in (11.11) holds for some $\mu_c > 0$, there exists a positive definite matrix $P^c \triangleq P^c(A_{0\sim n-1})$ satisfying

$$\Phi^{\mathrm{T}}\left(A_{0\sim n-1}\right)P^c + P^c\Phi\left(A_{0\sim n-1}\right) \leq -\mu_c P^c. \tag{11.12}$$

Similarly, due to Assumption 11.3, there exists a matrix $L \in \mathbb{R}^{nr\times m}$ satisfying

$$\text{Re}\ \lambda_i\left(\Phi\left(0_{0\sim n-1}\right) + LC_{0\sim n-1}\right) \leq -\frac{\mu_o}{2},\ i = 1, 2, \cdots, nr. \tag{11.13}$$

Then it follows Lemma 11.1 again that there exists a positive definite matrix P^o satisfying

$$\left(\Phi\left(0_{0\sim n-1}\right) + LC_{0\sim n-1}\right)^{\mathrm{T}}P^o + P^o\left(\Phi\left(0_{0\sim n-1}\right) + LC_{0\sim n-1}\right) \leq -\mu_o P^o. \tag{11.14}$$

The augmented system

Define the observation error vector

$$\tilde{x}^{(0\sim n-1)} = x^{(0\sim n-1)} - \hat{x}^{(0\sim n-1)}. \tag{11.15}$$

Then by Assumption 11.2, we have

$$\left\| f\left(x^{(0\sim n-1)}, t\right) - f\left(\hat{x}^{(0\sim n-1)}, t\right)\right\| \leqslant \rho\left\|\tilde{x}^{(0\sim n-1)}\right\|, \tag{11.16}$$

and the second equation of (11.3) can be written as

$$\dot{\hat{x}}^{(0\sim n-1)} = \Phi\left(A_{0\sim n-1}\right)\hat{x}^{(0\sim n-1)} - LC_{0\sim n-1}\tilde{x}^{(0\sim n-1)}. \tag{11.17}$$

Since

$$\Phi\left(A_{0\sim n-1}\right) = \Phi\left(0_{0\sim n-1}\right) - B_c A_{0\sim n-1}, \tag{11.18}$$

the observer equation (11.17) can be rewritten as

$$\dot{\hat{x}}^{(0\sim n-1)} = \Phi\left(0_{0\sim n-1}\right)\hat{x}^{(0\sim n-1)} - LC_{0\sim n-1}\tilde{x}^{(0\sim n-1)} - B_c A_{0\sim n-1}\hat{x}^{(0\sim n-1)}. \tag{11.19}$$

Meanwhile, it follows from (11.6) and (11.19) that

$$\dot{\tilde{x}}^{(0\sim n-1)} = \left(\Phi\left(0_{0\sim n-1}\right) + LC_{0\sim n-1}\right)\tilde{x}^{(0\sim n-1)} + B_c\Delta f\left(\cdot\right). \tag{11.20}$$

By combining the above equation with (11.17), we obtain the following closed-loop system:

$$\begin{cases} \dot{\hat{x}}^{(0\sim n-1)} = \Phi\left(A_{0\sim n-1}\right)\hat{x}^{(0\sim n-1)} - LC_{0\sim n-1}\tilde{x}^{(0\sim n-1)} \\ \dot{\tilde{x}}^{(0\sim n-1)} = \left(\Phi\left(0_{0\sim n-1}\right) + LC_{0\sim n-1}\right)\tilde{x}^{(0\sim n-1)} + B_c\Delta f\left(\cdot\right). \end{cases} \tag{11.21}$$

Define

$$X = \begin{bmatrix} \hat{x}^{(0\sim n-1)} \\ \tilde{x}^{(0\sim n-1)} \end{bmatrix}, \tag{11.22}$$

then we can obtain the following augmented system:

$$\dot{X} = \begin{bmatrix} \Phi(A_{0\sim n-1}) & -LC_{0\sim n-1} \\ 0 & \Phi(0_{0\sim n-1}) + LC_{0\sim n-1} \end{bmatrix} X + \begin{bmatrix} 0 \\ B_c \Delta f(\cdot) \end{bmatrix}, \tag{11.23}$$

where $\Delta f(\cdot)$ satisfies (11.16).

Uniformly exponential stability

Regarding the stability of the above system (11.23), we have the following result.

Theorem 11.1 *Suppose that Assumptions 11.1∼11.3 hold. Let*
1. μ_{c} *and* μ_{o} *be two positive numbers;*
2. $A_{0\sim n-1} \in \mathbb{R}^{r\times nr}$ *and* $L \in \mathbb{R}^{nr\times m}$ *be two matrices satisfying (11.11) and (11.13), respectively; and*
3. P^{c} *and* P^{o} *be two matrices satisfying (11.12), (11.14) and*

$$\begin{cases} \mu_{\mathrm{c}} - \lambda_{\max}(P^{\mathrm{c}}) > 0 \\ \mu_{\mathrm{o}} - \dfrac{2\rho \|P^{\mathrm{o}} B_c\|}{\lambda_{\min}(P^{\mathrm{o}})} - \dfrac{\|LC_{0\sim n-1}\|^2}{\lambda_{\min}(P^{\mathrm{o}})} > 0. \end{cases} \tag{11.24}$$

Then, the observer-based controller (11.3) guarantees that system (11.23) is uniformly exponentially stable.

For a proof of the above result, please refer to the appendix.

Remark 11.1 If Assumption 11.2 is assumed to be globally satisfied, then the closed-loop system is uniformly globally exponentially stable, which is distinct from other reported boundedness in various senses such as semi-global boundedness (Chen et al. (2016)) and global ultimately uniformly boundedness (Cai et al. (2023b)).

11.3 Reduced-order Observer-based Control

11.3.1 Problem Formulation

Consider the following FAS with an output equation:

$$\begin{cases} \begin{bmatrix} x_1^{(n_1)} \\ x_2^{(n_2)} \end{bmatrix} = f(x_1^{(0\sim n_1-1)}, x_2^{(0\sim n_2-1)}) + B(x_1^{(0\sim n_1-1)}, t)u \\ y = x_1^{(0\sim n_1-1)}, \end{cases} \tag{11.25}$$

where $x_1 \in \mathbb{R}^{r_1}$ and $x_2 \in \mathbb{R}^{r_2}$ are the states, $u \in \mathbb{R}^{r_1+r_2}$ is the input, $y \in \mathbb{R}^{n_1 r_1}$ is the output, $f(\cdot,\cdot) : \mathbb{R}^{n_1 r_1} \times \mathbb{R}^{n_2 r_2} \to \mathbb{R}^{r_1+r_2}$ is a sufficiently smooth (vector) function, $B(\cdot,t) : \mathbb{R}^{n_1 r_1} \times [0,\infty) \to \mathbb{R}^{(r_1+r_2)\times(r_1+r_2)}$ is a sufficiently smooth (matrix) function. Assume that FAS (11.25) satisfies the following assumption.

Assumption 11.4 $\det B(z,t) \neq 0$, $\forall z \in \mathbb{R}^{n_1 r_1}$ *and* $t \geq 0$.

Without loss of generality, the function $f(\cdot,\cdot)$ can be always decomposed as follows:

$$f(z_1, z_2) = F_D \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + f_0(z_1, z_2), \tag{11.26}$$

where $f_0(\cdot,\cdot) : \mathbb{R}^{n_1 r_1} \times \mathbb{R}^{n_2 r_2} \to \mathbb{R}^{r_1 + r_2}$ is a known nonlinear function, and

$$F_D = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}, \tag{11.27}$$

with $F_{11} \in \mathbb{R}^{r_1 \times n_1 r_1}$, $F_{12} \in \mathbb{R}^{r_1 \times n_2 r_2}$, $F_{21} \in \mathbb{R}^{r_2 \times n_1 r_1}$, $F_{22} \in \mathbb{R}^{r_2 \times n_2 r_2}$ being known matrices. In the special case of $f_0(z_1, z_2) = 0$, system (11.25) reduces to

$$\begin{cases} \begin{bmatrix} x_1^{(n_1)} \\ x_2^{(n_2)} \end{bmatrix} = F_D \begin{bmatrix} x_1^{(0\sim n_1-1)} \\ x_2^{(0\sim n_2-1)} \end{bmatrix} + B(x_1^{(0\sim n_1-1)}, t)u \\ y = x_1^{(0\sim n_1-1)}. \end{cases} \tag{11.28}$$

When both $x_1^{(0\sim n_1-1)}$ and $x_2^{(0\sim n_2-1)}$ are measurable, we can design for system (11.25) the following state feedback controller:

$$u = -B^{-1}(x_1^{(0\sim n_1-1)}, t)\left(f_0(x_1^{(0\sim n_1-1)}, x_2^{(0\sim n_2-1)}) + A_F \begin{bmatrix} x_1^{(0\sim n_1-1)} \\ x_2^{(0\sim n_2-1)} \end{bmatrix} \right), \tag{11.29}$$

where

$$A_F = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

is a feedback gain with $A_{11} \in \mathbb{R}^{r_1 \times n_1 r_1}$, $A_{12} \in \mathbb{R}^{r_1 \times n_2 r_2}$, $A_{21} \in \mathbb{R}^{r_2 \times n_1 r_1}$, and $A_{22} \in \mathbb{R}^{r_2 \times n_2 r_2}$. Under this controller, the closed-loop system is given by

$$\begin{bmatrix} x_1^{(n_1)} \\ x_2^{(n_2)} \end{bmatrix} = \begin{bmatrix} F_{11} - A_{11} & F_{12} - A_{12} \\ F_{21} - A_{21} & F_{22} - A_{22} \end{bmatrix} \begin{bmatrix} x_1^{(0\sim n_1-1)} \\ x_2^{(0\sim n_2-1)} \end{bmatrix},$$

whose state-space form is

$$\begin{bmatrix} \dot{x}_1^{(0\sim n_1-1)} \\ \dot{x}_2^{(0\sim n_2-1)} \end{bmatrix} = \Phi_{FA} \begin{bmatrix} x_1^{(0\sim n_1-1)} \\ x_2^{(0\sim n_2-1)} \end{bmatrix}, \tag{11.30}$$

where

$$\begin{cases} \Phi_{FA} = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \\ \Phi_{11} = \Phi(0_{r_1 \times n_1 r_1}) + B_1 (F_{11} - A_{11}) \\ \Phi_{12} = B_1 (F_{12} - A_{12}) \\ \Phi_{21} = B_2 (F_{21} - A_{21}) \\ \Phi_{22} = \Phi(0_{r_2 \times n_2 r_2}) + B_2 (F_{22} - A_{22}), \end{cases} \tag{11.31}$$

with

$$B_i = \left[\begin{array}{cccc} 0 & \cdots & 0 & I_{r_i} \end{array}\right]^{\mathrm{T}} \in \mathbb{R}^{n_i r_i},\ i = 1, 2.$$

Clearly, the controller parameter matrices A_{11}, A_{12}, A_{21}, and A_{22} are so chosen to make the closed-loop system (11.30) stable. We point out that such matrices A_{11}, A_{12}, A_{21}, and A_{22} always exist. For example, we can let $A_{21} = F_{21}$, A_{12} be an arbitrary matrix, and A_{11} and A_{22} be such that Φ_{11} and Φ_{22} are Hurwitz. A general solution is given in Subsection 11.3.2.

However, in the case that only $y = x_1^{(0\sim n_1-1)}$ is available for feedback, the state feedback controller (11.29) is no longer implementable. In this case, we may propose an observer to construct the state $x_2^{(0\sim n_2-1)}$, which, together with $x_1^{(0\sim n_1-1)}$, will then be used to stabilize the system.

Recalling that controller (11.29) produces the closed-loop system (11.30), instead of designing an observer for the nonlinear system (11.25), we can design an observer for the linear system (11.30) with the same observation equation in (11.25), that is,

$$\left\{\begin{array}{l} \left[\begin{array}{c} \dot{x}_1^{(0\sim n_1-1)} \\ \dot{x}_2^{(0\sim n_2-1)} \end{array}\right] = \left[\begin{array}{cc} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{array}\right] \left[\begin{array}{c} x_1^{(0\sim n_1-1)} \\ x_2^{(0\sim n_2-1)} \end{array}\right] \\ y = x_1^{(0\sim n_1-1)}. \end{array}\right. \tag{11.32}$$

In the case that Φ_{FA} is made stable, the above system is obviously detectable.

By standard reduced-order observer design for linear systems, the associated observer can be designed based on the above system (11.32) as

$$\left\{\begin{array}{l} \dot{z} = (\Phi_{22} + L\Phi_{12})z + (L\Phi_{11} - (\Phi_{22} + L\Phi_{12})L + \Phi_{21})y \\ \hat{x}_2^{(0\sim n_2-1)} = z - Ly, \end{array}\right. \tag{11.33}$$

where $L \in \mathbb{R}^{n_2 r_2 \times n_1 r_1}$ is the observer gain to be determined. Then, by replacing $x_2^{(0\sim n_2-1)}$ in (11.29) with $\hat{x}_2^{(0\sim n_2-1)}$, we can design the output feedback controller as

$$u = -B^{-1}\left(x_1^{(0\sim n_1-1)}, t\right)\left(f_0(x_1^{(0\sim n_1-1)}, \hat{x}_2^{(0\sim n_2-1)}) + A_F \left[\begin{array}{c} x_1^{(0\sim n_1-1)} \\ \hat{x}_2^{(0\sim n_2-1)} \end{array}\right]\right). \tag{11.34}$$

Applying this controller to FAS (11.25), yields

$$\begin{aligned} \left[\begin{array}{c} x_1^{(n_1)} \\ x_2^{(n_2)} \end{array}\right] &= \left[\begin{array}{cc} F_{11} - A_{11} & F_{12} \\ F_{21} - A_{21} & F_{22} \end{array}\right] \left[\begin{array}{c} x_1^{(0\sim n_1-1)} \\ x_2^{(0\sim n_2-1)} \end{array}\right] \\ &\quad + \left[\begin{array}{c} -A_{12}\hat{x}_2^{(0\sim n_2-1)} \\ -A_{22}\hat{x}_2^{(0\sim n_2-1)} \end{array}\right] + \Delta f_0, \end{aligned} \tag{11.35}$$

where

$$\Delta f_0 \triangleq f_0\left(x_1^{(0\sim n_1-1)}, x_2^{(0\sim n_2-1)}\right) - f_0\left(x_1^{(0\sim n_1-1)}, \hat{x}_2^{(0\sim n_2-1)}\right). \tag{11.36}$$

Based on the above description, the problem to be solved in this section can be stated as follows:

Problem 11.2 *Given system (11.25) satisfying Assumption 11.4, find the feedback controller gain matrix A_F and the observer gain L, such that the closed-loop system formed by (11.33) and (11.35) is globally exponentially stable.*

11.3.2 Case of $f_0(z_1, z_2)$ Independent on z_2

In this section, let us consider the case that the nonlinear term $f_0(z_1, z_2)$ is independent on z_2, that is,

$$f_0(z_1, z_2) = f_0(z_1),\ \forall (z_1, z_2) \in \mathbb{R}^{n_1 r_1} \times \mathbb{R}^{n_2 r_2}. \tag{11.37}$$

The closed-loop system

It follows from (11.36) and (11.37) that $\Delta f_0 \equiv 0$. In this case, we have the following result regarding the closed-loop system.

Theorem 11.2 *Let condition (11.37) be met. Then the closed-loop system consisting of system (11.25) and reduced-order observer-based output feedback (11.33)-(11.34) is given by*

$$\begin{bmatrix} \dot{x}_1^{(0\sim n_1-1)} \\ \dot{x}_2^{(0\sim n_2-1)} \\ \dot{e} \end{bmatrix} = A_C \begin{bmatrix} x_1^{(0\sim n_1-1)} \\ x_2^{(0\sim n_2-1)} \\ e \end{bmatrix}, \tag{11.38}$$

where

$$A_C = \begin{bmatrix} \Phi_{11} & \Phi_{12} & B_1 A_{12} \\ \Phi_{21} & \Phi_{22} & B_2 A_{22} \\ 0 & 0 & \Phi_{33} \end{bmatrix}, \tag{11.39}$$

and

$$e = x_2^{(0\sim n_2-1)} - \hat{x}_2^{(0\sim n_2-1)}, \tag{11.40}$$

$$\Phi_{33} = \Phi(0_{r_2 \times n_2 r_2}) + B_2 F_{22} + L B_1 F_{12}. \tag{11.41}$$

Proof. Since $\Delta f_0 \equiv 0$, equation (11.35) reduces to

$$\begin{bmatrix} x_1^{(n_1)} \\ x_2^{(n_2)} \end{bmatrix} = \begin{bmatrix} F_{11} - A_{11} & F_{12} \\ F_{21} - A_{21} & F_{22} \end{bmatrix} \begin{bmatrix} x_1^{(0\sim n_1-1)} \\ x_2^{(0\sim n_2-1)} \end{bmatrix} + \begin{bmatrix} -A_{12}\hat{x}_2^{(0\sim n_2-1)} \\ -A_{22}\hat{x}_2^{(0\sim n_2-1)} \end{bmatrix},$$

whose state-space form is given by

$$\begin{aligned} \begin{bmatrix} \dot{x}_1^{(0\sim n_1-1)} \\ \dot{x}_2^{(0\sim n_2-1)} \end{bmatrix} &= \Phi_{FA} \begin{bmatrix} x_1^{(0\sim n_1-1)} \\ x_2^{(0\sim n_2-1)} \end{bmatrix} \\ &\quad + \begin{bmatrix} B_1 A_{12}(x_2^{(0\sim n_2-1)} - \hat{x}_2^{(0\sim n_2-1)}) \\ B_2 A_{22}(x_2^{(0\sim n_2-1)} - \hat{x}_2^{(0\sim n_2-1)}) \end{bmatrix}. \end{aligned} \tag{11.42}$$

In view of (11.33) and (11.42), the dynamics of the observer error e can be obtained as

$$\begin{aligned}
\dot{e} &= \dot{x}_2^{(0\sim n_2-1)} - \dot{\hat{x}}_2^{(0\sim n_2-1)} \\
&= \dot{x}_2^{(0\sim n_2-1)} - \dot{z} + L\dot{y} \\
&= \Phi_{21}x_1^{(0\sim n_1-1)} + \Phi_{22}x_2^{(0\sim n_2-1)} + B_2A_{22}\left(x_2^{(0\sim n_2-1)} - \hat{x}_2^{(0\sim n_2-1)}\right) \\
&\quad -(\Phi_{22}+L\Phi_{12})\left(\hat{x}_2^{(0\sim n_2-1)} + Ly\right) - (L\Phi_{11} - ((\Phi_{22}+L\Phi_{12})L + \Phi_{21})y \\
&\quad +L\left(\Phi_{11}x_1^{(0\sim n_1-1)} + \Phi_{12}x_2^{(0\sim n_2-1)} + B_1A_{12}\left(x_2^{(0\sim n_2-1)} - \hat{x}_2^{(0\sim n_2-1)}\right)\right) \\
&= \Phi_{22}x_2^{(0\sim n_2-1)} + B_2A_{22}e - \Phi_{22}\hat{x}_2^{(0\sim n_2-1)} \\
&\quad -L\Phi_{12}\hat{x}_2^{(0\sim n_2-1)} + L\Phi_{12}x_2^{(0\sim n_2-1)} + LB_1A_{12}e \\
&= \Phi_{22}e + B_2A_{22}e + L\Phi_{12}e + LB_1A_{12}e \\
&= \left(\Phi(0_{r_2\times n_2r_2}) + B_2F_{22} - B_2A_{22}\right)e + B_2A_{22}e + LB_1\left(F_{12} - A_{12}\right)e + LB_1A_{12}e \\
&= \left(\Phi(0_{r_2\times n_2r_2}) + B_2F_{22} + LB_1F_{12}\right)e \\
&= \Phi_{33}e. \qquad (11.43)
\end{aligned}$$

Combining (11.42) and (11.43) gives (11.38). The proof is finished. ■

Clearly, it follows from Theorem 11.2 that the closed-loop system (11.38) is asymptotically stable if and only if A_C is Hurwitz. This is equivalent to requiring

- the matrix Φ_{FA} is Hurwitz; and
- the matrix $\Phi_{33} = \Phi(0_{r_2\times n_2r_2}) + B_2F_{22} + LB_1F_{12}$ is Hurwitz.

These two subproblems are addressed below, respectively.

Solution of matrix A_F

Define

$$\begin{aligned}
\Phi_F &\triangleq \begin{bmatrix} \Phi_{11}^f & \Phi_{12}^f \\ \Phi_{21}^f & \Phi_{22}^f \end{bmatrix} \qquad (11.44) \\
&= \begin{bmatrix} \Phi(0_{r_1\times n_1r_1}) + B_1F_{11} & B_1F_{12} \\ B_2F_{21} & \Phi(0_{r_2\times n_2r_2}) + B_2F_{22} \end{bmatrix}, \qquad (11.45) \\
B_E &= \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}. \qquad (11.46)
\end{aligned}$$

Then, the following result shows the controllability of (Φ_F, B_E).

Theorem 11.3 *The matrix pair (Φ_F, B_E) defined by (11.45)-(11.46) is controllable.*

Proof. Observe

$$\begin{aligned}
&\operatorname{rank}\begin{bmatrix} sI_{n_1r_1+n_2r_2}-\Phi_F & B_E \end{bmatrix}\\
=\ &\operatorname{rank}\begin{bmatrix} sI_{n_1r_1}-\Phi(0_{r_1\times n_1r_1})-B_1F_{11} & -B_1F_{12} & B_1 & 0\\ -B_2F_{21} & sI_{n_2r_2}-\Phi(0_{r_2\times n_2r_2})-B_2F_{22} & 0 & B_2 \end{bmatrix}\\
=\ &\operatorname{rank}\begin{bmatrix} sI_{n_1r_1}-\Phi(0_{r_1\times n_1r_1}) & -B_1F_{12} & B_1 & 0\\ -B_2F_{21} & sI_{n_2r_2}-\Phi(0_{r_2\times n_2r_2}) & 0 & B_2 \end{bmatrix}\\
=\ &\operatorname{rank}\begin{bmatrix} sI_{n_1r_1}-\Phi(0_{r_1\times n_1r_1}) & 0 & B_1 & 0\\ 0 & sI_{n_2r_2}-\Phi(0_{r_2\times n_2r_2}) & 0 & B_2 \end{bmatrix}\\
=\ &\operatorname{rank}\begin{bmatrix} sI_{n_1r_1}-\Phi(0_{r_1\times n_1r_1}) & B_1 & 0 & 0\\ 0 & 0 & sI_{n_2r_2}-\Phi(0_{r_2\times n_2r_2}) & B_2 \end{bmatrix}.
\end{aligned}$$

Since $\big(\Phi(0_{r_1\times n_1r_1}), B_1\big)$ and $\big(\Phi(0_{r_2\times n_2r_2}), B_2\big)$ are both controllable, we then have

$$\operatorname{rank}\begin{bmatrix} sI-\Phi_F & B_E \end{bmatrix} = n_1r_1+n_2r_2.$$

Therefore, the conclusion holds. ■

In view of

$$\begin{aligned}
\Phi_{FA} &= \begin{bmatrix} \Phi(0_{r_1\times n_1r_1})+B_1F_{11}-B_1A_{11} & B_1(F_{12}-A_{12})\\ B_2(F_{21}-A_{21}) & \Phi(0_{r_2\times n_2r_2})+B_2F_{22}-B_2A_{22} \end{bmatrix}\\
&= \Phi_F-B_EA_F, \qquad (11.47)
\end{aligned}$$

and the fact of (Φ_F, B_E) being controllable, the solution of A_F is turned into an ESA problem, while the controllability of (Φ_F, B_E) ensures arbitrary assignment of the eigenvalues of Φ_{FA}.

Solution of observer gain L

Recalling (11.44)-(11.45), we have

$$\Phi_{22}^f = \Phi(0_{r_2\times n_2r_2})+B_2F_{22} = \left[\begin{array}{cccc} 0 & I_{r_2} & & \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & I_{r_2}\\ \hline & & F_{22} & \end{array}\right],$$

$$\Phi_{12}^f = B_1F_{12} = \left[\begin{array}{c} 0\\ \vdots\\ 0\\ \hline F_{12} \end{array}\right].$$

Further, define

$$C_M \triangleq \begin{bmatrix} I_{n_1r_1} & 0 \end{bmatrix}.$$

Then, regarding solution of the observer gain L, we have the following results.

Theorem 11.4 *There exists an observer gain L such that Φ_{33} is Hurwitz if and only if one of the following conditions holds:*

1. *the matrix pair* $\left(\Phi_{22}^f, \Phi_{12}^f\right)$ *is detectable;*
2. *the matrix pair* (Φ_F, C_M) *is detectable; and*
3. *there exists*

$$\begin{bmatrix} F_{12} \\ F_{22} \end{bmatrix} = \begin{bmatrix} F_{12}^{[0]} & F_{12}^{[1]} & \cdots & F_{12}^{[n_2-1]} \\ F_{22}^{[0]} & F_{22}^{[1]} & \cdots & F_{22}^{[n_2-1]} \end{bmatrix},$$

with $F_{12}^{[i]} \in \mathbb{R}^{r_1 \times r_2}$, $F_{22}^{[i]} \in \mathbb{R}^{r_2 \times r_2}$, $i = 1, 2, \ldots, n_2 - 1$, *satisfying*

$$\operatorname{rank} \begin{bmatrix} -\sum_{i=0}^{n_2-1} s^i F_{12}^{[i]} \\ s^{n_2} I_{r_2} - \sum_{i=0}^{n_2-1} s^i F_{22}^{[i]} \end{bmatrix} = r_2, \ \forall s \notin \mathbb{C}^-. \tag{11.48}$$

Proof. Firstly, the detectability of the matrix pair $\left(\Phi_{22}^f, \Phi_{12}^f\right)$ obviously follows from well-known linear system theories.

Secondly, noting

$$\begin{aligned} & \operatorname{rank} \begin{bmatrix} sI_{(n_1 r_1 + n_2 r_2)} - \Phi_F \\ C_M \end{bmatrix} \\ = \ & \operatorname{rank} \begin{bmatrix} sI_{n_1 r_1} - \Phi(0_{r_1 \times n_1 r_1}) - B_1 F_{11} & -B_1 F_{12} \\ -B_2 F_{21} & sI_{n_2 r_2} - \Phi(0_{r_2 \times n_2 r_2}) - B_2 F_{22} \\ I_{n_1 r_1} & 0 \end{bmatrix} \\ = \ & \operatorname{rank} \begin{bmatrix} sI_{n_2 r_2} - \Phi(0_{r_2 \times n_2 r_2}) - B_2 F_{22} \\ -B_1 F_{12} \end{bmatrix} + n_1 r_1 \\ = \ & \operatorname{rank} \begin{bmatrix} sI_{n_2 r_2} - \Phi_{22}^f \\ \Phi_{12}^f \end{bmatrix} + n_1 r_1, \end{aligned}$$

we know immediately that the detectability of (Φ_F, C_M) is equivalent to that of $(\Phi_{22}^f, \Phi_{12}^f)$.

Finally, by the well-known PBH criterion, the last condition follows from

$$\begin{aligned} \operatorname{rank} \begin{bmatrix} sI_{n_2 r_2} - \Phi_{22}^f \\ \Phi_{12}^f \end{bmatrix} &= \operatorname{rank} \left[\begin{array}{cccc} sI_{r_2} & -I_{r_2} & & \\ & \ddots & \ddots & \\ & & sI_{r_2} & -I_{r_2} \\ \hline -F_{22}^{[0]} & \cdots & -F_{22}^{[n_2-2]} & sI_{r_2} - F_{22}^{[n_2-1]} \\ \hline \multicolumn{4}{c}{0_{(n_1-1)r_1 \times n_2 r_2}} \\ \multicolumn{4}{c}{-F_{12}} \end{array}\right] \\ &= \operatorname{rank} \left[\begin{array}{cccc} sI_{r_2} & -I_{r_2} & & \\ & \ddots & \ddots & \\ & & sI_{r_2} & -I_{r_2} \\ \hline -F_{12}^{[0]} & \cdots & -F_{12}^{[n_2-2]} & -F_{12}^{[n_2-1]} \\ -F_{22}^{[0]} & \cdots & -F_{22}^{[n_2-2]} & sI_{r_2} - F_{22}^{[n_2-1]} \end{array}\right] \end{aligned}$$

$$= \operatorname{rank} \left[\begin{array}{c|ccc} 0 & -I_{r_2} & & \\ \vdots & & \ddots & \\ 0 & & & -I_{r_2} \\ \hline -\sum_{i=0}^{n_2-1} F_{12}^{[i]} s^{n_i-i} & * & \cdots & * \\ s^{n_2} I_{r_2} - \sum_{i=0}^{n_2-1} F_{22}^{[i]} s^{n_i-i} & * & \cdots & * \end{array}\right],$$

where we have used the elementary transformation, and $*$ symbols represent some matrices that are not of interest. ■

11.3.3 Case of $f_0(z_1, z_2)$ Dependent on z_2

In the case that $f_0(z_1, z_2)$ is dependent on z_2, we impose the following assumption.

Assumption 11.5 *The function $f_0(\cdot,\cdot) : \mathbb{R}^{n_1 r_1} \times \mathbb{R}^{n_2 r_2} \to \mathbb{R}^{r_1+r_2}$ is α-Lipschitz with respect to its second argument, that is, there exists a scalar $\alpha > 0$ such that*

$$\| f_0(X, Y_1) - f_0(X, Y_2)\| \le \alpha \|Y_1 - Y_2\|.$$

Closed-loop system

In this case, partition

$$\begin{aligned} \Delta f_0 &= f_0\left(x_1^{(0\sim n_1-1)}, x_2^{(0\sim n_2-1)}\right) - f_0\left(x_1^{(0\sim n_1-1)}, \hat{x}_2^{(0\sim n_2-1)}\right) \\ &= \left[\begin{array}{c} \Delta f_1\left(x_1^{(0\sim n_1-1)}, x_2^{(0\sim n_2-1)}, \hat{x}_2^{(0\sim n_2-1)}\right) \\ \Delta f_2\left(x_1^{(0\sim n_1-1)}, x_2^{(0\sim n_2-1)}, \hat{x}_2^{(0\sim n_2-1)}\right) \end{array}\right], \end{aligned} \tag{11.49}$$

where $\Delta f_1 : \mathbb{R}^{n_1 r_1} \times \mathbb{R}^{n_2 r_2} \to \mathbb{R}^{r_1}$, $\Delta f_2 : \mathbb{R}^{n_1 r_1} \times \mathbb{R}^{n_2 r_2} \to \mathbb{R}^{r_2}$. Then all Δf_0, Δf_1 and Δf_2 are present, and now Theorem 11.2 becomes the following.

Theorem 11.5 *The closed-loop system consisting of system (11.25) and reduced-order observer-based output feedback (11.33)-(11.34) is given by*

$$\left[\begin{array}{c} \dot{x}_1^{(0\sim n_1-1)} \\ \dot{x}_2^{(0\sim n_2-1)} \\ \dot{e} \end{array}\right] = A_C \left[\begin{array}{c} x_1^{(0\sim n_1-1)} \\ x_2^{(0\sim n_2-1)} \\ e \end{array}\right] + \left[\begin{array}{c} B_1 \Delta f_1 \\ B_2 \Delta f_2 \\ L B_1 \Delta f_1 + B_2 \Delta f_2 \end{array}\right], \tag{11.50}$$

where A_C, e, and Φ_{33} are given by (11.39)-(11.41).

Proof. Write the state-space form of (11.35) as

$$\begin{aligned} \left[\begin{array}{c} \dot{x}_1^{(0\sim n_1-1)} \\ \dot{x}_2^{(0\sim n_2-1)} \end{array}\right] &= \Phi_{FA} \left[\begin{array}{c} x_1^{(0\sim n_1-1)} \\ x_2^{(0\sim n_2-1)} \end{array}\right] \\ &\quad + \left[\begin{array}{c} B_1 A_{12}(x_2^{(0\sim n_2-1)} - \hat{x}_2^{(0\sim n_2-1)}) \\ B_2 A_{22}(x_2^{(0\sim n_2-1)} - \hat{x}_2^{(0\sim n_2-1)}) \end{array}\right] + \left[\begin{array}{c} B_1 \Delta f_1 \\ B_2 \Delta f_2 \end{array}\right]. \end{aligned} \tag{11.51}$$

In view of (11.33) and (11.51), the dynamics of the observer error e can be obtained as

$$\begin{aligned}
\dot{e} &= \dot{x}_2^{(0\sim n_2-1)} - \dot{\hat{x}}_2^{(0\sim n_2-1)} \\
&= \dot{x}_2^{(0\sim n_2-1)} - \dot{z} + L\dot{y} \\
&= \Phi_{21}x_1^{(0\sim n_1-1)} + \Phi_{22}x_2^{(0\sim n_2-1)} + B_2A_{22}\left(x_2^{(0\sim n_2-1)} - \hat{x}_2^{(0\sim n_2-1)}\right) + B_2\Delta f_2 \\
&\quad -(\Phi_{22}+L\Phi_{12})\left(\hat{x}_2^{(0\sim n_2-1)} + Ly\right) - (L\Phi_{11} - (\Phi_{22}+L\Phi_{12})L + \Phi_{21})\,y \\
&\quad +L\left(\Phi_{11}x_1^{(0\sim n_1-1)} + \Phi_{12}x_2^{(0\sim n_2-1)} + B_1A_{12}\left(x_2^{(0\sim n_2-1)} - \hat{x}_2^{(0\sim n_2-1)}\right) + B_1\Delta f_1\right) \\
&= \Phi_{22}x_2^{(0\sim n_2-1)} + B_2A_{22}e + B_2\Delta f_2 - \Phi_{22}\hat{x}_2^{(0\sim n_2-1)} - L\Phi_{12}\hat{x}_2^{(0\sim n_2-1)} \\
&\quad +L\Phi_{12}x_2^{(0\sim n_2-1)} + LB_1A_{12}e + LB_1\Delta f_1 \\
&= \Phi_{22}e + B_2A_{22}e + B_2\Delta f_2 + L\Phi_{12}e + LB_1A_{12}e + LB_1\Delta f_1 \\
&= \left(\Phi(0_{r_2\times n_2r_2}) + B_2F_{22} - B_2A_{22}\right)e + B_2A_{22}e \\
&\quad +LB_1(F_{12}-A_{12})e + LB_1A_{12}e + LB_1\Delta f_1 + B_2\Delta f_2 \\
&= \left(\Phi(0_{r_2\times n_2r_2}) + B_2F_{22} + LB_1F_{12}\right)e + LB_1\Delta f_1 + B_2\Delta f_2 \\
&= \Phi_{33}e + LB_1\Delta f_1 + B_2\Delta f_2. \qquad (11.52)
\end{aligned}$$

Combining (11.51) and (11.52) gives (11.50). The proof is finished. ■

Robust stability

Now we study the exponential stability of the overall nonlinear system (11.50). Denote

$$n \triangleq n_1r_1 + 2n_2r_2,\ r \triangleq r_1 + r_2,$$

and

$$\xi \triangleq \begin{bmatrix} x_1^{(0\sim n_1-1)} \\ x_2^{(0\sim n_2-1)} \\ e \end{bmatrix} \in \mathbb{R}^n,\ B_C \triangleq \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \\ LB_1 & B_2 \end{bmatrix} \in \mathbb{R}^{n\times r}. \qquad (11.53)$$

Then the closed-loop system (11.50) can be rewritten in the compact form:

$$\dot{\xi} = A_C\xi + B_C\Delta f_0, \qquad (11.54)$$

for which the following simple result can be obtained.

Theorem 11.6 *Let Assumptions 11.4 and 11.5 be met, condition (11.48) be satisfied, and the feedback and observer gains A_F and L be designed such that A_C is Hurwitz. Then the closed-loop system (11.54) is globally exponentially stable if there exist scalars $k > 0$, $\varepsilon > 0$, and a positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that the following LMI*

$$\begin{bmatrix} A_C^{\mathrm{T}}P + PA_C + k\alpha^2 I_n & PB_c \\ B_c^{\mathrm{T}}P & -kI_r \end{bmatrix} < -\varepsilon I \qquad (11.55)$$

is satisfied.

Proof. Define a candidate Lyapunov function as

$$V(\xi) = \xi^{\mathrm{T}} P \xi,$$

whose time derivative along the trajectory of system (11.54) is

$$\dot{V}(\xi) = \xi^{\mathrm{T}} \left(A_C^{\mathrm{T}} P + P A_C\right) \xi + 2\xi^{\mathrm{T}} P B_c \Delta f_0.$$

By Assumption 11.5, we have

$$\|\Delta f_0\| \leq \alpha \left\| x_2^{(0\sim n_2-1)} - \hat{x}_2^{(0\sim n_2-1)} \right\| = \alpha \|e\| \leq \alpha \|\xi\|,$$

which is equivalent to

$$(\Delta f_0)^{\mathrm{T}} \Delta f_0 - \alpha^2 \xi^{\mathrm{T}} \xi \leq 0. \tag{11.56}$$

Therefore, we have, using (11.55) and (11.56), the following:

$$\begin{aligned}
\dot{V}(\xi) &\leq \xi^{\mathrm{T}} \left(A_C^{\mathrm{T}} P + P A_C\right) \xi + 2\xi^{\mathrm{T}} P B_c \Delta f_0 - k \left((\Delta f_0)^{\mathrm{T}} \Delta f_0 - \alpha^2 \xi^{\mathrm{T}} \xi\right) \\
&= \begin{bmatrix} \xi \\ \Delta f_0 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} A_C^{\mathrm{T}} P + P A_C + k\alpha^2 I_n & P B_c \\ B_c^{\mathrm{T}} P & -k I_r \end{bmatrix} \begin{bmatrix} \xi \\ \Delta f_0 \end{bmatrix} \\
&\leq -\varepsilon \begin{bmatrix} \xi \\ \Delta f_0 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \xi \\ \Delta f_0 \end{bmatrix} \\
&= -\varepsilon \left(\|\xi\|^2 + \|\Delta f_0\|^2\right) \\
&\leq -\varepsilon \|\xi\|^2.
\end{aligned}$$

This immediately completes the proof. ■

We point out that the inequality (11.55) is linear if the observer gain L is prescribed. However, it might fail to hold for a given L. In practice, a trial-and-error approach can be employed to identify a feasible L. First, we choose an observer gain L such that Φ_{33} is Hurwitz. Next, we attempt to solve the LMI (11.55). If this LMI is solvable, the chosen L is valid; otherwise, we refine our selection of L and repeat the process.

11.3.4 Case of $A_F = 0$

In this section, the case of $A_F = 0$, or equivalently, $A_{ij} = 0$, $i, j = 1, 2$, is specifically considered. In this case, system (11.32) reduces into the following one:

$$\begin{cases} \begin{bmatrix} \dot{x}_1^{(0\sim n_1-1)} \\ \dot{x}_2^{(0\sim n_2-1)} \end{bmatrix} = \Phi_F \begin{bmatrix} x_1^{(0\sim n_1-1)} \\ x_2^{(0\sim n_2-1)} \end{bmatrix} \\ y = x_1^{(0\sim n_1-1)}, \end{cases} \tag{11.57}$$

where Φ_F is given by (11.45). Likewise, instead of designing an observer for the nonlinear system (11.25), we can design for the linear system (11.57) the following reduced-order observer:

$$\begin{cases} \dot{z} = \left(\Phi(0_{r_2 \times n_2 r_2}) + B_2 F_{22} + L B_1 F_{12}\right) z \\ \quad + L \left(\Phi(0_{r_1 \times n_1 r_1}) + B_1 F_{11}\right) y \\ \quad - \left[\left(\Phi(0_{r_2 \times n_2 r_2}) + B_2 F_{22} + L B_1 F_{12}\right) L - B_2 F_{21}\right] y \\ \hat{x}_2^{(0\sim n_2-1)} = z - Ly, \end{cases} \tag{11.58}$$

where $L \in \mathbb{R}^{n_2 r_2 \times n_1 r_1}$ is the observer gain to be determined. Then, we can design the output feedback controller as

$$u = -B^{-1}\left(x_1^{(0\sim n_1-1)}, t\right) f_0(x_1^{(0\sim n_1-1)}, \hat{x}_2^{(0\sim n_2-1)}). \tag{11.59}$$

Applying the above controller (11.59) to FAS (11.25), yields

$$\begin{bmatrix} x_1^{(n_1)} \\ x_2^{(n_2)} \end{bmatrix} = F_D \begin{bmatrix} x_1^{(0\sim n_1-1)} \\ x_2^{(0\sim n_2-1)} \end{bmatrix} + \Delta f_0, \tag{11.60}$$

where F_D is given by (11.27), and Δf_0 is given by (11.36). Please note that there is quite some DOFs to choose F_D via the decomposition (11.26). The stability of the closed-loop system can be guaranteed by adjusting, to a certain extent, the matrix F_D.

Based on the above description, similarly, two cases where $f_0(z_1, z_2)$ is independent on z_2 and $f_0(z_1, z_2)$ is dependent on z_2 are considered in the following subsections. Let us start with the simple case where $f_0(z_1, z_2)$ is independent on z_2.

Case of $f_0(z_1, z_2)$ independent on z_2

Letting $A_{ij} = 0$, $i, j = 1, 2$, in the above Theorem 11.2, immediately gives the following result.

Corollary 11.1 *Let condition (11.37) be met. Then the closed-loop system consisting of system (11.25) and reduced-order observer-based output feedback (11.58)-(11.59) is given by*

$$\begin{bmatrix} \dot{x}_1^{(0\sim n_1-1)} \\ \dot{x}_2^{(0\sim n_2-1)} \end{bmatrix} = \Phi_F \begin{bmatrix} x_1^{(0\sim n_1-1)} \\ x_2^{(0\sim n_2-1)} \end{bmatrix}, \tag{11.61}$$

and

$$\dot{e} = \Phi_{33} e, \tag{11.62}$$

where Φ_F and Φ_{33} are given by (11.45) and (11.41), respectively.

The above result indicates that, in this case, the closed-loop system is stable if and only if both Φ_F and Φ_{33} are Hurwitz.

Firstly, regarding the requirement that the matrix Φ_F is Hurwitz, we observe

$$\begin{aligned} \Phi_F &= \begin{bmatrix} \Phi(0_{r_1 \times n_1 r_1}) + B_1 F_{11} & B_1 F_{12} \\ B_2 F_{21} & \Phi(0_{r_2 \times n_2 r_2}) + B_2 F_{22} \end{bmatrix} \\ &= \Phi_0 + B_E F_D, \end{aligned} \tag{11.63}$$

where

$$\Phi_0 = \begin{bmatrix} \Phi(0_{r_1 \times n_1 r_1}) & 0 \\ 0 & \Phi(0_{r_2 \times n_2 r_2}) \end{bmatrix}.$$

Since it is easily shown that (Φ_0, B_E) is controllable, we can always find the matrix F_D such that Φ_F has arbitrarily prescribed eigenvalues. Therefore, we can often choose a proper F_D matrix by solving the pole assignment problem (11.63) and adjusting the decomposition (11.26).

Secondly, regarding the existence of an observer gain L such that Φ_{33} is Hurwitz, the problem has been completely solved in Subsection 11.3.2.

Case of $f_0(z_1, z_2)$ dependent on z_2

Letting $A_{ij} = 0$, $i, j = 1, 2$, in the above Theorem 11.5, immediately gives the following result.

Corollary 11.2 *The closed-loop system consisting of system (11.25) and reduced-order observer-based output feedback (11.58)-(11.59) is given by*

$$\begin{bmatrix} \dot{x}_1^{(0\sim n_1-1)} \\ \dot{x}_2^{(0\sim n_2-1)} \end{bmatrix} = \Phi_F \begin{bmatrix} x_1^{(0\sim n_1-1)} \\ x_2^{(0\sim n_2-1)} \end{bmatrix} + B_E \Delta f_0, \tag{11.64}$$

and

$$\dot{e} = \Phi_{33} e + B_L \Delta f_0, \tag{11.65}$$

where

$$B_L = \begin{bmatrix} L B_1 & B_2 \end{bmatrix}. \tag{11.66}$$

and Φ_F and B_E are given by (11.45) and (11.46), respectively, and e, Φ_{33}, and Δf_0 are given by (11.40), (11.41), and (11.36), respectively.

Regarding the exponential stability of the closed-loop system composed of (11.64)-(11.65), obviously, Theorem 11.6 can be readily utilized to solve the problem, but setting $A_{ij} = 0$, $i, j = 1, 2$. However, in this special case, a more convenient result can be provided.

Theorem 11.7 *Let Assumptions 11.4 and 11.5 be met, condition (11.48) be satisfied, and the feedback and observer gains A_F and L be designed such that Φ_F and Φ_{33} are Hurwitz. Then the closed-loop system (11.64)-(11.65) is globally exponentially stable if there exist positive constants k and ε and a positive definite matrix $P \in \mathbb{R}^{m_0 \times m_0}$, $m_0 = n_2 r_2$, such that the following LMI*

$$\begin{bmatrix} \Phi_{33}^{\mathrm{T}} P + P \Phi_{33} + k\alpha^2 I_n & P B_L \\ B_L^{\mathrm{T}} P & -k I_r \end{bmatrix} < -\varepsilon I \tag{11.67}$$

is satisfied.

The proof of the above result is given in the appendix.

Please note that the usage of the above result needs only to check a single LMI of order $n_2 r_2$.

11.4 Example: Full-order Observer-based Control

The thermoacoustic instability system in Zhao et al. (2023b) is used to illustrate the effectiveness of the proposed observer-based controller. Consider a Rijke tube without time delays

$$\begin{cases} \ddot{\eta}_1 = f_1\left(\eta_{1\sim 2}^{(0\sim 1)}(t)\right) + q_1\left(\eta_1^{(0\sim 1)}(t)\right) + u_1 \\ \ddot{\eta}_2 = f_2\left(\eta_{1\sim 2}^{(0\sim 1)}(t)\right) + q_2\left(\eta_2^{(0\sim 1)}(t)\right) + u_2, \end{cases} \tag{11.68}$$

where $\eta = \begin{bmatrix} \eta_1 & \eta_2 \end{bmatrix}^{\mathrm{T}}$, with $\eta_i, \dot{\eta}_i$ denoting the amplitude of velocity and pressure uctuation of the i-th order acoustic mode, respectively, u_1, u_2 are the control inputs, and

$$\begin{cases} f_i\left(\eta_{1\sim2}^{(0\sim1)}(t)\right) = -2i\pi\left(\gamma-1\right)\dot{Q}_{si}(t)\sin\left(i\pi x_f\right) \\ q_i\left(\eta_i^{(0\sim1)}(t)\right) = -\begin{bmatrix} (i\pi)^2 & \epsilon_i \end{bmatrix}\eta_i^{(0\sim1)},\ i=1,2, \end{cases}$$

with γ being the specific heat ratio of the medium, x_f being the location of the heat source, and $\epsilon_i,\ i=1,2$ denoting the damping coefficients. We assume the output of system (11.68) to be

$$\begin{cases} y = C_{0\sim1}\eta^{(0\sim1)} \\ C_{0\sim1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \end{cases}$$

Additionally, $\dot{Q}_{si}(t),\ i=1,2,$ are the nondimensional heat release rates in the following form of

$$\dot{Q}_{si}(t) = \frac{2L_\omega\left(T_\omega-\bar{T}_0\right)}{\sqrt{3\upsilon_0}\gamma P_0 S}\sqrt{\pi\lambda C_\upsilon\rho_0\frac{d_\omega}{2}}\left(\sqrt{\left|\frac{1}{3}+\upsilon_{fi}\right|}-\sqrt{\frac{1}{3}}\right), \tag{11.69}$$

where $\upsilon_{fi} = \eta_i\cos\left(\pi x_f\right)$, L_ω, d_ω and T_ω are the dimensionless length, diameter and temperature of the heating coil, respectively, $\bar{T}_0$ is the average temperature of the surrounding air, υ_0, P_0 are the mean value of velocity and pressure, C_υ is the specific heat capacity, S denotes the cross-sectional area of the tube, ρ_0 represents the average density of the gas, and λ is the thermal conductivity of air. It can be easily verified that the matrix pair $(\Phi(0_{0\sim1}), C_{0\sim1})$ is observable and Assumption 11.2 also holds.

In the simulation, the physical parameters listed below are considered (Zhao et al. (2022)):

$$\begin{aligned} L_\omega &= 2.5\mathrm{m},\ S = 0.00156\mathrm{m}^2,\ d_\omega = 0.0005\mathrm{m},\ T_\omega = 1680\mathrm{K}, \\ \rho_0 &= 1.225\mathrm{kg/m},\ C_\upsilon = 719\mathrm{J/(kg\cdot K)},\ \lambda = 0.0328\mathrm{W/(m\cdot K)}, \\ x_f &= 0.25\mathrm{m},\ T_0 = 295\mathrm{K},\ \upsilon_0 = 0.3\mathrm{m/s},\ P_0 = 86900\mathrm{Pa}, \\ \epsilon_1 &= 0.044,\ \epsilon_2 = 0.1657,\ \gamma = 1.4. \end{aligned}$$

Applying Theorem 11.1, we choose the following control parameters:

$$\begin{aligned} A_{0\sim1} &= \begin{bmatrix} 3 & 0 & 4 & 0 \\ 0 & 8 & 0 & 6 \end{bmatrix}, \\ L &= \begin{bmatrix} -17 & 0 & -70 & 0 \\ 0 & -8 & 0 & -15 \end{bmatrix}^{\mathrm{T}}. \end{aligned}$$

Besides, the initial values are selected as

$$\begin{aligned} \eta^{(0\sim1)}(0) &= \begin{bmatrix} 0.5 & 0.2 & 0.05 & -0.1 \end{bmatrix}^{\mathrm{T}}, \\ \hat{\eta}^{(0\sim1)}(0) &= \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}. \end{aligned}$$

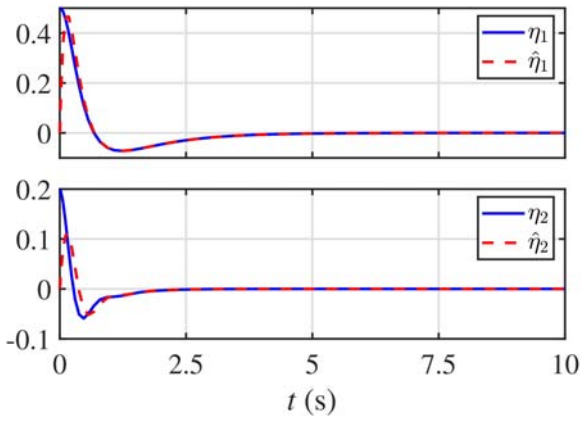


Figure 11.1: Response of η and $\hat{\eta}$

The results are shown in Figs. 11.1-11.3. Figs. 11.1-11.2 depict the time responses of velocity amplitude and pressure amplitude, where $\hat{\eta}^{(0\sim1)}$ is the estimation of $\eta^{(0\sim1)}$. Commensurate with the aforementioned analysis, the proposed observer-based controller ensures that $\hat{\eta}^{(0\sim1)}$ tracks the actual state, and $\eta^{(0\sim1)}$ converges to zero in about 5 seconds. Also, it is seen from Fig. 11.3 that the control magnitudes required are still in a reasonable range.

11.5 Example: Reduced-order Observer-based Control

Consider the following FAS:

$$\begin{cases} \begin{bmatrix} x_1^{(3)} \\ \dot{x}_2 \end{bmatrix} = f(x_1^{(0\sim2)}, x_2^{(0\sim1)}) + B(x_1^{(0\sim2)}, t)u \\ y = x_1^{(0\sim2)}, \end{cases} \tag{11.70}$$

where

$$x_1 = \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} \in \mathbb{R}^2, \ x_2 \in \mathbb{R},$$

are the states, and

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \in \mathbb{R}^3$$

is the input, $y \in \mathbb{R}^6$ is the output. Furthermore, function $f(\cdot)$ possesses the following decomposition:

$$f(x_1^{(0\sim2)}, x_2^{(0\sim1)}) = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} x_1^{(0\sim2)} \\ x_2^{(0\sim1)} \end{bmatrix} + f_0(x_1^{(0\sim2)}, x_2^{(0\sim1)}), \tag{11.71}$$

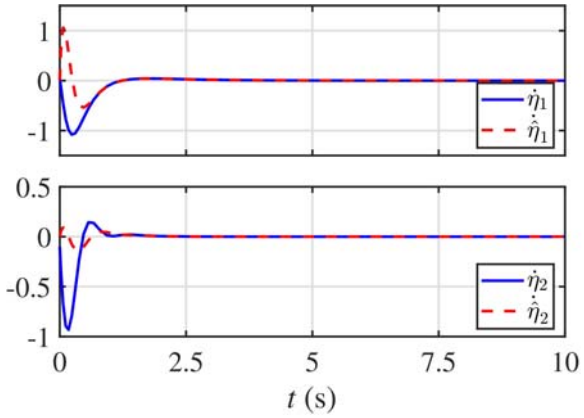


Figure 11.2: Response of $\dot{\eta}$ and $\hat{\dot{\eta}}$

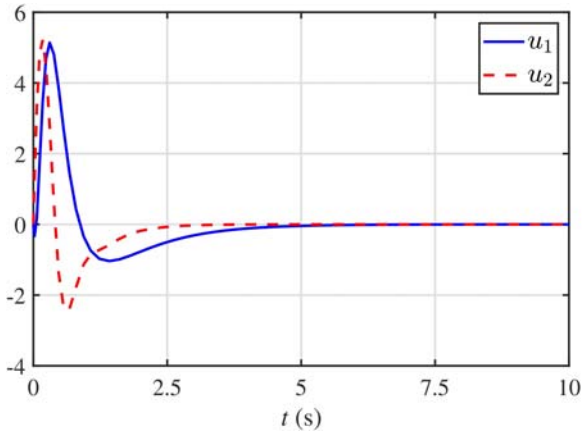


Figure 11.3: Control inputs

with

$$f_0(x_1^{(0\sim 2)}, x_2^{(0\sim 1)}) = \frac{1}{10}\begin{bmatrix}\cos\left(x_1^{\mathrm{T}}\dot{x}_1\right) & 0\\ 0 & 1\\ \sin\left(x_1^{\mathrm{T}}\ddot{x}_1\right) & 0\end{bmatrix} x_2^{(0\sim 1)}, \tag{11.72}$$

$$F_{11} = \begin{bmatrix}0 & 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 2 & 0 & 0\end{bmatrix}, \tag{11.73}$$

$$F_{12} = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \tag{11.74}$$

$$F_{21} = 0_{1\times 6}, \tag{11.75}$$

$$F_{22} = \begin{bmatrix}1 & -2\end{bmatrix}, \tag{11.76}$$

and

$$B(x_1^{(0\sim 2)}, t) = \begin{bmatrix}1 & 0 & \dot{x}_1^{\mathrm{T}}\ddot{x}_1\\ 0 & -1 & \sin t\\ 0 & 0 & 1\end{bmatrix}. \tag{11.77}$$

Step 1 Verify Assumptions 11.4 and 11.5 and condition (11.48)

Obviously, system (11.70)-(11.77) is in the form of (11.25)-(11.26). Specifically, we have $r_1 = 2$, $r_2 = 1$, $n_1 = 3$, and $n_2 = 2$. Noting (11.77), we have

$$\det B(x_1^{(0\sim 2)}, t) = -1.$$

Thus Assumption 11.4 holds.

Denote $Y = x_2^{(0\sim 1)}$. Then according to the vector function $f_0(x_1^{(0\sim 2)}, x_2^{(0\sim 1)})$ shown in (11.72), we can easily deduce

$$\begin{aligned}
&\left\| f_0(x_1^{(0\sim 2)}, Y_1) - f_0(x_1^{(0\sim 2)}, Y_2)\right\| \\
=\ &\left\| \frac{1}{10}\begin{bmatrix}\cos\left(x_1^{\mathrm{T}}\dot{x}_1\right) & 0\\ 0 & 1\\ \sin\left(x_1^{\mathrm{T}}\ddot{x}_1\right) & 0\end{bmatrix} Y_1 - \frac{1}{10}\begin{bmatrix}\cos\left(x_1^{\mathrm{T}}\dot{x}_1\right) & 0\\ 0 & 1\\ \sin\left(x_1^{\mathrm{T}}\ddot{x}_1\right) & 0\end{bmatrix} Y_2\right\| \\
\le\ &\left\| \frac{1}{10}\begin{bmatrix}\cos\left(x_1^{\mathrm{T}}\dot{x}_1\right) & 0\\ 0 & 1\\ \sin\left(x_1^{\mathrm{T}}\ddot{x}_1\right) & 0\end{bmatrix}\right\|_{\mathrm{F}} \|Y_1 - Y_2\| \\
=\ &\frac{1}{10}\sqrt{\cos^2\left(x_1^{\mathrm{T}}\dot{x}_1\right) + 1 + \sin^2\left(x_1^{\mathrm{T}}\ddot{x}_1\right)}\,\|Y_1 - Y_2\| \\
\le\ &\frac{\sqrt{3}}{10}\|Y_1 - Y_2\|.
\end{aligned}$$

Thus, Assumption 11.5 holds with

$$\alpha = \frac{\sqrt{3}}{10}.$$

Further, noting that

$$F_{12}^{[0]}=\begin{bmatrix}1\\0\end{bmatrix},\ F_{12}^{[1]}=\begin{bmatrix}0\\0\end{bmatrix},\ F_{22}^{[0]}=1,\ F_{22}^{[1]}=-2,$$

we obtain

$$\begin{aligned}\operatorname{rank}\begin{bmatrix}-F_{12}^{[1]}s-F_{12}^{[0]}\\ s^2-F_{22}^{[1]}s-F_{22}^{[0]}\end{bmatrix} &= \operatorname{rank}\begin{bmatrix}-1\\0\\ s^2+2s-1\end{bmatrix}\\ &= 1,\ \forall s\notin\mathbb{C}^-,\end{aligned}$$

which means that condition (11.48) holds.

The above assumptions and condition permit the following reduced-order observer-based controller:

$$\begin{cases}\dot{z}=(\Phi_{22}+L\Phi_{12})z+(L\Phi_{11}-(\Phi_{22}+L\Phi_{12})L+\Phi_{21})y\\ \hat{x}_2^{(0\sim1)}=z-Ly\\ u=-B^{-1}(x_1^{(0\sim2)})\left(f_0(x_1^{(0\sim2)},\hat{x}_2^{(0\sim1)})+\begin{bmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{bmatrix}\begin{bmatrix}x_1^{(0\sim2)}\\ \hat{x}_2^{(0\sim1)}\end{bmatrix}\right),\end{cases}$$

where the parameter matrices $\Phi_{ij},\ i=1,2,\ j=1,2$, and the feedback gain matrices $A_{ij},\ i=1,2,\ j=1,2$, as well as the gain matrix L will be specified below.

Step 2 Design the feedback and observer gains A_{11}, A_{12}, A_{21}, A_{22}, and L

In order to make matrix Φ_{FA} Hurwitz, for simplicity, we let

$$\Phi_{12}=0,\ \Phi_{21}=0,$$

which obviously give

$$A_{21}=F_{21}=0_{1\times6},\ A_{12}=F_{12}=\begin{bmatrix}1&0\\0&0\end{bmatrix}. \tag{11.78}$$

In this case, Φ_{FA} is Hurwitz if and only if Φ_{11} and Φ_{22} are Hurwitz. When the set of eigenvalues of matrix

$$\Phi_{11}=\Phi(0_{2\times6})-B_1(A_{11}-F_{11})$$

is specified as $\{-1\pm j,-2\pm2j,-3\pm3j\}$, via certain pole assignment technique, a solution is obtained as

$$A_{11}-F_{11}=\begin{bmatrix}-12&-12&0&-22&6&-6\\12&-12&22&0&6&6\end{bmatrix},$$

which gives

$$A_{11}=\begin{bmatrix}-12&-12&-1&-22&6&-6\\12&-12&22&2&6&6\end{bmatrix}. \tag{11.79}$$

Meanwhile, choosing the set of eigenvalues of matrix

$$\Phi_{22} = \Phi(0_{1\times 2}) - B_2(A_{22} - F_{22})$$

to be $\{-5 \pm 5j\}$, gives the following unique solution:

$$A_{22} - F_{22} = \begin{bmatrix} 50 & 10 \end{bmatrix},$$

which gives

$$A_{22} = \begin{bmatrix} 51 & 8 \end{bmatrix}. \tag{11.80}$$

Similarly, choose the set of eigenvalues of matrix

$$\Phi_{33} = \Phi(0_{1\times 2}) + B_2 F_{22} + L B_1 F_{12}$$

to be $\{-1 \pm 1j\}$. Then, via a pole assignment technique, an L matrix is obtained as

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 \end{bmatrix}. \tag{11.81}$$

Step 3 Verify the closed-loop stability

Solving inequality (11.55), we obtain $k = 2.6938$, $\varepsilon = 0.0387$, and the positive definite matrix

$$P = \begin{bmatrix} P_1 & P_2 \end{bmatrix},$$

where

$$P_1 = \begin{bmatrix} 4.3027 & -0.0027 & 1.5814 & 1.6222 & 0.2217 \\ -0.0027 & 4.3148 & -1.6292 & 1.5834 & -0.3028 \\ 1.5814 & -1.6292 & 2.3413 & -0.0022 & 0.3065 \\ 1.6222 & 1.5834 & -0.0022 & 2.3378 & -0.1264 \\ 0.2217 & -0.3028 & 0.3065 & -0.1264 & 0.1044 \\ 0.3020 & 0.2227 & 0.1258 & 0.3066 & 0 \\ -0.0226 & 0.0685 & -0.0716 & 0.0559 & -0.0067 \\ -0.0047 & 0.0015 & -0.0057 & -0.0006 & -0.0023 \\ 0.0627 & -0.1048 & 0.0742 & -0.0401 & 0.0077 \\ 0.0413 & -0.0739 & 0.0538 & -0.0247 & 0.0033 \end{bmatrix},$$

and

$$P_2 = \begin{bmatrix} 0.3020 & -0.0226 & -0.0047 & 0.0627 & 0.0413 \\ 0.2227 & 0.0685 & 0.0015 & -0.1048 & -0.0739 \\ 0.1258 & -0.0716 & -0.0057 & 0.0742 & 0.0538 \\ 0.3066 & 0.0559 & -0.0006 & -0.0401 & -0.0247 \\ 0 & -0.0067 & -0.0023 & 0.0077 & 0.0033 \\ 0.1045 & 0.0123 & -0.0007 & -0.0026 & -0.0061 \\ 0.0123 & 1.9010 & 0.0134 & -1.8239 & -0.0339 \\ -0.0007 & 0.0134 & 0.0384 & -0.0072 & -0.0347 \\ -0.0026 & -1.8239 & -0.0072 & 2.7871 & 0.2393 \\ -0.0061 & -0.0339 & -0.0347 & 0.2393 & 0.5614 \end{bmatrix}.$$

Thus, by Theorem 11.6, the designed controller is a globally exponentially stabilizing one.

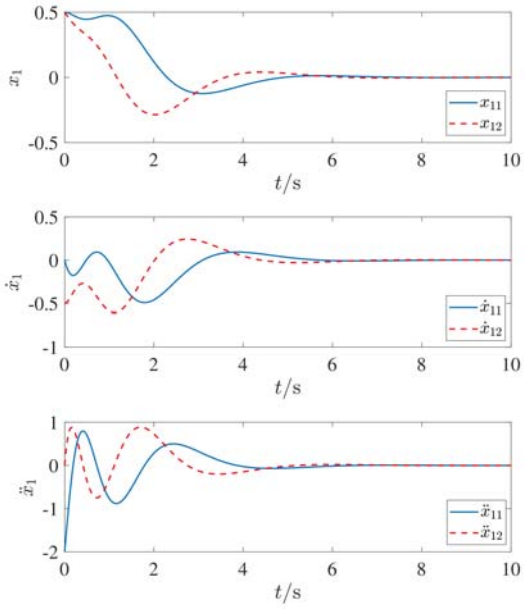


Figure 11.4: Responses of state vector $x_1^{(0\sim2)}$

Step 4 Carry out the simulation and analyze the results

Choose the following initial values:

$$x_1^{(0\sim2)}(0)=\begin{bmatrix}0.5\\0.5\\0\\-0.5\\-2\\0\end{bmatrix},\ x_2^{(0\sim1)}(0)=\begin{bmatrix}0.5\\-0.5\end{bmatrix},\ z(0)=\begin{bmatrix}1.5\\-1.5\end{bmatrix}.$$

The simulation of the designed system is carried out and the results are shown in Figures 11.4-11.6.

As shown in Figures 11.4-11.6, states $x_1^{(0\sim2)}$ and $x_2^{(0\sim1)}$ and the observation error $e=\begin{bmatrix}e_1 & e_2\end{bmatrix}^{\mathrm{T}}$ converge smoothly to the origin with a settling time of about 8 seconds, which shows the effectiveness of the proposed method. In addition, the absolute values of the control inputs are always less than 17, which are relatively small and within an acceptable range.

11.6 Further Notes

In this chapter, two observer-based control methods are proposed for FASs, namely, the full-order observer-based control and the reduced-order observer-based control.

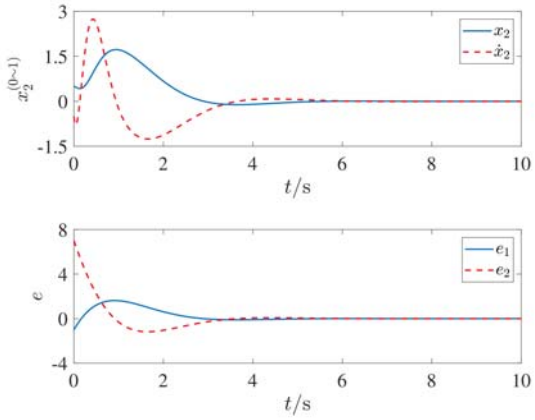


Figure 11.5: Responses of state vector $x_2^{(0\sim1)}$ and the observer error e

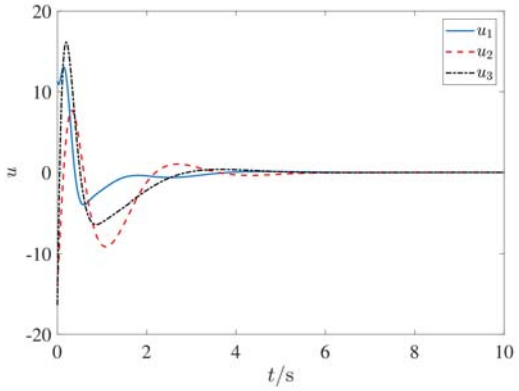


Figure 11.6: Control input u

Firstly, the full-order observer-based control scheme for a type of global FASs with external disturbances is proposed. It consists of an observer equation using output information and a feedback controller with the full-actuation property. The application of the parametric approach provides rich design DOFs for the selection of control parameters. The uniformly exponential stability of the augmented system of the closed-loop system is strictly proven by Lyapunov stability analysis. Finally, an illustrative example of thermoacoustic instability systems is employed to show the effectiveness of the proposed approach.

We point out that the corresponding results can be extended to multi-order FASs, even sub-FASs, which only partially satisfy the full-actuation condition.

Secondly, the reduced-order observer-based output feedback stabilizing controller for a class of FASs is proposed. Two cases are treated:

1. In the case that the system nonlinearity $f_0(z_1, z_2)$ is independent on z_2, the considered stabilization problem is solved by giving the solutions of the control gains and the observer gain such that the matrices Φ_{FA} and Φ_{33} are Hurwitz. The solution of the control gains A_{ij}, $i, j = 1, 2$, is demonstrated to be turned into an ESA problem, and three necessary and sufficient conditions for the existence of the solution of L are also presented.

2. In the case of $f_0(z_1, z_2)$ dependent on z_2, with a partial Lipschitz condition imposed on $f_0(z_1, z_2)$, the conditions for globally exponential stability of the closed-loop system are given. Subsequently, the specific case where the control gain matrix A_F is taken as a zero matrix is considered, for which parallel results are shown. An illustrative example is worked out to demonstrate the effectiveness of the developed method.

We also point out that the reduced-order observer-based control can be generalized to more general FASs, such as FASs with time-varying delays or uncertain FASs.

Chapter 12

Robust Stabilization of Type I Systems

The FAS approach is a general framework for control systems design. It not only treats FASs, but also solves the control of systems which can not be completely converted into FASs. In this chapter, the so-called Type I systems are firstly defined. Each of them is composed of two subsystems: one is a fully actuated subsystem representing the controllable part in the system, and the other is an isolated autonomous subsystem which represents the uncontrollable part in the system. Obviously, to control such a system, it suffices to control the fully actuated subsystem. However, when the fully actuated subsystem is subject to uncertainty involving information of the second subsystem, the control of the whole system can be no longer realized by controlling the fully actuated subsystem.

The main task of this chapter is to realize control of a general time-varying nonlinear uncertain system, which is composed of two subsystems: one is an uncertain fully actuated subsystem representing the controllable part in the system, the other is an isolated globally uniformly asymptotically (GUA) stable autonomous subsystem which represents the uncontrollable part in the system. Both the single-order FAS and the multi-order FAS representations of the controllable subsystem are introduced. The problem of robust stabilization of such a compound system with full-state feedback can be converted into an input-to-state GUA stabilization problem of the fully actuated subsystem, which is solved for both types of single- and multi-order FASs with very general assumptions on the uncertain perturbed functions. Under certain conditions, the solution reduces to that for robust stabilization with partial state feedback, and naturally reduces to that for robust stabilization of FASs. Two illustrative examples demonstrate both the effect and the application procedure of the proposed robust stabilization approach.

12.1 Type I System

12.1.1 The Linear Case

Example 12.1 (Uncontrollable LTI Systems) It is well-known that uncontrollable linear systems have controllability decompositions. More specifically, an uncontrollable linear system in a state-space representation

$$\dot{x} = Ax + Bu, \tag{12.1}$$

can be equivalently transformed via some state transformation into the following controllability decomposition form:

$$\begin{cases} \dot{x}_1 = A_1 x_1 + A_{12} x_2 + B_1 u \\ \dot{x}_2 = A_2 x_2, \end{cases} \tag{12.2}$$

where the first subsystem is the controllable subsystem, with (A_1, B_1) controllable, and the second one is the uncontrollable subsystem. This decomposition form falls into the format of the so-called Type I systems to be discussed in this chapter.

Furthermore, system (12.1) is called stabilizable if and only if the uncontrollable subsystem

$$\dot{x}_2 = A_2 x_2 \tag{12.3}$$

is stable. Only in this case, system (12.1) can be stabilized by designing a stabilizing controller for the following system

$$\dot{x}_1 = A_1 x_1 + B_1 u,$$

which can be equivalently converted into an LTI FAS due to the controllability of (A_1, B_1).

Example 12.2 Consider the following system:

$$\begin{cases} \ddot{x} = \dot{x} + \sin(x - y) + u \\ \ddot{y} = \dot{y} + u, \end{cases} \tag{12.4}$$

where u is a scalar control input. Introducing the variable transformation

$$z = x - y,$$

we can equivalently convert system (12.4) into the following form:

$$\begin{cases} \ddot{x} = \dot{x} + \sin z + u \\ \ddot{z} = \dot{z} + \sin z, \end{cases} \tag{12.5}$$

which also falls into a Type I system since the second subsystem in (12.5) is autonomous (free of control). Since the uncontrollable subsystem in (12.5) can be easily proved to be unstable, the original system (12.4) is not stabilizable.

12.1.2 Type I Systems

Let us first recall the following multi-order FAS form:

$$x_k^{(\mu_k)}|_{k=1\sim\eta} = \tilde{f}\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, t\right) + \tilde{B}\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, t\right)u, \tag{12.6}$$

where $u \in \mathbb{R}^r$ is the control vector, μ_k, $k = 1, 2, \ldots, \eta$, are a set of positive integers, $x_k \in \mathbb{R}^{r_k}$, $k = 1, 2, \ldots, \eta$, are a set of vectors, with r_k, $k = 1, 2, \ldots, \eta$, being a set of positive integers satisfying

$$r_1 + r_2 + \cdots + r_\eta = r, \tag{12.7}$$

$\tilde{f}_k(\cdot) \in \mathbb{R}^{r_k}$, $k = 1, 2, \ldots, \eta$, are a set of nonlinear vector functions, and $\tilde{B}(\cdot) \in \mathbb{R}^{r\times r}$ is a matrix function satisfying the following full-actuation assumption.

Assumption 12.1 $\det \tilde{B}(X, t) \neq 0$ *or* ∞, *for all* $X \in \mathbb{R}^{\varkappa_0}$, *and* $t \geq 0$, *where*

$$\varkappa_0 = \sum_{i=1}^{\eta} \mu_i r_i.$$

Clearly, a special case of the above general FAS (12.6) is the following single-order one:

$$x^{(n)} = f\left(x^{(0\sim n-1)}, t\right) + B\left(x^{(0\sim n-1)}, t\right)u, \tag{12.8}$$

and the above Assumption 12.1 becomes

Assumption 12.2 $\det B(X, t) \neq 0$ *or* ∞, *for all* $X \in \mathbb{R}^{nr}$, *and* $t \geq 0$.

It is well-known that a stabilizable linear system can be decomposed into a controllable subsystem and a stable uncontrollable subsystem. A similar fact holds also for nonlinear systems.

According to Duan (2021g), nonlinear (affine) systems which are not completely controllable generally fall into two main categories, one is the type of sub-controllable systems or sub-FASs in which the full-actuation assumption is not met completely but partially, the other is the Type I system in the following form:

$$\begin{aligned} x_k^{(\mu_k)}|_{k=1\sim\eta} &= \tilde{f}\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, z_k^{(0\sim\gamma_k-1)}|_{k=1\sim\xi}, t\right) \\ &\quad + \tilde{B}\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, z_k^{(0\sim\gamma_k-1)}|_{k=1\sim\xi}, t\right)u, \qquad (12.9) \\ z_k^{(\gamma_k)}|_{k=1\sim\xi} &= \tilde{g}\left(z_k^{(0\sim\gamma_k-1)}|_{k=1\sim\xi}, t\right), \qquad (12.10) \end{aligned}$$

where γ_k, $k = 1, 2, \ldots, \xi$, are a set of positive integers, $z_k \in \mathbb{R}^{l_k}$, $k = 1, 2, \ldots, \xi$, are a set of vectors of proper dimensions, $\tilde{f}(\cdot)$ and $\tilde{g}(\cdot)$ are two nonlinear vector functions, while the other variables are as stated before. It is required that $z_k^{(0\sim\gamma_k-1)}|_{k=1\sim\xi}$ and $x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}$ are independent from each other. Under this requirement, the extra subsystem (12.10) is obviously an isolated unforced one.

The stability of the subsystem (12.10), when it is nonlinear, is complicated. Note that a system in the form of (12.10) can be always converted into a state-space form. Regarding the stability of a high-order system in the form of (12.10), we make the following convention.

Convention I: A high-order nonlinear system in the form of (12.10) is said to be GUA stable if its state-space form is GUA stable.

Without loss of generality, in this chapter, the following stability assumption is proposed:

Assumption 12.3 *Subsystem (12.10) is GUA stable at the origin.*

Furthermore, the following full-actuation assumption is also imposed.

Assumption 12.4 $\det \tilde{B}\left(X, z_k^{(0\sim\gamma_k-1)}|_{k=1\sim\xi}, t\right) \neq 0$ *or* ∞*, for all* $X \in \mathbb{R}^{x_0}$, $t \geq 0$, *and* $z_k^{(0\sim\gamma_k-1)}|_{k=1\sim\xi}$*, with* $z_k^{(0\sim\gamma_k-1)}|_{k=1\sim\xi}$ *being the solution to system (12.10).*

According to Duan (2021g) and Duan (2023b), under Assumption 12.4, if in addition, the whole Type I system (12.9)-(12.10) can not be further converted into another Type I system in which the fully actuated subsystem is of a higher order than that of FAS (12.9), then the FAS in (12.9) represents the controllable subsystem, while (12.10) represents the uncontrollable subsystem, and the variables $z_k(t)$, $k = 1, 2, \ldots, \xi$, are called the uncontrollable variables. Furthermore, the whole system (12.9)-(12.10) is called stabilizable in a certain sense if the uncontrollable subsystem (12.10) is stable in the same sense. Please note that, in certain cases, a Type I system may be still completely controllable, that is, it can be still further converted into a FAS.

When $\eta = 1$, the above system (12.9)-(12.10) obviously reduces to the following:

$$x^{(n)} = f\left(x^{(0\sim n-1)}, z^{(0\sim m-1)}, t\right) + B\left(x^{(0\sim n-1)}, z^{(0\sim m-1)}, t\right)u, \quad (12.11)$$

$$z^{(m)} = g\left(z^{(0\sim m-1)}, t\right). \quad (12.12)$$

where n and m are two positive integers.

In this circumstance, the above Assumptions 12.3 and 12.4 respectively become

Assumption 12.5 *System (12.12) is GUA stable at the origin.*

Assumption 12.6 $\det B\left(X, z^{(0\sim m-1)}, t\right) \neq 0$ *or* ∞ *for all* $X \in \mathbb{R}^{nr}$, $t \geq 0$, *and* $z^{(0\sim m-1)}$, *with* $z^{(0\sim m-1)}$ *being the solution to system (12.12).*

When Assumption 12.4 is met, we can design the following controller for the Type I system (12.9)-(12.10) based on only FAS (12.9) as follows (see Duan (2021g, 2023b)):

$$\begin{cases} u = -\tilde{B}^{-1}\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, z_k^{(0\sim\gamma_k-1)}|_{k=1\sim\xi}, t\right)u' \\ u' = \tilde{f}\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, z_k^{(0\sim\gamma_k-1)}|_{k=1\sim\xi}, t\right)+u^* \\ u^* = \begin{bmatrix} u_1^* \\ u_2^* \\ \vdots \\ u_\eta^* \end{bmatrix},\ u_k^* = [A_k]_{0\sim\mu_k-1}\, x_k^{(0\sim\mu_k-1)}, \end{cases} \tag{12.13}$$

and the following closed-loop system is obtained:

$$\begin{cases} x_k^{(\mu_k)} + [A_k]_{0\sim\mu_k-1} x_k^{(0\sim\mu_k-1)} = 0, \quad k = 1,2,\ldots,\eta \\ z_k^{(\gamma_k)}|_{k=1\sim\xi} = g\left(z_k^{(0\sim\gamma_k-1)}|_{k=1\sim\xi}, t\right), \end{cases} \tag{12.14}$$

which is clearly GUA stable if the coefficient matrices $[A_k]_{0\sim\mu_k-1}$, $k = 1,2,\ldots,\eta$, are so chosen that the linear systems in (12.14) are asymptotically stable. However, this controller (12.13) may not work when FAS (12.9) is subject to uncertainties, since in this case the obtained first subsystem in the closed-loop system (12.14) is no longer an independent linear one, but may be affected by $z_k^{(0\sim\gamma_k-1)}$. This gives rise to the robust stabilization problem to be formulated in the next section.

12.2 Robust Control Problem Formulation

In Duan (2021g), the so-called Type I system is introduced, which is composed of a fully actuated subsystem and an unforced stable subsystem. It is shown in Duan (2021g) that stabilization of the whole system can be realized by only finding a stabilizing controller for the fully actuated subsystem. What is more, the closed-loop system is composed of two decoupled subsystems with the first one being constant and linear. However, when the first subsystem, that is, the FAS, is subject to uncertainties, a closed-loop system with two completely decoupled subsystems can no longer be obtained, and in general none of the two subsystems in the closed-loop system can be linear. This gives rise to a problem of robust stabilization in the FAS framework. Without consideration of both the time-varying terms and the time-delay terms, in the state-space formulation the corresponding problem is to stabilize the following system:

$$\begin{cases} \dot{x} = f(x,z) + \Delta f(x,z) + B(x,z)u \\ \dot{z} = g(z), \end{cases} \tag{12.15}$$

where $x \in \mathbb{R}^{n_1}$ and $z \in \mathbb{R}^{n_2}$ are the system state vectors, and $u \in \mathbb{R}^r$, $(r < n_1)$, is the system input vector. As it is well-known, the problem is not solvable unless a series of strict conditions on the functions $f(x,z)$, $g(z)$ and $B(x,z)$, as well as on the perturbation $\Delta f(x,z)$ are imposed.

In the rest of this chapter, stabilization of a time-varying nonlinear uncertain Type I system with time-varying delays is further considered using full-state feedback. As in Duan (2021g), the considered uncertain Type I system is also composed of a controllable subsystem and an uncontrollable subsystem, but with the controllable subsystem modeled by either a single-order or a multi-order time-varying nonlinear time-delay FAS with uncertainties. The uncontrollable subsystem, that is, the unforced one, is assumed to be GUA stable.

Firstly, it is shown that robust GUA stabilization of the proposed uncertain Type I system with full-state feedback can be converted into a robust input-to-state stabilization of the time-varying nonlinear uncertain time-delay FAS with a controller containing a full-state feedback term plus an external input feedforward term. More specifically, this particular robust input-to-state stabilizing controller for the FAS simply turns out to be a full-state feedback GUA stabilizing controller for the original uncertain Type I system.

Secondly, a stabilizing controller for the uncertain Type I system is designed based on the full-actuation feature of the first subsystem and an input-to-state stability result under some very weak assumptions. A parameter matrix in the controller can be systematically solved by using the parametric eigenstructure assignment technique, and the degrees of design freedom existing in the solution to this parameter matrix can be further utilized to achieve some additional system performance.

Thanks to the full-actuation feature of the FAS models, the proposed approach for robust control of time-varying nonlinear uncertain systems with time-varying delays is generally simpler and guarantees GUA stability. As special cases, the problem of robust stabilization of a Type I system with partial state feedback and that of robust stabilization of general FASs are also solved. Two illustrative examples demonstrate the advantages of the proposed approach.

12.2.1 Perturbed Type I System

Before formulating the problem, we make the following convention to be used throughout the rest of this chapter.

Convention II: In the following sections as well as the appendix, the notations

$$\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, z_k^{(0\sim\gamma_k-1)}|_{k=1\sim\xi}, t\right)$$

and

$$\left(x^{(0\sim n-1)}, z^{(0\sim m-1)}, t\right)$$

are both simply denoted by $(\cdot)$, since both can be easily distinguished in the specific situations.

Also, as preliminaries, let us quote some basic definitions (Khalil (2002a)).

Definition 12.1 *A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$.*

Definition 12.2 *A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$.*

Definition 12.3 *A nonlinear function $h(z) : \mathbb{R}^n \to \mathbb{R}^m$ is called a $\mathcal{K}_\infty$-bounded function if there exists a $\mathcal{K}_\infty$ function $\kappa_h(\cdot)$ such that*

$$\|h(z)\| \leq \kappa_h(\|z\|).$$

Consider stabilization of the following uncertain Type I system:

$$\begin{cases} x_k^{(\mu_k)}|_{k=1\sim\eta} = \tilde{f}(\cdot) + \Delta\tilde{f}(\cdot) + \tilde{B}(\cdot)u + \tilde{h}\left(z_k^{(0\sim\gamma_k-1)}|_{k=1\sim\xi}\right) \\ z_k^{(\gamma_k)}|_{k=1\sim\xi} = \tilde{g}\left(z_k^{(0\sim\gamma_k-1)}|_{k=1\sim\xi}, t\right), \end{cases} \tag{12.16}$$

where

$$\Delta\tilde{f}(\cdot) = \begin{bmatrix} \Delta\tilde{f}_1(\cdot) \\ \Delta\tilde{f}_2(\cdot) \\ \vdots \\ \Delta\tilde{f}_\eta(\cdot) \end{bmatrix}, \ \Delta\tilde{f}_i \in \mathbb{R}^{r_i}, \tag{12.17}$$

is the nonlinear uncertainty in the system, and is subject to the following assumption:

Assumption 12.7 *There exist two sets of nonnegative functions $\tilde{\rho}_i(\cdot)$ and $\tilde{\sigma}_i(\cdot)$, $i = 1, 2, \ldots, \eta$, such that the nonlinear uncertainty $\Delta\tilde{f}_i(\cdot) \in \mathbb{R}^{r_i}$ satisfies*

$$\left\|\Delta\tilde{f}_i(\cdot)\right\| \leqslant \tilde{\rho}_i(\cdot)\left\|x_i^{(0\sim\mu_i-1)}\right\| + \tilde{\sigma}_i(\cdot)\kappa_i\left(\left\|z_k^{(0\sim\gamma_k-1)}|_{k=1\sim\xi}\right\|\right), \tag{12.18}$$
$$i = 1, 2, \ldots, \eta,$$

where $\kappa_i(\cdot)$, $i = 1, 2, \ldots, \eta$, are a set of properly chosen $\mathcal{K}_\infty$ functions.

The nonlinear function

$$\tilde{h}(\cdot) = \begin{bmatrix} \tilde{h}_1(\cdot) \\ \tilde{h}_2(\cdot) \\ \vdots \\ \tilde{h}_\eta(\cdot) \end{bmatrix}, \ \tilde{h}_i \in \mathbb{R}^{r_i}, \tag{12.19}$$

in (12.16) is required to satisfy the following assumption.

Assumption 12.8 *$\tilde{h}_i\left(z_k^{(0\sim\gamma_k-1)}|_{k=1\sim\xi}\right)$, $i = 1, 2, \ldots, \eta$, are a series of $\mathcal{K}_\infty$-bounded functions.*

Based on the above description, the main problem to be solved in this chapter can be described as follows.

Problem 12.1 *Let Assumptions 12.3∼12.8 be met. Find a full-state feedback controller in the form of*

$$u = \tilde{\alpha}\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, z_k^{(0\sim\gamma_k-1)}|_{k=1\sim\xi}, t\right), \tag{12.20}$$

that robustly GUA stabilizes the uncertain Type I system (12.16).

Obviously, an extremely special form of the uncertain Type I system (12.16) is the following:

$$\begin{cases} x^{(n)} = f(\cdot) + \Delta f(\cdot) + B(\cdot)u + h\left(z^{(0\sim m-1)}\right) \\ z^{(m)} = g\left(z^{(0\sim m-1)}, t\right), \end{cases} \tag{12.21}$$

and now controller (12.20) becomes

$$u = \alpha\left(x^{(0\sim n-1)}, z^{(0\sim m-1)}, t\right). \tag{12.22}$$

Moreover, in this circumstance, Assumptions 12.7 and 12.8 become

Assumption 12.9 *There exist two nonnegative functions $\rho(\cdot)$, $\sigma(\cdot)$, such that*

$$\|\Delta f(\cdot)\| \leqslant \rho(\cdot)\left\|x^{(0\sim n-1)}\right\| + \sigma(\cdot)\kappa\left(\left\|z^{(0\sim m-1)}\right\|\right),$$

where $\kappa(\cdot)$ is a $\mathcal{K}_\infty$ function.

Assumption 12.10 *$h\left(z^{(0\sim m-1)}\right)$ is a $\mathcal{K}_\infty$-bounded function.*

Moreover, Problem 12.1 reduces to the following one.

Problem 12.2 *Let Assumptions 12.5∼12.10 be met. Find a full-state feedback controller in the form of (12.22) such that the uncertain Type I system (12.21) is robustly GUA stabilized.*

12.2.2 Input-to-State Stability

Consider the system

$$\dot{x} = f(x, u, t), \tag{12.23}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^r$ are the state vector and input vector, respectively, and the input $u(t)$ is a piecewise continuous, bounded function of t for all $t \geq 0$. $f(x, u, t) \in \mathbb{R}^n$ is a piecewise continuous function in t and locally Lipschitz in x and u. Suppose the unforced system

$$\dot{x} = f(x, 0, t), \tag{12.24}$$

has a GUA stable equilibrium point at the origin $x = 0$. Then we have the following definition (Khalil (2002a)).

Definition 12.4 *System (12.23) is said to be input-to-state stable if there exist a class $\mathcal{KL}$ function $\beta(\cdot)$ and a class $\mathcal{K}$ function $\gamma(\cdot)$ such that for any initial state $x(t_0)$ and any bounded input $u(t)$, the solution $x(t)$ exists for all $t \geq t_0$ and satisfies*

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) + \gamma\left(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|\right). \tag{12.25}$$

Obviously, inequality (12.25) guarantees that, for any bounded input $u(t)$, the state $x(t)$ will be bounded. Furthermore, as t increases, the state $x(t)$ will be ultimately bounded by a class $\mathcal{K}$ function of $\sup_{t \geq t_0} \|u(t)\|$.

The following lemma gives a sufficient condition for input-to-state stability (Sontag (1990)).

Lemma 12.1 *If there exists a continuously differentiable scalar function $V(t, x)$ such that*

$$\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|), \tag{12.26}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x, u, t) \leq -\alpha_3(\|x\|) + \alpha_4(\|u\|), \tag{12.27}$$

hold for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^r$ and $t \geq 0$, where α_i, $i = 1 \sim 4$, are class $\mathcal{K}_\infty$ functions, then system (12.23) is input-to-state stable.

An interesting application of the concept of input-to-state stability arises in the stability analysis of the cascade system

$$\dot{x} = f_1(x, z, t), \tag{12.28}$$
$$\dot{z} = f_2(z, t), \tag{12.29}$$

where $f_1(x, z, t) \in \mathbb{R}^n$ and $f_2(z, t) \in \mathbb{R}^m$ are piecewise continuous functions in t and locally Lipschitz with respect to x and z. Suppose both

$$\dot{x} = f_1(x, 0, t)$$

and (12.29) have GUA stable equilibrium points at their respective origins. Under what condition will the origin $(x, z) = (0, 0)$ of the compound system (12.28)-(12.29) possess the same property? The next lemma given in Khalil (2002a) shows that this will be the case if (12.28), with z viewed as input, is input-to-state stable.

Lemma 12.2 *Let $f_1(x, z, t) \in \mathbb{R}^n$ and $f_2(z, t) \in \mathbb{R}^m$ be piecewise continuous in t and locally Lipschitz with respect to x and z. If system (12.28), with z as input, is input-to-state stable and the origin of (12.29) is GUA stable, then the origin of the cascade system (12.28)-(12.29) is GUA stable.*

12.2.3 Problem Conversion

When controller (12.20) is applied to the uncertain Type I system (12.16), the closed-loop system is

$$\begin{cases} x_k^{(\mu_k)}|_{k=1\sim\eta} = \tilde{f}(\cdot) + \Delta\tilde{f}(\cdot) + \tilde{B}(\cdot)\tilde{\alpha}(\cdot) + \tilde{h}\left(z_k^{(0\sim\gamma_k-1)}|_{k=1\sim\xi}\right) \\ z_k^{(\gamma_k)}|_{k=1\sim\xi} = \tilde{g}\left(z_k^{(0\sim\gamma_k-1)}|_{k=1\sim\xi}, t\right). \end{cases} \tag{12.30}$$

Then, according to Lemma 12.2, and also in view of Convention I, we immediately have the following result regarding robust stabilizability of the uncertain Type I system (12.16).

Proposition 12.1 *Let Assumptions 12.3∼12.8 be met. Then, there exists a controller in the form of (12.20) that GUA stabilizes the uncertain Type I system (12.16) if there exists a controller in the form of (12.20) that robustly input-to-state stabilizes, with respect to the external input* $z_k^{(0\sim\gamma_k-1)}|_{k=1\sim\xi}$*, the uncertain multi-order FAS in (12.16).*

The above proposition clearly implies the following corollary corresponding to the uncertain single-order FAS (12.21).

Corollary 12.1 *Let Assumptions 12.5∼12.10 be met. Then, there exists a controller in the form of (12.22) that GUA stabilizes the uncertain Type I system (12.21) if there exists a controller in the form of (12.22) that robustly input-to-state stabilizes, with respect to the external input* $z^{(0\sim m-1)}$*, the uncertain single-order FAS in (12.21).*

Due to the above Proposition 12.1, it is clear to see that the robust GUA stabilizing controller for the uncertain Type I system (12.16) is the same as the robustly input-to-state stabilizing controller for the uncertain FAS in (12.16). Therefore, a solution to Problem 12.1 is immediately provided if the following robustly input-to-state stabilization problem is solved.

Problem 12.3 *Let Assumptions 12.4∼12.8 be met. Find a full-state feedback controller in the form of (12.20) such that the uncertain multi-order FAS in the Type I system (12.16) is robustly input-to-state stabilized.*

Parallelly, we also have the following problem corresponding to Problem 12.2.

Problem 12.4 *Let Assumptions 12.6∼12.10 be met. Find a full-state feedback controller in the form of (12.22) such that the uncertain single-order FAS in (12.21) is robustly input-to-state stabilized.*

Remark 12.1 In the special case that the functions $\tilde{f}(\cdot)$ and $\tilde{B}(\cdot)$ are independent on $z_k^{(0\sim\gamma_k-1)}|_{k=1\sim\xi}$, instead of the full-state feedback controller (12.20), the following partial state feedback controller can be considered:

$$u = \tilde{\alpha}\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, t\right). \tag{12.31}$$

Similarly, in the special case that the functions $f(\cdot)$ and $B(\cdot)$ are independent on $z^{(0\sim m-1)}$, instead of the full-state feedback controller (12.22), the following partial state feedback controller can be considered:

$$u = \alpha\left(x^{(0\sim n-1)}, t\right). \tag{12.32}$$

These two cases will be treated in Section 12.4.1.

12.3 Robust Stabilization with Full State Feedback

The aim of this section is to solve Problems 12.3 and 12.4. For simplicity, let us start with the single-order FAS case.

12.3.1 Case of Single-order FASs

In order to provide the main results in this subsection, let us first recall from Lemma 11.2 that, for any $\mu > 0$, there exist a set of matrices $A_i \in \mathbb{R}^{r\times r}$, $i = 0, 1, \ldots, n-1$, satisfying

$$\mathrm{Re}\lambda_i\left(\Phi\left(A_{0\sim n-1}\right)\right) \leq -\frac{\mu}{2},\ i = 1, 2, \ldots, nr. \tag{12.33}$$

The following defines a procedure to obtain a matrix $P_L(A_{0\sim n-1})$ from a given matrix $A_{0\sim n-1}$.

It is known from Lemma 11.1 that, when the condition in (12.33) holds for some $\mu > 0$, there exists a positive definite matrix $P \triangleq P(A_{0\sim n-1})$ satisfying the following Lyapunov inequality:

$$\Phi^{\mathrm{T}}(A_{0\sim n-1})P + P\Phi(A_{0\sim n-1}) \leq -\mu P. \tag{12.34}$$

Partition $P(A_{0\sim n-1})$ as

$$P(A_{0\sim n-1}) = \left[\begin{array}{cccc} P_1 & P_2 & \cdots & P_n \end{array}\right],\ P_i \in \mathbb{R}^{nr\times r}. \tag{12.35}$$

We can further introduce the following notation which will be frequently used in the the sequel:

$$P_L(A_{0\sim n-1}) = P(A_{0\sim n-1})\left[\begin{array}{c} 0 \\ I_r \end{array}\right] = P_n. \tag{12.36}$$

Based on the above preparations, we can now give the following result.

Theorem 12.1 *Let $A_{0\sim n-1} \in \mathbb{R}^{r\times nr}$ be a matrix satisfying (12.33), and Assumptions 12.6∼12.10 be met. Then, a control law that robustly input-to-state stabilizes the uncertain FAS in (12.21), with $z^{(0\sim n-1)}$ being viewed as the system external input, is given by*

$$u = -B^{-1}(\cdot)\left(f(\cdot) + (A_{0\sim n-1} + \Theta)\, x^{(0\sim n-1)}\right), \tag{12.37}$$

with

$$\Theta = \frac{1}{\varepsilon\sigma^2_{\min}(P_L)}\left[\sigma_{\max}(P_L)\,\rho(\cdot) + \frac{1}{2}\sigma^2(\cdot)\right]P_L^{\mathrm{T}}, \tag{12.38}$$

where $\varepsilon \in (0, 1)$ is an arbitrarily chosen number.

Substituting control law (12.37) into the FAS in (12.21), gives the closed-loop system

$$x^{(n)} = -(A_{0\sim n-1} + \Theta)\, x^{(0\sim n-1)} + \Delta f(\cdot) + h\left(z^{(0\sim m-1)}\right), \tag{12.39}$$

which has the following equivalent state-space form:

$$\dot{x}^{(0\sim n-1)} = \Phi(A_{0\sim n-1} + \Theta)x^{(0\sim n-1)} + \begin{bmatrix} 0_{(n-1)r} \\ \Delta f(\cdot) + h\left(z^{(0\sim m-1)}\right) \end{bmatrix}. \tag{12.40}$$

Therefore, the above Theorem 12.1 immediately follows from the following technical lemma.

Lemma 12.3 *Let $A_{0\sim n-1} \in \mathbb{R}^{r\times nr}$ be a matrix satisfying (12.33), and Assumptions 12.9 and 12.10 be met. Then, system (12.39) or (12.40), is input-to-state stable, with $z^{(0\sim m-1)}$ being viewed as the system external input, if Θ is given by (12.38) with $\varepsilon \in (0,1)$.*

A proof of the above lemma is given in the appendix.

Further, if $\rho(\cdot) = \sigma(\cdot) \equiv 0$, then $\Theta = 0$, and $\Delta f(\cdot)$ vanishes. Thus the above lemma immediately gives the following corollary.

Corollary 12.2 *Let $A_{0\sim n-1} \in \mathbb{R}^{r\times nr}$ be a matrix satisfying (12.33), and $h\left(z^{(0\sim m-1)}\right)$ is a $\mathcal{K}_\infty$-bounded function. Then the following system*

$$x^{(n)} = -A_{0\sim n-1}x^{(0\sim n-1)} + h\left(z^{(0\sim m-1)}\right)$$

is input-to-state stable, with $z^{(0\sim m-1)}$ being viewed as the system external input.

Further applying Corollary 12.1, we immediately have the following result, which gives the solution to Problem 12.2.

Corollary 12.3 *Let $A_{0\sim n-1} \in \mathbb{R}^{r\times nr}$ be a matrix satisfying (12.33), and Assumptions 12.5∼12.10 hold. Furthermore, let $\varepsilon \in (0,1)$ be an arbitrarily given number. Then, control law (12.37)-(12.38) robustly GUA stabilizes the uncertain system (12.21).*

12.3.2 Case of Multi-order FASs

Let $[A_i]_{0\sim\mu_i-1} \in \mathbb{R}^{r_i\times r_i\mu_i}$, $i = 1,2,\ldots,\eta$, be a set of given matrices satisfying

$$\begin{aligned} \mathrm{Re}\lambda_j\left(\Phi\left([A_i]_{0\sim\mu_i-1}\right)\right) &\leq -\frac{\mu}{2}, \\ j &= 1,2,\ldots,\mu_i r_i,\ i = 1,2,\ldots,\eta. \end{aligned} \tag{12.41}$$

Then it follows from Lemma 11.1 again that there exists a set of positive definite matrices

$$P_i \triangleq P([A_i]_{0\sim\mu_i-1}),\ i = 1,2,\ldots,\eta,$$

satisfying

$$\Phi^{\mathrm{T}}([A_i]_{0\sim\mu_i-1})P_i + P_i\Phi([A_i]_{0\sim\mu_i-1}) \le -\mu P_i, \quad i=1,2,\ldots,\eta. \tag{12.42}$$

Then, following the procedure given in Subsection 12.3.1, we can define the matrices

$$P_{Li} \triangleq P_L\left([A_i]_{0\sim\mu_i-1}\right),\ i=1,2,\ldots,\eta.$$

Based on the defined matrices $P_{Li},\ i=1,2,\ldots,\eta$, the following controller for the uncertain multi-order FAS in (12.16) can be designed:

$$\begin{cases} u = -\tilde{B}^{-1}(\cdot)\left(\tilde{f}(\cdot)+u^*\right) \\ u^* = \begin{bmatrix} u_1^* \\ u_2^* \\ \vdots \\ u_\eta^* \end{bmatrix},\ u_i^* = \left([A_i]_{0\sim\mu_i-1}+\Theta_i\right)x_i^{(0\sim\mu_i-1)}, \end{cases} \tag{12.43}$$

with

$$\begin{aligned} &\Theta_i = \frac{1}{\varepsilon\sigma_{\min}^2(P_{Li})}\left[\sigma_{\max}^2(P_{Li})\,\tilde{\rho}_i(\cdot)+\tfrac{1}{2}\tilde{\sigma}_i^2(\cdot)\right]P_{Li}^{\mathrm{T}}, \\ &i=1,2,\ldots,\eta, \end{aligned} \tag{12.44}$$

where ε is a positive number less than 1.

Substituting the above control law (12.43) into the FAS in (12.16), gives the following closed-loop system:

$$x_i^{(\mu_i)} = -[A_i]_{0\sim\mu_i-1}x_i^{(0\sim\mu_i-1)} + \phi_i(\cdot) + \tilde{h}_i(\cdot),\ i=1,2,\ldots,\eta, \tag{12.45}$$

where

$$\phi_i(\cdot) = -\Theta_i x_k^{(0\sim\mu_k-1)} + \Delta\tilde{f}_i(\cdot),\ i=1,2,\ldots,\eta.$$

The closed-loop system (12.45) can be written in the following state-space form:

$$\begin{aligned} \dot{x}_i^{(0\sim\mu_i-1)} &= \Phi\left([A_i]_{0\sim\mu_i-1}\right)x_i^{(0\sim\mu_i-1)} + \begin{bmatrix} 0_{(\mu_i-1)r_i} \\ \phi_i(\cdot)+\tilde{h}_i(\cdot) \end{bmatrix}, \\ &i=1,2,\ldots,\eta. \end{aligned} \tag{12.46}$$

Since $[A_i]_{0\sim\mu_i-1}\in\mathbb{R}^{r_i\times r_i\mu_i},\ i=1,2,\ldots,\eta$, satisfy condition (12.41), there exists a set of positive definite matrices $P_i,\ i=1,2,\ldots,\eta$, satisfying the Lyapunov inequality (12.42). Then the following Lyapunov function can be chosen for each subsystem in (12.45):

$$V_i = \frac{1}{2}\left(x_i^{(0\sim\mu_i-1)}\right)^{\mathrm{T}}P_i x_i^{(0\sim\mu_i-1)},\ i=1,2,\ldots,\eta.$$

Following the same procedure in the proof of Lemma 12.3 (see the Appendix section), we can show that each subsystem in (12.45) is input-to-state stable with respect to the external input $z_k^{(0\sim\gamma_k-1)}|_{k=1\sim\xi}$. To summarize, we have the following result about the robustly input-to-state stabilization of the uncertain FAS in (12.16) with the external input $z_k^{(0\sim\gamma_k-1)}|_{k=1\sim\xi}$.

Theorem 12.2 *Let $[A_i]_{0\sim\mu_i-1} \in \mathbb{R}^{r_i\times r_i\mu_i}$, $i=1,2,\ldots,\eta$, be a set of matrices satisfying (12.41), Assumptions 12.4∼12.8 be met. Then, a control law that robustly input-to-state stabilizes the uncertain FAS in (12.16), with $z_k^{(0\sim\gamma_k-1)}|_{k=1\sim\xi}$ being viewed as the system input, is given by (12.43)-(12.44) with $\varepsilon\in(0,1)$ being an arbitrarily given number.*

In view of Proposition 12.1, we immediately have from the above result the following corollary, which gives the solution to Problem 12.1.

Corollary 12.4 *Let $[A_i]_{0\sim\mu_i-1} \in \mathbb{R}^{r_i\times r_i\mu_i}$, $i=1,2,\ldots,\eta$, be a set of matrices satisfying (12.41), and Assumptions 12.3∼12.8 be satisfied. Further, let $\varepsilon\in(0,1)$ be an arbitrarily given number. Then, control law (12.43)-(12.44) robustly GUA stabilizes the uncertain Type I system (12.16).*

Remark 12.2 The above design is called the decoupled design since η number of decoupled subsystems (with respect to the internal variables) are obtained in the closed-loop system (12.45). In Chapter 3 (see also Duan (2021g)), a coupled design for controllers of nonlinear FASs is also given, and we point out that the above approach can also be extended to the coupled design case, in which a whole coupled closed-loop system with an external input $z_k^{(0\sim\gamma_k-1)}|_{k=1\sim\xi}$ is derived (see, also, Duan (2022e, 2023c)).

Remark 12.3 In Chapter 2 (see also Duan (2020a, 2021c,d,e,f,g)), nonlinear FASs in non-affine form are also introduced, and they are then generalized to the discrete-time case (Duan (2022f)), the discrete-time delay system case (Duan (2022b,c)) and also the continuous-time delay system case (Duan (2023b,c)). We point out that the affine FAS in the uncertain Type I system proposed in this chapter can also be substituted by a general non-affine FAS, and the corresponding robust stabilization problem can be also similarly treated.

12.4 Further Issues

12.4.1 Robust Stabilization with Partial State Feedback

In this subsection, we further show that, in certain special cases, controllers (12.37)-(12.38) and (12.43)-(12.44) turn to be partial state feedback ones. Here we only demonstrate this with the case of single-order FASs.

Let us restrict the function $f(\cdot)$ and matrix $B(\cdot)$ to be independent on $z^{(0\sim m-1)}$ and then system (12.21) becomes

$$\begin{cases} x^{(n)} = f\left(x^{(0\sim n-1)},t\right)+\Delta f(\cdot)+B\left(x^{(0\sim n-1)},t\right)u+h\left(z^{(0\sim m-1)}\right) \\ z^{(m)} = g\left(z^{(0\sim m-1)},t\right), \end{cases} \tag{12.47}$$

where the matrix $B\left(x^{(0\sim n-1)},t\right)$ satisfies the full-actuation Assumption 12.2, the functions $g\left(z^{(0\sim m-1)},t\right)$ and $h\left(z^{(0\sim m-1)}\right)$ still satisfy Assumptions 12.5 and 12.10, respectively, while Assumption 12.9 on the nonlinear uncertainty $\Delta f(\cdot)$ becomes

Assumption 12.11 *There exist two nonnegative scalar functions $\rho\left(x^{(0\sim n-1)},t\right)$ and $\sigma\left(x^{(0\sim n-1)},t\right)$ such that the nonlinear uncertainty $\Delta f(\cdot)\in\mathbb{R}^r$ satisfies*

$$\|\Delta f(\cdot)\|\leqslant\rho\left(x^{(0\sim n-1)},t\right)\left\|x^{(0\sim n-1)}\right\|+\sigma\left(x^{(0\sim n-1)},t\right)\kappa\left(\left\|z^{(0\sim m-1)}\right\|\right), \quad (12.48)$$

where $\kappa(\cdot)$ is a $\mathcal{K}_\infty$ function.

When Assumption 12.9 is substituted with Assumption 12.11, Theorem 12.1 becomes the following.

Theorem 12.3 *Let $A_{0\sim n-1}\in\mathbb{R}^{r\times nr}$ be a matrix satisfying (12.33), and Assumptions 12.2, 12.10, and 12.11 hold. Then a controller that robustly input-to-state stabilizes the uncertain FAS in (12.47), with $z^{(0\sim m-1)}$ being viewed as the system external input, is given by*

$$\begin{cases} u=-B^{-1}\left(x^{(0\sim n-1)},t\right)\left(f\left(x^{(0\sim n-1)},t\right)+u^*\right) \\ u^*=\left[A_{0\sim n-1}+\Theta\right]x^{(0\sim n-1)}, \end{cases} \quad (12.49)$$

with

$$\begin{cases} \Theta=\frac{1}{\varepsilon\sigma^2_{\min}(P_L)}\vartheta P_L^{\mathrm{T}}(A_{0\sim n-1}) \\ \vartheta=\sigma^2_{\max}(P_L)\rho\left(x^{(0\sim n-1)},t\right)+\frac{1}{2}\sigma^2\left(x^{(0\sim n-1)},t\right), \end{cases} \quad (12.50)$$

where $\varepsilon\in(0,1)$ is an arbitrarily given number.

The following result immediately follows from the above Theorem 12.3 and Corollary 12.1.

Corollary 12.5 *Let $A_{0\sim n-1}\in\mathbb{R}^{r\times nr}$ be a matrix satisfying (12.33), and Assumptions 12.2, 12.5, 12.10, and 12.11 be met. Furthermore, let $\varepsilon\in(0,1)$ be an arbitrarily given number. Then controller (12.49)-(12.50) robustly GUA stabilizes the uncertain Type I system (12.47).*

To end this subsection, we make the following important remark.

Remark 12.4 The function $\Delta f(\cdot)$ in system (12.47) may be also looked upon as a known function satisfying condition (12.48). In this case, the above system (12.47) is no longer uncertain, but a deterministic one. The problem turns to be a stabilization one for the deterministic Type I system (12.47) using output feedback, with the output variables being the partial states $x^{(0\sim n-1)}$. It is clearly seen from the above results that this stabilization problem is solvable as long as $\Delta f(\cdot)$ satisfies condition (12.48), which divides out the information about $x^{(0\sim n-1)}$ that is used for feedback.

12.4.2 Case of Perturbed $B(\cdot)$ Matrix

For simplicity and space limit, here we only consider the single-order case, where the perturbed FAS is given by

$$x^{(n)}=f(\cdot)+\Delta f(\cdot)+[B(\cdot)+\Delta B(\cdot)]u+h\left(z^{0\sim m-1}\right), \quad (12.51)$$

where the matrix $B(\cdot)$ and the function $\Delta f(\cdot)$ still satisfy Assumptions 12.6 and 12.9, and the function $h\left(z^{0\sim m-1}\right)$ satisfies Assumption 12.10. In addition, $\Delta B(\cdot)$ is a perturbed matrix satisfying the following two assumptions.

Assumption 12.12 *The following condition*

$$\left\| \Delta B(\cdot) B^{-1}(\cdot) \right\| \leq c < 1, \tag{12.52}$$

holds for all the variables $x^{(0\sim n-1)}$, $z^{(0\sim m-1)}$, *and* $t \geq 0$.

Assumption 12.13 *There exist two nonnegative functions* $\rho_b(\cdot)$ *and* $\sigma_b(\cdot)$, *such that*

$$\|\Delta B(\cdot)\| \leqslant \rho_b(\cdot) \left\| x^{(0\sim n-1)} \right\| + \sigma_b(\cdot) \kappa_b \left(\left\| z^{(0\sim m-1)} \right\| \right),$$

where $\kappa_b(\cdot)$ *is a* $\mathcal{K}_\infty$ *function.*

Still, as in Subsection 12.3.1, let the controller be in the form of

$$u = -B^{-1}(\cdot) \left[f(\cdot) + \left(A_{0\sim n-1} + \frac{1}{\varepsilon \sigma^2_{\min}(P_L)} \vartheta P_L^{\mathrm{T}} \right) x^{(0\sim n-1)} \right]. \tag{12.53}$$

Then the closed-loop system is

$$\begin{aligned} x^{(n)} &= -\left(A_{0\sim n-1} + \frac{1}{\varepsilon \sigma^2_{\min}(P_L)} \vartheta P_L^{\mathrm{T}} \right) x^{(0\sim n-1)} \\ &\quad + h\left(z^{0\sim m-1}\right) + \Delta f(\cdot) + \Delta f_b(\cdot), \end{aligned} \tag{12.54}$$

where

$$\begin{aligned} \Delta f_b(\cdot) &= -\Delta B(\cdot) B^{-1}(\cdot) \left(f(\cdot) + A_{0\sim n-1} x^{(0\sim n-1)} \right) \\ &\quad - \frac{1}{\varepsilon \sigma^2_{\min}(P_L)} \vartheta \Delta B(\cdot) B^{-1}(\cdot) P_L^{\mathrm{T}} x^{(0\sim n-1)}. \end{aligned} \tag{12.55}$$

Thus it can be shown that

$$\begin{aligned} \|\Delta f_b(\cdot)\| &\leq \|\Delta B(\cdot)\| \left\| B^{-1}(\cdot) \left(f(\cdot) + A_{0\sim n-1} x^{(0\sim n-1)} \right) \right\| \\ &\quad + \frac{1}{\varepsilon \sigma^2_{\min}(P_L)} \vartheta \left\| \Delta B(\cdot) B^{-1}(\cdot) \right\| \left\| P_L^{\mathrm{T}} x^{(0\sim n-1)} \right\| \\ &\leq \left\| B^{-1}(\cdot) \left(f(\cdot) + A_{0\sim n-1} x^{(0\sim n-1)} \right) \right\| \\ &\quad \times \left(\rho_b(\cdot) \left\| x^{(0\sim n-1)} \right\| + \sigma_b(\cdot) \kappa_b \left(\left\| z^{(0\sim m-1)} \right\| \right) \right) \\ &\quad + \frac{c}{\varepsilon} \frac{1}{\sigma^2_{\min}(P_L)} \vartheta \left\| P_L^{\mathrm{T}} x^{(0\sim n-1)} \right\|. \end{aligned} \tag{12.56}$$

Now, let $\hat{\sigma}(\cdot)$ be a positive scalar function, and $\hat{\kappa}\left(\left\|z^{(0\sim m-1)}\right\|\right)$ be a $\mathcal{K}_\infty$-function satisfying

$$\begin{aligned} & \left\|B^{-1}(\cdot)\left(f(\cdot)+A_{0\sim n-1}x^{(0\sim n-1)}\right)\right\|\sigma_b(\cdot)\kappa_b\left(\left\|z^{(0\sim m-1)}\right\|\right) \\ & +\sigma(\cdot)\kappa\left(\left\|z^{(0\sim m-1)}\right\|\right) \\ \leq\ & \hat{\sigma}(\cdot)\hat{\kappa}\left(\left\|z^{(0\sim m-1)}\right\|\right), \end{aligned} \tag{12.57}$$

and define

$$\hat{\rho}(\cdot)=\rho(\cdot)+\left\|B^{-1}(\cdot)\left(f(\cdot)+A_{0\sim n-1}x^{(0\sim n-1)}\right)\right\|\rho_b(\cdot). \tag{12.58}$$

Then, as a generalization to Lemma 12.3, we have the following lemma.

Lemma 12.4 *Let $A_{0\sim n-1}\in\mathbb{R}^{r\times nr}$ be a matrix satisfying (12.33), and Assumptions 12.6∼12.13 be met. Then, system (12.54) is input-to-state stable, with $z^{(0\sim m-1)}$ being viewed as the system external input, if $\varepsilon\in(0,1-c)$, and*

$$\vartheta=\sigma_{\max}^2(P_L)\hat{\rho}(\cdot)+\frac{1}{2}\hat{\sigma}^2(\cdot), \tag{12.59}$$

where $\hat{\sigma}(\cdot)$ and $\hat{\rho}(\cdot)$ are given by (12.57) and (12.58), respectively.

A proof of the above lemma is also given in Appendix B.

With the help of the above lemma, the following result can be immediately deduced.

Theorem 12.4 *Let $A_{0\sim n-1}\in\mathbb{R}^{r\times nr}$ be a matrix satisfying (12.33), and Assumptions 12.6∼12.13 be met. Then, a control law that robustly input-to-state stabilizes the uncertain FAS in (12.51), with $z^{(0\sim m-1)}$ being viewed as the system external input, is given by (12.53) and (12.59), with $\varepsilon\in(0,1-c)$ being an arbitrary scalar.*

To end this subsection, we point out that the above results can be also generalized into the multi-order FAS case.

12.4.3 Robust Control of FASs

Again here we only demonstrate with the case of single-order FASs.

When the second subsystem in (12.47) does not exist, the Type I system (12.47) becomes the following uncertain FAS:

$$x^{(n)}=f\left(x^{(0\sim n-1)},t\right)+\Delta f\left(x^{(0\sim n-1)},t\right)+B\left(x^{(0\sim n-1)},t\right)u, \tag{12.60}$$

where the matrix $B\left(x^{(0\sim n-1)},t\right)$ still satisfies the full-actuation Assumption 12.2, while Assumption 12.11 on the nonlinear uncertainty $\Delta f(\cdot)$ becomes

Assumption 12.14 *There exist a nonnegative function $\rho\left(x^{(0\sim n-1)},t\right)$ such that*

$$\|\Delta f(\cdot)\|\leqslant\rho\left(x^{(0\sim n-1)},t\right)\left\|x^{(0\sim n-1)}\right\|.$$

As a direct corollary of Theorem 12.3, the following result holds.

Theorem 12.5 *Let $A_{0\sim n-1} \in \mathbb{R}^{r\times nr}$ be a matrix satisfying (12.33), and Assumptions 12.2 and 12.14 be met. Then a controller that robustly and globally stabilizes the uncertain FAS in (12.60) is given by (12.49)-(12.50) with $\sigma\left(x^{(0\sim n-1)},t\right)$ being set to zero.*

It is easy to note that, in the case of $\rho\left(x^{(0\sim n-1)},t\right)\equiv 0$, the uncertainty $\Delta f(\cdot)$ does not exist, and FAS (12.60) becomes the following deterministic one:

$$x^{(n)} = f(\cdot) + B(\cdot)u, \tag{12.61}$$

while the controller simply reduces to

$$u = -B^{-1}(\cdot)\left(f(\cdot) + A_{0\sim n-1}x^{(0\sim n-1)}\right). \tag{12.62}$$

As a consequence, controller (12.62) serves as a controller for the above FAS (12.61). This is clearly a direct extension of the Proposition 3.1 in Duan (2021g), and also a restatement of the result in Duan (2023b). Meantime, it follows from Corollary 12.1 that control law (12.62) GUA stabilizes the Type I system (12.11)-(12.12). Such a fact gives a clear explanation of the concept of stabilizability proposed in the Subsection 4.1 in Duan (2021g).

It is seen from the above sections that our solutions to the formulated robust stabilization problems depend heavily on the parameter matrix $A_{0\sim n-1}$ satisfying (12.33), or the series of parameter matrices $[A_i]_{0\sim\mu_i-1}\in\mathbb{R}^{r_i\times r_i\mu_i}$, $i=1,2,\ldots,\eta$, satisfying (12.41). Now in this subsection let us state an existing parametric solution to these parameter matrices.

It is seen from the above result that in order to meet condition (12.33), it suffices only to select a matrix F satisfying

$$\mathrm{Re}\lambda_i(F) \le -\frac{\mu}{2},\ i=1,2,\ldots,nr. \tag{12.63}$$

Clearly, there are quite some degrees of freedom (DOFs) in the selection of F, while on the other hand we also have the parameter matrix Z, which provides r^2n DOFs. All these DOFs can be further utilized to achieve additional performance of the system (see, e.g., Duan (1992a, 1993a)).

12.5 Two Illustrative Examples

12.5.1 Example 1: Full State Feedback

Consider the following system:

$$\begin{cases} \ddot{x} = \mathrm{tg}\theta \\ \ddot{\theta} = h(z) + \Delta\left(x,\dot{x},\dot{\theta},z,\dot{z}\right) + u \\ \ddot{z} = -5\dot{z} - 6z, \end{cases} \tag{12.64}$$

where x and θ are scalar variables, with θ satisfying $|\theta| < \frac{\pi}{2}$, u is a scalar input. The nonlinear function $\Delta(x, \dot{x}, z, \dot{z})$ is an uncertainty, which is in fact given as

$$\Delta(\cdot) = 10^{-2}\left(\delta_1\dot{\theta} - (\delta_2 - 2\delta_1)x^2 z + 3\delta_2 \dot{x}\dot{z}^2\right),$$

with $0 \leq \delta_1 \leq 2$, $-1 \leq \delta_2 \leq 2$.

Following the procedure in the treatment of the Example 3.4 in Duan (2021g), we can obtain, from the first equation of (12.64),

$$\begin{aligned}
\theta &= \text{arctg}(\ddot{x}), \\
\dot{\theta} &= \frac{1}{1+\ddot{x}^2}\dddot{x}, \\
\ddot{\theta} &= \left(\frac{1}{1+\ddot{x}^2}\right)'\dddot{x} + \frac{1}{1+\ddot{x}^2}x^{(4)} \\
&= \frac{1}{1+\ddot{x}^2}x^{(4)} - \frac{2}{\left(1+\ddot{x}^2\right)^2}\ddot{x}\dddot{x}^2.
\end{aligned}$$

Substituting the above expression of $\ddot{\theta}$ into the second equation of (12.64), gives

$$x^{(4)} = \frac{2}{\left(1+\ddot{x}^2\right)}\ddot{x}\dddot{x}^2 + \left(1+\ddot{x}^2\right)h(z) + \left(1+\ddot{x}^2\right)\Delta(x, \dot{x}, \dot{\theta}, z, \dot{z}) + \left(1+\ddot{x}^2\right)u.$$

Therefore, we can equivalently convert system (12.64) into the following uncertain Type I system:

$$\begin{cases} x^{(4)} = f(\ddot{x}, \dddot{x}, z) + \Delta f(x, z) + \left(1+\ddot{x}^2\right)u \\ \ddot{z} = -5\dot{z} - 6z, \end{cases} \tag{12.65}$$

where

$$\begin{aligned}
f(\cdot) &= \frac{2}{1+\ddot{x}^2}\ddot{x}\,\dddot{x}^2 + \left(1+\ddot{x}^2\right)h(z), \\
\Delta f(\cdot) &= \left(1+\ddot{x}^2\right)\Delta\left(x, \dot{x}, \dot{\theta}, z, \dot{z}\right).
\end{aligned}$$

Using Proposition 1.1, we obtain

$$a_{0\sim 3} = \begin{bmatrix} 24 & 50 & 35 & 10 \end{bmatrix}, \tag{12.66}$$

which makes

$$\text{eig}[\Phi(a_{0\sim 3})] = \{-1, -2, -3, -4\},$$

where $\text{eig}(A)$ represents the set of eigenvalues of the matrix A. Further, it can be verified that the following

$$P = \begin{bmatrix} 195.14 & 206.39 & 69.75 & 7.34 \\ 206.39 & 232.40 & 80.83 & 8.32 \\ 69.75 & 80.83 & 31.28 & 3.13 \\ 7.34 & 8.32 & 3.13 & 0.40 \end{bmatrix}$$

satisfies the Lyapunov inequality (12.34) with $\mu = 1.8$. Therefore,

$$P_L^{\mathrm{T}}(a_{0\sim 3}) = [7.34 \qquad 8.32 \qquad 3.13 \qquad 0.40]. \tag{12.67}$$

Further, noting

$$\theta = \operatorname{arctg} \ddot{x}, \ \ \dot{\theta} = \frac{1}{1+\ddot{x}^2}\dddot{x}, \tag{12.68}$$

we have

$$\begin{aligned}
\Delta f(\cdot) &= 10^{-2}\left(1+\ddot{x}^2\right)\left(\delta_1\dot{\theta} - (\delta_2 - 2\delta_1)x^2 z + 3\delta_2 \dot{x}\dot{z}^2\right) \\
&= 10^{-2}\left(1+\ddot{x}^2\right)\left(\delta_1 \frac{1}{1+\ddot{x}^2}\dddot{x} - (\delta_2 - 2\delta_1)x^2 z + 3\delta_2 \dot{x}\dot{z}^2\right) \\
&= 10^{-2}\left(1+\ddot{x}^2\right)\left(3\delta_2 \dot{x}\dot{z}^2 - (\delta_2 - 2\delta_1)x^2 z\right) + 10^{-2}\delta_1 \dddot{x} \\
&= 10^{-2}\left[\begin{array}{cccc} 0 & 3\delta_2\left(1+\ddot{x}^2\right)\dot{z}^2 & 0 & \delta_1 \end{array}\right] x^{(0\sim 3)} \\
&\quad -10^{-2}\left[\begin{array}{cc} (\delta_2 - 2\delta_1)\left(1+\ddot{x}^2\right)x^2 & 0 \end{array}\right] z^{(0\sim 1)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|\Delta f(\cdot)\| &\le 10^{-2}\left\|\left[\begin{array}{cccc} 0 & 3\delta_2\left(1+\ddot{x}^2\right)\dot{z}^2 & 0 & \delta_1 \end{array}\right]\right\| \left\|x^{(0\sim 3)}\right\| \\
&\quad +10^{-2}\left\|\left[\begin{array}{cc} (\delta_2 - 2\delta_1)\left(1+\ddot{x}^2\right)x^2 & 0 \end{array}\right]\right\| \left\|z^{(0\sim 1)}\right\| \\
&= 10^{-2}\left[9\left(\delta_2\left(1+\ddot{x}^2\right)\dot{z}^2\right)^2 + \delta_1^2\right]^{\frac{1}{2}} \left\|x^{(0\sim 3)}\right\| \\
&\quad +10^{-2}\left|(\delta_2 - 2\delta_1)\left(1+\ddot{x}^2\right)x^2\right| \left\|z^{(0\sim 1)}\right\| \\
&\le \rho(\ddot{x},\dot{z})\left\|x^{(0\sim 3)}\right\| + \sigma(x,\ddot{x})\left\|z^{(0\sim 1)}\right\|, \\
&\quad 0 \le \delta_1 \le 2, \ -1 \le \delta_2 \le 2.
\end{aligned}$$

with

$$\begin{aligned}
\rho(\ddot{x},\dot{z}) &= 2\times 10^{-2}\left[9\left(1+\ddot{x}^2\right)^2 \dot{z}^4 + 1\right]^{\frac{1}{2}}, \\
\sigma(x,\ddot{x}) &= 5\times 10^{-2}\left(1+\ddot{x}^2\right)x^2.
\end{aligned}$$

Therefore, Assumption 12.9 holds, and the controller can be designed as

$$\left\{\begin{array}{l} u = -\frac{1}{1+\ddot{x}^2}\left[f(\ddot{x},\dddot{x},z) + \left(a_{0\sim 3} + \frac{1}{\varepsilon\sigma_{\min}^2(P_L)}\vartheta P_L^{\mathrm{T}}(a_{0\sim 3})\right)x^{(0\sim 3)}\right] \\ \vartheta = \sigma_{\max}^2(P_L)\rho(\ddot{x},\dot{z}) + \frac{1}{2}\sigma^2(x,\ddot{x}). \end{array}\right. \tag{12.69}$$

When $\varepsilon = 0.96$, $\delta_1 = \delta_2 = 1$, and $h(z) = \sin z$, the system initial values are chosen as

$$\left[\begin{array}{c} x(0) \\ \dot{x}(0) \\ \ddot{x}(0) \\ \dddot{x}(0) \end{array}\right] = \left[\begin{array}{c} 1 \\ -1 \\ -2 \\ 3 \end{array}\right],$$

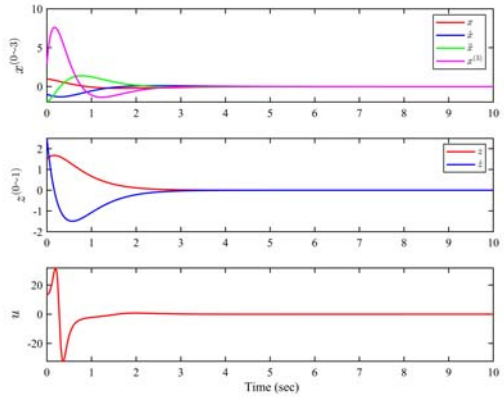


Figure 12.1: Simulation results for Example 1

$$z(0) = 1.5, \dot{z}(0) = 2.5,$$

the simulation results are carried out and shown in Fig. 12.1.

It is reported that, corresponding to the following extreme choices of $(\delta_1,\ \delta_2)$:

$$(0,-1),\ (2,2),\ (2,-1),\ (0,2),$$

the simulations are also carried out and the results turn out to be very similar to those in Fig. 12.1.

12.5.2 Example 2: Partial State Feedback

Consider the following system slightly altered from Treangle et al. (2019):

$$\begin{cases} \dot{x}_1 = -0.1x_1 + x_3 + u_1 + u_2 \\ \dot{x}_2 = -x_3 - 0.3x_2 + 0.1\tanh x_2 \\ \dot{x}_3 = -x_1 - 0.01x_3 - 0.5x_3^3 + x_2 + u_2, \end{cases} \tag{12.70}$$

where x_i, $i = 1,2,3$, are the state variables, and u_i, $i = 1,2$, are the control variables.

To derive the FAS model of the system, let us repeat the process in Duan (2023b). Taking differential of the second equation of (12.70), and into the results submitting the third equation of (12.70), yield

$$\begin{aligned} \ddot{x}_2 &= -\dot{x}_3 - 0.3\dot{x}_2 + 0.1(1-\tanh^2 x_2)\dot{x}_2 \\ &= x_1 + 0.01x_3 + 0.5x_3^3 - x_2 - u_2 \\ &\quad -0.3\dot{x}_2 + 0.1(1-\tanh^2 x_2)\dot{x}_2. \end{aligned} \tag{12.71}$$

Meanwhile, from the second equation of (12.70) we get

$$x_3 = -\dot{x}_2 - 0.3x_2 + 0.1\tanh x_2.$$

Submitting this into (12.71), gives an expression of $\ddot{x}_2$ expressed with terms only related to x_1 and x_2 and their derivatives. Combining this expression with the first equation of (12.70), produces the following FAS of the above system (12.70) as

$$\begin{bmatrix} \dot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1\left(x_1, x_2^{(0\sim1)}\right) \\ f_2\left(x_1, x_2^{(0\sim1)}\right) \end{bmatrix} + Bu, \tag{12.72}$$

where

$$\begin{aligned} f_1(\cdot) &= -0.1x_1 - 0.3x_2 + 0.1\tanh x_2 - \dot{x}_2, & (12.73)\\ f_2(\cdot) &= -\left(0.2 + 0.1\tanh^2 x_2\right)\dot{x}_2 + 0.01\left(-0.3x_2 + 0.1\tanh x_2 - \dot{x}_2\right) \\ &\quad +0.5\left(-0.3x_2 + 0.1\tanh x_2 - \dot{x}_2\right)^3 + x_1 - x_2, & (12.74) \end{aligned}$$

and

$$B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix},\ u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{12.75}$$

Adding an uncertain term and an exosystem to the above FAS (12.72), gives the following uncertain Type I system:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = f(\cdot) + \Delta f(\cdot) + Bu \\ \dot{z} = -z, \end{cases} \tag{12.76}$$

where

$$f(\cdot) = \begin{bmatrix} f_1(\cdot) \\ f_2(\cdot) \end{bmatrix},\ \Delta f(\cdot) = \begin{bmatrix} x_1 \\ x_1^2 + x_2 \end{bmatrix} z.$$

Therefore, we can take $\rho_i(\cdot) = 0$, $i = 1, 2$, and

$$\sigma_1 = |x_1|,\ \sigma_2 = \left|x_1^2 + x_2\right|.$$

Applying Theorem 12.2 to the above uncertain Type I system (12.76), gives the following controller

$$\begin{cases} u = -B^{-1}\left(\begin{bmatrix} f_1(\cdot) \\ f_2(\cdot) \end{bmatrix} + \begin{bmatrix} u_1^* \\ u_2^* \end{bmatrix}\right) \\ u_1^* = ax_1 + \frac{1}{2\varepsilon\sigma_{\min}^2(P_{L1})}\sigma_1^2 P_{L1}^{\mathrm{T}} x_1 \\ u_2^* = a_0 x_2 + a_1\dot{x}_2 + \frac{1}{2\varepsilon\sigma_{\min}^2(P_{L2})}\sigma_2^2 P_{L2}^{\mathrm{T}} x_2^{(0\sim1)}, \end{cases} \tag{12.77}$$

where a_0, a_1 and a are negative scalars. Specially choosing $a = a_0 = a_1 = -1$, we can easily get

$$P_1 = P_{L1} = 1.$$

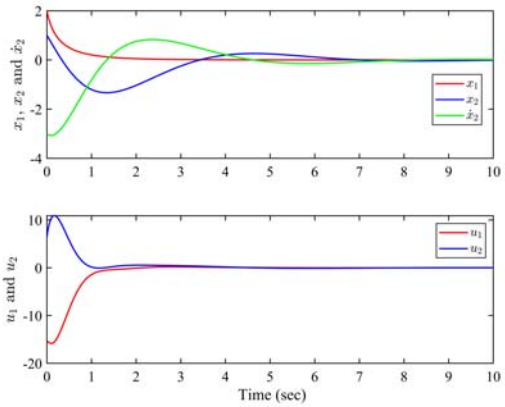


Figure 12.2: Simulation results for Example 2

Further, with $\mu = 0.8$, we obtain

$$P_2 = \begin{bmatrix} 6.9737 & 3.2895 \\ 3.2895 & 6.3158 \end{bmatrix}, \; P_{L2} = \begin{bmatrix} 3.2895 \\ 6.3158 \end{bmatrix}.$$

When the system initial values are taken as

$$x_1(0) = 2, \; x_2(0) = 1, \; \dot{x}_2(0) = -3, \; z(0) = -1.5,$$

the simulation of the system is carried out and shown in Fig. 12.2.

12.6 Notes and References

Two references closely related to this chapter are Duan (2021g, 2023d). The concept of Type I systems is introduced in Duan (2021g), and robust stabilization of a type of uncertain Type I systems is studied in Duan (2023d).

12.6.1 Comments on Results

It is shown that the Type I system introduced in Duan (2021g) can be generalized to a much more complicated one, which is composed of two subsystems, with the first one being a time-varying nonlinear FAS with both uncertainties, and the second one being an isolated unforced time-varying nonlinear system. Two presentations of the FAS in the proposed Type I system are used, namely, the single-order FAS and the multi-order FAS.

Robust stabilization of the whole system is effectively solved using full and partial state feedback. The problem is firstly shown to possess the same solution as the robust input-to-state stabilization of the nonlinear FAS in the considered Type I system, and

then solved using an input-to-state stability criterion for a cascade system and also the full-actuation assumption of the FAS, and simple GUA stabilizing controllers are proposed.

It is well-known that the problem of robust stabilization of a nonlinear system is generally hard in the state-space approaches, while that of a time-varying nonlinear system turns out to be unimaginably difficult. The results of this chapter have demonstrated the great advantages of the proposed FAS approach.

Sub-FASs represent a type of systems in which the full-actuation assumption is not completely, but only partially, satisfied. It will be a great challenge if the FAS in the considered Type I system is replaced with a sub-FAS. Such a problem will be further addressed in the future.

12.6.2 Relevant References

Stability and stabilization of dynamical systems are crucial problems in control systems theory, and have gained enormous attention in the past century. The book, Chen and Huang (2015), deals with the stabilization and output regulation problems in nonlinear control theory and applications, and also lays a special emphasis on the global robust stabilization problem for uncertain nonlinear control systems, and Yang et al. (2014) reviews some developments on stabilization of switched nonlinear systems with unstable modes.

Studies on the whole topic of stabilization of nonlinear systems are carried out mainly in the state-space framework, and as a consequence, most of the obtained results give only the local stability of the considered systems (e.g., Mazenc and Malisoff (2014), Lu et al. (2020)). Specifically, Mazenc and Malisoff (2014) adopts the reduction model approach to prove the local asymptotic stability of the closed-loop input delayed systems, using feedback that may be nonlinear, and Lu et al. (2020) investigates the domain of attraction and the local stabilization problem under the nonlinear parameter-varying framework.

The problem of stabilization of nonlinear systems is very difficult in nature. It is revealed in Brockett (1983) that many nonholonomic and underactuated systems cannot be stabilized by continuously differentiable, time-invariant, state feedback control laws, and the challenge of the problem of static output stabilization of an LTI system with a linear time-varying feedback proposed in Brockett (1999) has been fully recognized and has stimulated numerous further investigations.

Global stabilization of nonlinear systems is generally very hard to achieve in the state-space framework. As stated in Li and Muldowney (1996), the difficulty associated with the global stabilization problem is largely due to the lack of practical tools. The method of Lyapunov functions is almost the only one, whose application is often hindered by the fact that in many cases global Lyapunov functions are difficult to construct and there is practically no general approach to the construction of such functions. In general, global stabilization can only be achieved in the state-space approach for certain special types of nonlinear systems or systems satisfying certain strict conditions, e.g., systems in lower triangular forms (Ma and Zhao (2010); Karafyllis and Jiang (2010)), uncertain feedforward nonlinear systems (Ding et al. (2010); Marconi and Isidori (2000)), and nonlinear cascade systems satisfying an appropriate input-to-

state stability condition (Jankovic et al. (1996); Sussmann and Kokotovic (1991); Lin and Polycarpou (2003)). For general nonlinear systems, the problem of global stabilization has already been a long-standing insurmountable one in the literature, not to mention that for more complicated time-varying nonlinear systems.

For performance control of nonlinear systems, often more complicated types of controllers have to be designed, e.g., adaptive fuzzy control (Yin et al. (2016)), switched control (Li et al. (2022a)), observer-based control (Liu et al. (2023c); Li et al. (2019)), and finite-time command filtered backstepping control (Yu et al. (2018, 2021)).

Factually, state-space models are not the best ones to handle control. They are convenient for solving out the state vectors (responses), but not for the control vectors. While there is no doubt that a model from which the control vector can be explicitly solved out would best facilitate controller design. Fortunately, such a system model has been found very recently, which happens to be the type of so-called FAS models motivated strongly by the physical background in mechanical engineering. A great advantage of this approach is that a controller for a deterministic FAS can be easily designed such that a linear closed-loop system is obtained with an arbitrarily assignable eigenstructure. In the above mentioned two series of papers, several nonlinear control problems have been well addressed, including robust control, adaptive control, disturbance rejection, optimal control and tracking control. Different from the state-space approach, the closed-loop systems derived in these designs using the FAS approach are mostly made globally exponentially or uniformly asymptotically stable under very mild conditions. Furthermore, the FAS approach has also been generalized to discrete-time and continuous-time nonlinear time-varying systems with time-varying delays (Duan (2022b,c, 2023b,c)), and has also been well tested by some long-standing subtle problems in the nonlinear control literature (Duan (2022g,a, 2023a)).

Part V

Appendix

Appendix A

Specific Solutions for Controller Parameterization

It has been sufficiently observed that the control problems of both linear and most nonlinear systems based on the FAS approach are finally turned into a specific eigenstructure assignment (ESA) problem of a particular matrix pair. Therefore, this problem is essentially elementary in the FAS approach. For this consideration, this problem has been investigated in detail in Duan et al. (2024). This chapter represents some of the main results in Duan et al. (2024), including a general parametric solution to the fundamental problem in iterative form, which provides all the degrees of freedom (DOFs). Special solutions for some commonly used cases are also given. These solutions provide for the various design problems some ready-to-use results, and are therefore of fundamental importance. Illustrative examples demonstrate the usage of the proposed solutions.

A.1 Problem and Solution

As shown in Chapter 3 (see also, e.g., Duan (2021a,g)), after completely canceling the nonlinearity of the system, the stabilization problem of the system is immediately transformed into a specific ESA problem of a certain matrix pair, regardless of whether the open-loop system is linear or not, and no matter how complex the structure of the open-loop system is. In addition, it has been proven that this specific ESA problem, termed as "fundamental problem", also plays an important role in many other control problems throughout this book and also the rest two volumes, involving adaptive control (Duan (2021d)), robust control (Duan (2021c)), disturbance attenuation and decoupling (Duan (2021f)), tracking control (Duan (2022d)), optimal control (Duan (2022e)), etc. Therefore, this fundamental problem is of great importance, to which many existing ESA methods can be readily used (see Apkarian et al. (2001); Patton and Chen (2000); Andry et al. (1983); Sobel et al. (1994) for instance). However, considering that this issue is of particular importance, and it only involves the ESA of a certain matrix pair instead of a general one, we specifically present some results for this

problem in some detail in this appendix, and establish complete parametric solutions for different cases of desired eigenstructures. The aim of this appendix is to provide a database in which one can find matching parametric solutions directly.

A.1.1 Fundamental Problem

It can be clearly seen from Chapters 1 and 3 that many control problems in the FAS approach can be reduced to the following problem.

Problem A.1 *For given integers n and r, find a parameter matrix $A_{0\sim n-1}$ such that the following system is asymptotically stable:*

$$x^{(n)} + A_{0\sim n-1} x^{(0\sim n-1)} = 0, \tag{A.1}$$

where $x \in \mathbb{R}^r$.

To solve Problem A.1, we rewrite system (A.1) into the following state-space form:

$$\dot{x}^{(0\sim n-1)} = \Phi\left(A_{0\sim n-1}\right) x^{(0\sim n-1)}. \tag{A.2}$$

Thus we only need to find proper gain matrix $A_{0\sim n-1}$ such that $\Phi\left(A_{0\sim n-1}\right)$ is Hurwitz. Note that

$$\Phi\left(A_{0\sim n-1}\right) = \Phi\left(0_{0\sim n-1}\right) - B_{\mathrm{c}} A_{0\sim n-1}, \tag{A.3}$$

where

$$B_{\mathrm{c}} = \left[\begin{array}{cccc} 0 & 0 & \cdots & I_r \end{array}\right]^{\mathrm{T}},$$

and a matrix is Hurwitz if and only if it is similar to a Hurwitz matrix of the same size, the problem is finally transformed into the following one.

Problem A.2 *[The fundamental problem] For an arbitrarily chosen Hurwitz matrix $F \in \mathbb{R}^{nr\times nr}$, find all the parameter matrix $A_{0\sim n-1}$ such that the following equation holds:*

$$\Phi\left(0_{0\sim n-1}\right) - B_{\mathrm{c}} A_{0\sim n-1} = V F V^{-1}, \tag{A.4}$$

where $V \in \mathbb{R}^{nr\times nr}$ is some nonsingular matrix.

It is noted that, for at least the type of F having a set of distinct eigenvalues, there is always a solution to the above fundamental problem, because it is well-known that $\left(\Phi\left(0_{0\sim n-1}\right), B_{\mathrm{c}}\right)$ is controllable (see, e.g., Duan (2021g)).

Remark A.1 In fact, for the case of discrete-time systems (see Duan (2022f)), a parallel fundamental problem can be also derived, in which the matrix F needs to be Schur instead of Hurwitz. Furthermore, besides the basic stabilization problem of continuous- and discrete-time systems, some other problems can be also proven to be very closely related to this fundamental problem when the FAS approach is employed. These include, but are not limited to, adaptive control (Duan (2021d)), robust control (Duan (2021c)), disturbance attenuation and decoupling (Duan (2021f)), tracking control (Duan (2022d)), and optimal control (Duan (2022e)).

Remark A.2 This section has shown that many control problems can be reduced to such a specific fundamental problem. It is commonly seen in control system context that one problem is often converted into another. However, this particular conversion described above is special in two aspects:

1. the scope of the problems which can be converted is astonishingly wide, as we have seen; and
2. the final problems obtained after the conversion turn out to be not a class but a single one, namely, the ESA of a specific pair of matrices $(\Phi(0_{0\sim n-1}), B_c)$, which is determined by only two parameters n and r.

A.1.2 General Parametric Solution

As a matter of fact, a general solution to Problem A.2 is given by Proposition 1.1 which is restated below for convenience.

Lemma A.1 *For a given matrix $F \in \mathbb{R}^{nr\times nr}$, all the matrix $A_{0\sim n-1}$ and the nonsingular matrix $V \in \mathbb{R}^{nr\times nr}$ satisfying*

$$\Phi(A_{0\sim n-1}) = VFV^{-1} \tag{A.5}$$

are given by

$$\begin{cases} A_{0\sim n-1} = -ZF^nV^{-1}(Z,F) \\ V = V(Z,F) = \begin{bmatrix} Z \\ ZF \\ \vdots \\ ZF^{n-1} \end{bmatrix}, \end{cases} \tag{A.6}$$

where $Z \in \mathbb{R}^{r\times nr}$ is an arbitrary parameter matrix satisfying

$$\det V(Z,F) \neq 0. \tag{A.7}$$

In general, constraint (A.7) is very loose. The matrix $Z \in \mathbb{R}^{r\times nr}$ satisfying this constraint provides the DOFs in the design.

The aim of this appendix is to present several further specific versions of this result, which will be more directly applicable.

For robustness consideration, let us restrict the matrix F to be nondefective, hence we can generally express the matrix F in the following form:

$$F = \operatorname{diag}(\Lambda_1, \Lambda_2), \tag{A.8}$$

where

$$\Lambda_1 = \operatorname{diag}(s_k,\ k = 1\sim m_1), \tag{A.9}$$

$$\Lambda_2 = \operatorname{diag}\left(\begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix},\ i = 1\sim m_2\right), \tag{A.10}$$

with $s_k,\ k=1\sim m_1$, and $\alpha_i,\ \beta_i,\ i=1\sim m_2$, being three sets of negative scalars. Eventually, the matrix F possesses m_1 real eigenvalues and m_2 self-conjugate pairs of eigenvalues.

Based on Lemma A.1 and the above preparation, the following result can be derived.

Theorem A.1 *Let $F\in\mathbb{R}^{nr\times nr}$ be an arbitrarily chosen Hurwitz matrix in the form of (A.8) with $m_1+2m_2=nr$. Then, the solution to Problem A.2 is given by*

$$A_{0\sim n-1}=-\left[\begin{array}{cc}\Psi^{\mathrm{R}} & \Psi^{\mathrm{I}}\end{array}\right]\left[\begin{array}{cc}V^{\mathrm{R}} & V^{\mathrm{I}}\end{array}\right]^{-1}, \tag{A.11}$$

and

$$\left\{\begin{array}{l}\Psi^{\mathrm{R}}=\left[\begin{array}{ccc}\Psi_1^{\mathrm{R}} & \cdots & \Psi_{m_1}^{\mathrm{R}}\end{array}\right]\\ \Psi_k^{\mathrm{R}}=Z_{1,k}s_k^n,\ k=1\sim m_1,\end{array}\right. \tag{A.12}$$

$$\left\{\begin{array}{l}V^{\mathrm{R}}=\left[\begin{array}{ccc}V_1^{\mathrm{R}} & \cdots & V_{m_1}^{\mathrm{R}}\end{array}\right]\\ V_k^{\mathrm{R}}=\left[\begin{array}{c}Z_{1,k}\\ Z_{1,k}s_k\\ \vdots\\ Z_{1,k}s_k^{n-1}\end{array}\right],\ k=1\sim m_1,\end{array}\right. \tag{A.13}$$

$$\left\{\begin{array}{l}\Psi^{\mathrm{I}}=\left[\begin{array}{cccc}V_{a_1}^n & V_{b_1}^n & \cdots & V_{b_{m_2}}^n\end{array}\right]\\ V^{\mathrm{I}}=\left[\begin{array}{cccc}V_{a_1}^{\mathrm{I}} & V_{b_1}^{\mathrm{I}} & \cdots & V_{b_{m_2}}^{\mathrm{I}}\end{array}\right]\\ \left[\begin{array}{cc}V_{a_i}^{\mathrm{I}} & V_{b_i}^{\mathrm{I}}\end{array}\right]=\left[\begin{array}{cc}V_{a_i}^0 & V_{b_i}^0\\ V_{a_i}^1 & V_{b_i}^1\\ \vdots & \vdots\\ V_{a_i}^{n-1} & V_{b_i}^{n-1}\end{array}\right],\ i=1\sim m_2,\end{array}\right. \tag{A.14}$$

with $V_{a_i}^j,\ V_{b_i}^j,\ i=1\sim m_2,\ j=0\sim n$, determined by the iteration

$$\left\{\begin{array}{l}V_{a_i}^{j+1}=\alpha_i V_{a_i}^j-\beta_i V_{b_i}^j\\ V_{b_i}^{j+1}=\beta_i V_{a_i}^j+\alpha_i V_{b_i}^j,\ i=1\sim m_2,\ j=0\sim n-1,\end{array}\right. \tag{A.15}$$

and the initial values

$$V_{a_i}^0=Z_{2,i}^a,\ V_{b_i}^0=Z_{2,i}^b,\ i=1\sim m_2, \tag{A.16}$$

where $Z_{1,k},\ Z_{2,i}^a,\ Z_{2,i}^b\in\mathbb{R}^r,\ k=1\sim m_1,\ i=1\sim m_2$, are arbitrary parameter vectors which make the matrix V nonsingular.

Proof. It follows from Lemma A.1 that the solution to Problem A.2 is given by (A.11) with

$$\left[\begin{array}{cc}\Psi^{\mathrm{R}} & \Psi^{\mathrm{I}}\end{array}\right]=ZF^n, \tag{A.17}$$

and

$$\begin{bmatrix} V^{\mathrm{R}} & V^{\mathrm{I}} \end{bmatrix} = \begin{bmatrix} Z \\ ZF \\ \vdots \\ ZF^{n-1} \end{bmatrix}. \tag{A.18}$$

Define

$$\Lambda_{2,i} = \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix},\ i = 1 \sim m_2, \tag{A.19}$$

then the matrix F can be given by

$$F = \operatorname{diag}\left(\Lambda_1, \Lambda_{2,i},\ i = 1 \sim m_2\right). \tag{A.20}$$

In accordance with the structure of the above matrix F, let us denote the parameter matrix Z in the following form:

$$\begin{cases} Z = \begin{bmatrix} Z_1 & Z_2 \end{bmatrix} \\ Z_1 = \begin{bmatrix} Z_{1,1} & \cdots & Z_{1,m_1} \end{bmatrix} \\ Z_2 = \begin{bmatrix} Z_{2,1}^a & Z_{2,1}^b & \cdots & Z_{2,m_2}^a & Z_{2,m_2}^b \end{bmatrix}, \end{cases} \tag{A.21}$$

where $Z_{1,k}, Z_{2,i}^a, Z_{2,i}^b \in \mathbb{R}^r,\ k = 1 \sim m_1,\ i = 1 \sim m_2$. Then, it follows from (A.18) that the matrices V^{R} and V^{I} can be obtained by

$$V^{\mathrm{R}} = \begin{bmatrix} V_1^0 & \cdots & V_{m_1}^0 \\ V_1^1 & \cdots & V_{m_1}^1 \\ \vdots & \vdots & \vdots \\ V_1^{n-1} & \cdots & V_{m_1}^{n-1} \end{bmatrix}, \tag{A.22}$$

$$V^{\mathrm{I}} = \begin{bmatrix} V_{a_1}^0 & V_{b_1}^0 & \cdots & V_{a_{m_2}}^0 & V_{b_{m_2}}^0 \\ V_{a_1}^1 & V_{b_1}^1 & \cdots & V_{a_{m_2}}^1 & V_{b_{m_2}}^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ V_{a_1}^{n-1} & V_{b_1}^{n-1} & \cdots & V_{a_{m_2}}^{n-1} & V_{b_{m_2}}^{n-1} \end{bmatrix}, \tag{A.23}$$

where

$$V_k^j = Z_{1,k} s_k^j,\ k = 1 \sim m_1,\ j = 0 \sim n-1,$$

$$
\begin{aligned}
\begin{bmatrix} V_{a_i}^0 & V_{b_i}^0 \end{bmatrix} &= \begin{bmatrix} Z_{2,i}^a & Z_{2,i}^b \end{bmatrix}, \\
\begin{bmatrix} V_{a_i}^1 & V_{b_i}^1 \end{bmatrix} &= \begin{bmatrix} V_{a_i}^0 & V_{b_i}^0 \end{bmatrix} \Lambda_{2,i} \\
&= \begin{bmatrix} V_{a_i}^0 & V_{b_i}^0 \end{bmatrix} \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix} \\
&= \begin{bmatrix} \alpha_i V_{a_i}^0 - \beta_i V_{b_i}^0 & \beta_i V_{a_i}^0 + \alpha_i V_{b_i}^0 \end{bmatrix}, \\
&\vdots \\
\begin{bmatrix} V_{a_i}^{j+1} & V_{b_i}^{j+1} \end{bmatrix} &= \begin{bmatrix} V_{a_i}^j & V_{b_i}^j \end{bmatrix} \Lambda_{2,i} \\
&= \begin{bmatrix} V_{a_i}^j & V_{b_i}^j \end{bmatrix} \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix} \\
&= \begin{bmatrix} \alpha_i V_{a_i}^j - \beta_i V_{b_i}^j & \beta_i V_{a_i}^j + \alpha_i V_{b_i}^j \end{bmatrix}, \\
&\quad i = 1 \sim m_2,\ j = 0 \sim n-2.
\end{aligned}
$$

Then, using (A.17), the matrices Ψ^{R} and Ψ^{I} are, respectively, given by (A.12) and

$$
\begin{aligned}
\Psi^{\mathrm{I}} &= \left(Z_2 \Lambda_2^{n-1} \right) \Lambda_2 \\
&= \begin{bmatrix} V_{a_1}^{n-1} & V_{b_1}^{n-1} & \cdots & V_{a_{m_2}}^{n-1} & V_{b_{m_2}}^{n-1} \end{bmatrix} \Lambda_2 \\
&= \begin{bmatrix} \begin{bmatrix} V_{a_1}^{n-1} & V_{b_1}^{n-1} \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix} & \cdots \\
& \qquad \begin{bmatrix} V_{a_{m_2}}^{n-1} & V_{b_{m_2}}^{n-1} \end{bmatrix} \begin{bmatrix} \alpha_{m_2} & \beta_{m_2} \\ -\beta_{1m_2} & \alpha_{m_2} \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} V_{a_1}^n & V_{b_1}^n & \cdots & V_{a_{m_2}}^n & V_{b_{m_2}}^n \end{bmatrix}.
\end{aligned} \tag{A.24}
$$

The proof is completed. ■

A.2 Special Cases: Multi-input Systems

In this section, we give the solutions to some commonly used multi-input special cases. Let us start with the two-dimensional input case.

A.2.1 Case of Two-dimensional Input

In the case of two-dimensional input, $r = 2$, let us firstly consider the case of second-order systems, $n = 2$.

Case of second-order systems

For this special case, $n = 2$, system (A.1) is

$$\ddot{x} + A_1 \dot{x} + A_0 x = 0, \tag{A.25}$$

where $x \in \mathbb{R}^2$ and $A_0, A_1 \in \mathbb{R}^{2\times 2}$. There are three choices of the matrix F:

$$F_1 = \text{diag}\,(s_k, k = 1 \sim 4), \tag{A.26}$$

$$F_2 = \text{diag}\left(s_1, s_2, \begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix}\right), \tag{A.27}$$

$$F_3 = \text{diag}\left(\begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix}, \begin{bmatrix} \alpha_2 & \beta_2 \\ -\beta_2 & \alpha_2 \end{bmatrix}\right), \tag{A.28}$$

with $s_k, \alpha_i, \beta_i,\ k = 1, 2, 3, 4,\ i = 1, 2$, being a set of negative scalars. Based on Theorem A.1, the corresponding result can be immediately derived as follows.

Corollary A.1 *Let $n = r = 2$. Then the solution to Problem A.2 is given by (A.11), where the matrices V and Ψ are:*

1) in the case of $F = F_1$,

$$\Psi = \begin{bmatrix} Z_{1,1}s_1^2 & Z_{1,2}s_2^2 & Z_{1,3}s_3^2 & Z_{1,4}s_4^2 \end{bmatrix}, \tag{A.29}$$

$$V = \begin{bmatrix} Z_{1,1} & Z_{1,2} & Z_{1,3} & Z_{1,4} \\ Z_{1,1}s_1 & Z_{1,2}s_2 & Z_{1,3}s_3 & Z_{1,4}s_4 \end{bmatrix}; \tag{A.30}$$

2) in the case of $F = F_2$,

$$\Psi = \begin{bmatrix} Z_{1,1}s_1^2 & Z_{1,2}s_2^2 & \alpha_1 V_a^1 - \beta_1 V_b^1 & \beta_1 V_a^1 + \alpha_1 V_b^1 \end{bmatrix}, \tag{A.31}$$

$$V = \begin{bmatrix} Z_{1,1} & Z_{1,2} & Z_2^a & Z_2^b \\ Z_{1,1}s_1 & Z_{1,2}s_2 & V_a^1 & V_b^1 \end{bmatrix}, \tag{A.32}$$

with

$$V_a^1 = \alpha_1 Z_2^a - \beta_1 Z_2^b,\ V_b^1 = \beta_1 Z_2^a + \alpha_1 Z_2^b; \tag{A.33}$$

3) in the case of $F = F_3$,

$$\Psi = \begin{bmatrix} V_{a_1}^2 & V_{b_1}^2 & V_{a_2}^2 & V_{b_2}^2 \end{bmatrix}, \tag{A.34}$$

$$V = \begin{bmatrix} V_{a_1}^0 & V_{b_1}^0 & V_{a_2}^0 & V_{b_2}^0 \\ V_{a_1}^1 & V_{b_1}^1 & V_{a_2}^1 & V_{b_2}^1 \end{bmatrix}, \tag{A.35}$$

with

$$\begin{cases} V_{a_i}^0 = Z_{2,i}^a,\ V_{b_i}^0 = Z_{2,i}^b \\ V_{a_i}^{j+1} = \alpha_i V_{a_i}^j - \beta_i V_{b_i}^j \\ V_{b_i}^{j+1} = \beta_i V_{a_i}^j + \alpha_i V_{b_i}^j,\ i = 1, 2,\ j = 0, 1, \end{cases}$$

where in the above three cases the involved $Z_{1,i},\ i = 1 \sim 4$, and $Z_{2,i}^a$ and $Z_{2,i}^b,\ i = 1, 2$, are arbitrary parameter vectors which make the corresponding matrix V nonsingular.

Case of third-order systems

In the case of $n = 3$, system (A.1) is

$$\dddot{x} + A_2\ddot{x} + A_1\dot{x} + A_0 x = 0, \tag{A.36}$$

where $x \in \mathbb{R}^2$ and $A_i \in \mathbb{R}^{2\times 2}$, $i = 0, 1, 2$. In such a case, the relationship $m_1 + 2m_2 = 6$ holds. Therefore, there are four possible forms for the matrix F given in (A.8), and these correspond to $(m_1, m_2) = (6, 0)$, $(m_1, m_2) = (4, 1)$, $(m_1, m_2) = (2, 2)$ and $(m_1, m_2) = (0, 3)$.

As a demonstration only, the solutions to the cases of $(m_1, m_2) = (6, 0)$ and $(m_1, m_2) = (4, 1)$ are given based on Theorem A.1. The results for the other two cases can also be similarly derived.

Corollary A.2 *Let $n = 3, r = 2$. Then the solution to Problem A.2 is given by (A.11), where the matrices V and Ψ are:*

1) in the case of $(m_1, m_2) = (6, 0)$,

$$\Psi = \begin{bmatrix} Z_{1,1}s_1^3 & Z_{1,2}s_2^3 & \cdots & Z_{1,6}s_6^3 \end{bmatrix}, \tag{A.37}$$

$$V = \begin{bmatrix} Z_{1,1} & Z_{1,2} & \cdots & Z_{1,6} \\ Z_{1,1}s_1 & Z_{1,2}s_2 & \cdots & Z_{1,6}s_6 \\ Z_{1,1}s_1^2 & Z_{1,2}s_2^2 & \cdots & Z_{1,6}s_6^2 \end{bmatrix}; \tag{A.38}$$

2) in the case of $(m_1, m_2) = (4, 1)$,

$$\Psi = \begin{bmatrix} Z_{1,1}s_1^3 & Z_{1,2}s_2^3 & Z_{1,3}s_3^3 & Z_{1,4}s_4^3 & V_a^3 & V_b^3 \end{bmatrix},$$

$$V = \begin{bmatrix} Z_{1,1} & Z_{1,2} & Z_{1,3} & Z_{1,4} & V_a^0 & V_b^0 \\ Z_{1,1}s_1 & Z_{1,2}s_k & Z_{1,3}s_3 & Z_{1,4}s_4 & V_a^1 & V_b^1 \\ Z_{1,1}s_1^2 & Z_{1,2}s_2^2 & Z_{1,3}s_3^2 & Z_{1,4}s_4^2 & V_a^2 & V_b^2 \end{bmatrix},$$

with

$$\begin{cases} V_a^0 = Z_2^a,\ V_b^0 = Z_2^b \\ V_a^{j+1} = \alpha_1 V_a^j - \beta_1 V_b^j \\ V_b^{j+1} = \beta_1 V_a^j + \alpha_1 V_b^j,\ j = 0, 1, 2, \end{cases} \tag{A.39}$$

where in the above two cases the involved $Z_{1,k}$, $k = 1 \sim 6$, and Z_2^a, Z_2^b are arbitrary parameter vectors which make the corresponding matrix V nonsingular.

A.2.2 Case of Three-dimensional Input

In this subsection, only the case of $n = 2$, $r = 3$ is considered, in which system (A.1) is given by (A.25) again, but with $x \in \mathbb{R}^3$ and A_0, $A_1 \in \mathbb{R}^{3\times 3}$. Similarly, the relationship $m_1 + 2m_2 = 6$ still holds, thus there are four possible forms for the matrix F given in (A.8), these correspond to $(m_1, m_2) = (6, 0)$, $(m_1, m_2) = (4, 1)$, $(m_1, m_2) = (2, 2)$, and $(m_1, m_2) = (0, 3)$.

Here we just give the solutions to the cases of $(m_1, m_2) = (6, 0)$, $(m_1, m_2) = (4, 1)$, and $(m_1, m_2) = (2, 2)$ by using Theorem A.1 .

Corollary A.3 *Let $n = 2$, $r = 3$. Then the solution to Problem A.2 is given by (A.11), where the matrices V and Ψ are:*

1) in the case of $(m_1, m_2) = (6, 0)$,

$$\Psi = \begin{bmatrix} Z_{1,1}s_1^2 & Z_{1,2}s_2^2 & \cdots & Z_{1,6}s_6^2 \end{bmatrix}, \tag{A.40}$$

$$V = \begin{bmatrix} Z_{1,1} & Z_{1,2} & \cdots & Z_{1,6} \\ Z_{1,1}s_1 & Z_{1,2}s_2 & \cdots & Z_{1,6}s_6 \end{bmatrix}; \tag{A.41}$$

2) in the case of $(m_1, m_2) = (4, 1)$,

$$\Psi = \begin{bmatrix} Z_{1,1}s_1^2 & Z_{1,2}s_2^2 & Z_{1,3}s_3^2 & Z_{1,4}s_4^2 & V_a^2 & V_b^2 \end{bmatrix},$$

$$V = \begin{bmatrix} Z_{1,1} & Z_{1,2} & Z_{1,3} & Z_{1,4} & V_a^0 & V_b^0 \\ Z_{1,1}s_1 & Z_{1,2}s_2 & Z_{1,3}s_3 & Z_{1,4}s_4 & V_a^1 & V_b^1 \end{bmatrix},$$

with

$$\begin{cases} V_a^0 = Z_2^a,\ V_b^0 = Z_2^b \\ V_a^{j+1} = \alpha_1 V_a^j - \beta_1 V_b^j \\ V_b^{j+1} = \beta_1 V_a^j + \alpha_1 V_b^j,\ j = 0, 1; \end{cases} \tag{A.42}$$

3) in the case of $(m_1, m_2) = (2, 2)$,

$$\Psi = \begin{bmatrix} Z_{1,1}s_1^2 & Z_{1,2}s_2^2 & V_{a_1}^2 & V_{b_1}^2 & V_{a_2}^2 & V_{b_2}^2 \end{bmatrix},$$

$$V = \begin{bmatrix} Z_{1,1} & Z_{1,2} & V_{a_1}^0 & V_{b_1}^0 & V_{a_2}^0 & V_{b_2}^0 \\ Z_{1,1}s_1 & Z_{1,2}s_2 & V_{a_1}^1 & V_{b_1}^1 & V_{a_2}^1 & V_{b_2}^1 \end{bmatrix},$$

with

$$\begin{cases} V_{a_i}^0 = Z_{2,i}^a,\ V_{b_i}^0 = Z_{2,i}^b \\ V_{a_i}^{j+1} = \alpha_i V_{a_i}^j - \beta_i V_{b_i}^j \\ V_{b_i}^{j+1} = \beta_i V_{a_i}^j + \alpha_i V_{b_i}^j,\ i = 1, 2,\ j = 0, 1, \end{cases} \tag{A.43}$$

where in the above three cases the involved $Z_{1,k}$, $k = 1 \sim 6$, and $Z_{2,i}^a$, $Z_{2,i}^b$, $i = 1, 2$, are arbitrary parameter vectors which make the corresponding matrix V nonsingular.

A.3 Special Cases: Single-input Systems

Now let us consider the case of single-input systems in this section.

It is known that, under the requirement of the distinctness of the closed-loop poles, the solution to the eigenvalue assignment for single-input systems is unique regardless of the parameter matrix Z. Therefore, in this section, the matrix F is assumed to possess distinct eigenvalues, and meanwhile, we can set the parameter matrix Z to be a proper fixed one. Besides, the solution to Problem A.2 is turned into a parameter vector $a_{0\sim n-1}$ with a_i, $i = 0 \sim n-1$, being scalars.

According to Theorem A.1, the following result can be obtained.

Theorem A.2 *Let $r = 1$ and $F \in \mathbb{R}^{n\times n}$ be an arbitrarily chosen Hurwitz matrix in the form of (A.8) with distinct eigenvalues. Then, the solution to Problem A.2 is given by*

$$a_{0\sim n-1} = -\begin{bmatrix} \Psi_{\mathrm{R}} & \Psi_{\mathrm{I}} \end{bmatrix}\begin{bmatrix} V_{\mathrm{R}} & V_{\mathrm{I}} \end{bmatrix}^{-1}, \tag{A.44}$$

with

$$\Psi_{\mathrm{R}} = \begin{bmatrix} s_1^n & \cdots & s_{m_1}^n \end{bmatrix}, \tag{A.45}$$

$$\Psi_{\mathrm{I}} = \begin{bmatrix} v_{a_1}^n & v_{b_1}^n & \cdots & v_{a_{m_2}}^n & v_{b_{m_2}}^n \end{bmatrix}, \tag{A.46}$$

$$V_{\mathrm{R}} = \begin{bmatrix} 1 & \cdots & 1 \\ s_1 & \cdots & s_{m_1} \\ \vdots & \vdots & \vdots \\ s_1^{n-1} & \cdots & s_{m_1}^{n-1} \end{bmatrix}, \tag{A.47}$$

$$V_{\mathrm{I}} = \begin{bmatrix} v_{a_1}^0 & v_{b_1}^0 & \cdots & v_{a_{m_2}}^0 & v_{b_{m_2}}^0 \\ v_{a_1}^1 & v_{b_1}^1 & \cdots & v_{a_{m_2}}^1 & v_{b_{m_2}}^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{a_1}^{n-1} & v_{b_1}^{n-1} & \cdots & v_{a_{m_2}}^{n-1} & v_{b_{m_2}}^{n-1} \end{bmatrix}, \tag{A.48}$$

where $v_{a_i}^k$ and $v_{b_i}^k$, $i = 1 \sim m_2$, $k = 0 \sim n$, are given as follows:

$$\begin{aligned} \begin{bmatrix} v_{a_i}^0 & v_{b_i}^0 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \end{bmatrix}, \\ \begin{bmatrix} v_{a_i}^1 & v_{b_i}^1 \end{bmatrix} &= \begin{bmatrix} \alpha_i v_{a_i}^0 - \beta_i v_{b_i}^0 & \beta_i v_{a_i}^0 + \alpha_i v_{b_i}^0 \end{bmatrix}, \\ &\vdots \\ \begin{bmatrix} v_{a_i}^n & v_{b_i}^n \end{bmatrix} &= \begin{bmatrix} \alpha_i v_{a_i}^{n-1} - \beta_i v_{b_i}^{n-1} & \beta_i v_{a_i}^{n-1} + \alpha_i v_{b_i}^{n-1} \end{bmatrix}, \\ & \quad i = 1 \sim m_2. \end{aligned} \tag{A.49}$$

Proof. *Simply choosing*

$$\begin{aligned} Z_{1,k} &= 1,\ k = 1 \sim m_1, \\ Z_{2,i}^a &= Z_{2,i}^b = 1,\ i = 1 \sim m_2, \end{aligned}$$

in Theorem A.1 immediately gives equations (A.45)-(A.49). Furthermore, it can be also easily proven that in such a case the matrix V is nonsingular. ■

According to the number of self-conjugate poles, three specific cases, that is, $m_2 = 0, 1, 2$ in (A.8) are considered in the following. Based on Theorem A.2, the following corollary can be immediately derived.

Corollary A.4 *Let $r = 1$, and $F \in \mathbb{R}^{n\times n}$ be an arbitrarily chosen Hurwitz matrix in the form of (A.8) and with distinct eigenvalues. Then the solution to Problem A.2 is given by (A.44), where the matrices V and Ψ are:*

1) in the case of $m_2 = 0$,

$$\Psi = \left[\begin{array}{cccc} s_1^{m_1} & s_2^{m_1} & \cdots & s_{m_1}^{m_1} \end{array} \right], \tag{A.50}$$

$$V = \left[\begin{array}{cccc} 1 & 1 & \cdots & 1 \\ s_1 & s_2 & \cdots & s_{m_1} \\ \vdots & \vdots & \vdots & \vdots \\ s_1^{m_1-1} & s_2^{m_1-1} & \cdots & s_{m_1}^{m_1-1} \end{array} \right]; \tag{A.51}$$

2) in the case of $m_2 = 1$,

$$\Psi = \left[\begin{array}{ccccc} s_1^{m_1+2} & \cdots & s_{m_1}^{m_1+2} & v_a^{m_1+2} & v_b^{m_1+2} \end{array} \right], \tag{A.52}$$

$$V = \left[\begin{array}{ccccc} 1 & \cdots & 1 & v_a^0 & v_b^0 \\ s_1 & \cdots & s_{m_1} & v_a^1 & v_b^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ s_1^{m_1+1} & \cdots & s_{m_1}^{m_1+1} & v_a^{m_1+1} & v_b^{m_1+1} \end{array} \right], \tag{A.53}$$

with

$$\left\{ \begin{array}{l} v_a^0 = v_b^0 = 1 \\ v_a^{i+1} = \alpha_1 v_a^i - \beta_1 v_b^i \\ v_b^{i+1} = \beta_1 v_a^i + \alpha_1 v_b^i, \ i = 0 \sim m_1 + 1; \end{array} \right.$$

3) in the case of $m_2 = 2$,

$$\left\{ \begin{array}{l} \Psi = \left[\begin{array}{cc} \Psi^{\mathrm{R}} & \Psi^{\mathrm{I}} \end{array} \right] \\ \Psi^{\mathrm{R}} = \left[\begin{array}{ccc} s_1^{m_1+4} & \cdots & s_{m_1}^{m_1+4} \end{array} \right] \\ \Psi^{\mathrm{I}} = \left[\begin{array}{cccc} v_{a_1}^{m_1+4} & v_{b_1}^{m_1+4} & v_{a_2}^{m_1+4} & v_{b_2}^{m_1+4} \end{array} \right], \end{array} \right. \tag{A.54}$$

and

$$\left\{ \begin{array}{l} V = \left[\begin{array}{cc} V^{\mathrm{R}} & V^{\mathrm{I}} \end{array} \right] \\ V^{\mathrm{R}} = \left[\begin{array}{ccc} 1 & \cdots & 1 \\ s_1 & \cdots & s_{m_1} \\ \vdots & \vdots & \vdots \\ s_1^{m_1+3} & \cdots & s_{m_1}^{m_1+3} \end{array} \right] \\ V^{\mathrm{I}} = \left[\begin{array}{cccc} v_{a_1}^0 & v_{b_1}^0 & v_{a_2}^0 & v_{b_2}^0 \\ v_{a_1}^1 & v_{b_1}^1 & v_{a_2}^1 & v_{b_2}^1 \\ \vdots & \vdots & \vdots & \vdots \\ v_{a_1}^{m_1+3} & v_{b_1}^{m_1+3} & v_{a_2}^{m_1+3} & v_{b_2}^{m_1+3} \end{array} \right], \end{array} \right. \tag{A.55}$$

with

$$\left\{ \begin{array}{l} v_{a_i}^0 = v_{b_i}^0 = 1 \\ v_{a_i}^{j+1} = \alpha_i v_{a_i}^j - \beta_i v_{b_i}^j \\ v_{b_i}^{j+1} = \beta_i v_{a_i}^j + \alpha_i v_{b_i}^j, i = 1, 2, \ j = 0 \sim m_1 + 3. \end{array} \right.$$

A.4 Illustrative Examples

Example A.1 Consider the case of $r = n = 2$. The problem is to find the matrix $A_{0\sim1}$ to make the system in the form of (A.25) possess the following poles

$$\{-1,\ -2,\ -3 \pm \mathrm{i}\}. \tag{A.56}$$

Due to requirement (A.56), we clearly have $s_1 = -1$, $s_2 = -2$, $\alpha_1 = -3$, $\beta_1 = -1$, and the matrix F is given by

$$F = \mathrm{diag}\left(-1, -2, \begin{bmatrix} -3 & -1 \\ 1 & -3 \end{bmatrix}\right). \tag{A.57}$$

According to Corollary A.1, we have

$$\begin{aligned} \Psi &= \begin{bmatrix} Z_{1,1} & 4Z_{1,2} & 8Z_2^a - 6Z_2^b & 6Z_2^a + 8Z_2^b \end{bmatrix}, \\ V &= \begin{bmatrix} Z_{1,1} & Z_{1,2} & Z_2^a & Z_2^b \\ -Z_{1,1} & -2Z_{1,2} & -3Z_2^a + Z_2^b & -Z_2^a - 3Z_2^b \end{bmatrix}. \end{aligned}$$

When the following parameters

$$Z_{1,1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ Z_{1,2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{A.58}$$

$$Z_2^a = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ Z_2^b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{A.59}$$

are chosen, it is easy to verify that $\det V = 3 \neq 0$, thus it follows from (A.11) that the parameter matrix is obtained as

$$A_{0\sim1} = \frac{1}{3}\begin{bmatrix} 10 & 10 & 13 & 5 \\ -2 & 16 & -2 & 14 \end{bmatrix}. \tag{A.60}$$

Example A.2 In Duan (2021b), a robot system with an elastic joint is transformed into the following fifth-order fully-actuated quasi-linear system:

$$J_2 J_1 q_1^{(5)} + \sum_{i=0}^{4} l_i\left(q_1^{(1\sim4)}\right) q_1^{(i)} - k\varphi(\tau) = ku, \tag{A.61}$$

where q_1 is the angle of one joint, J_1 and J_2 are the moments of inertia of joints, k is the spring stiffness, τ is the output torque of the motor, u is the voltage input of the motor, $\varphi(\tau)$ is the piecewise continuous function, and $l_i\left(q_1^{(1\sim4)}\right)$, $i = 0 \sim 4$, are continuous functions given in Duan (2021b).

The control of the system is then converted into one of finding the vector $a_{0\sim4}$ to make the following system

$$q_1^{(5)} + a_{0\sim4} q_1^{(0\sim4)} = 0 \tag{A.62}$$

stable. Particularly, the desired closed-loop poles are chosen to be

$$\{-1,\ -2 \pm \mathrm{i},\ -3 \pm \mathrm{i}\}. \tag{A.63}$$

This gives $s_1 = -1$, $\alpha_1 = -2$, $\beta_1 = -1$, $\alpha_2 = -3$, $\beta_2 = -1$, and the corresponding matrix F is given by

$$F = \mathrm{diag}\left(-1, \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}, \begin{bmatrix} -3 & -1 \\ 1 & -3 \end{bmatrix}\right). \tag{A.64}$$

According to Corollary A.4, we can obtain

$$\Psi = \begin{bmatrix} -1 & 79 & -3 & 328 & -304 \end{bmatrix}, \tag{A.65}$$

$$V = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -3 & -2 & -4 \\ 1 & -1 & 7 & 2 & 14 \\ -1 & 9 & -13 & 8 & -44 \\ 1 & -31 & 17 & -68 & 124 \end{bmatrix}, \tag{A.66}$$

and it can be verified that $\det V = 200 \neq 0$. Hence the parameter vector is computed as

$$\begin{aligned} a_{0\sim 4} &= -\Psi V^{-1} \\ &= \begin{bmatrix} 50 & 120 & 109 & 49 & 11 \end{bmatrix}. \end{aligned} \tag{A.67}$$

It can be easily seen from these two examples that the formulas presented in this appendix can be indeed "directly" used in the computation of the gain matrix $A_{0\sim n-1}$ or the parameter vector $a_{0\sim n-1}$ when applying the FAS approach.

Finally, let us give a brief summarization. Due to the great superiority of the FAS approach, a constant linear closed-loop system can always be obtained with a desired eigenstructure for control of many types of systems, including but not limited to all controllable linear systems, strict-feedback nonlinear systems, all feedback linearizable systems and nonlinear systems in a kind of controllable canonical forms. Hence, the control problem of a dynamical system via the FAS approach can be often finally converted into a fundamental problem, that is, the ESA problem of the specific matrix pair $(\Phi(0_{0\sim n-1}), B_c)$. Due to the particular importance of this fundamental problem, this appendix represents a general and complete parametric solution in an explicit form Duan et al. (2024). A few special cases which are commonly encountered in practical applications are also treated, and ready-to-use results are presented for the convenience of the designers. Once a control problem is turned into the fundamental problem through the FAS approach, one can simply find the corresponding solution to use in one's design.

Appendix B

Proofs of Some Theorems

B.1 Proof of Theorem 2.3

Define two matrix operators as

$$\begin{cases} \Pi[H(t)] \triangleq A(t)H(t) - \frac{\mathrm{d}}{\mathrm{d}t}H(t) \\ \Xi[H(t)] \triangleq H(t)A(t) + \frac{\mathrm{d}}{\mathrm{d}t}H(t), \end{cases} \tag{B.1}$$

with $H(t)$ being any analytic matrix, and denote

$$\begin{aligned} \Pi^n[H(t)] &\triangleq \underbrace{\Pi\Pi\cdots\Pi}_{\text{n times}}[H(t)], \\ \Xi^n[H(t)] &\triangleq \underbrace{\Xi\Xi\cdots\Xi}_{\text{n times}}[H(t)],\ n \geq 1, \end{aligned}$$

and by default,

$$\Xi^0[H(t)] = \Pi^0[H(t)] = H(t).$$

With the above notations, we have the following proposition (Mutoh and Nakada (2009)).

Proposition B.1 *For the two matrix operators defined by (B.1), the following relations hold:*

$$\begin{aligned} \Xi^k\left[\bar{q}_{i,\mu_i-1}^{\mathrm{T}}\right]b_j &= \bar{q}_{i,\mu_i-1}^{\mathrm{T}}\Pi^k\left[b_j\right], \\ i,j &= 1,2,\ldots,r,\ k = 0,1,\ldots,\mu_i - 1. \end{aligned} \tag{B.2}$$

With the above proposition, we can further give the following result.

Proposition B.2 *Let Assumption 2.5 be satisfied, and*

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r \geq 1.$$

Then

$$\bar{q}_{i,\mu_i-1}^{\mathrm{T}}\Pi^k[b_i]=\Xi^k\left[\bar{q}_{i,\mu_i-1}^{\mathrm{T}}\right]b_i=0,\ k=0,1,\ldots,\mu_i-2, \tag{B.3}$$

and

$$\bar{q}_{i,\mu_i-1}^{\mathrm{T}}\Pi^{\mu_i-1}\left[b_j\right]=\Xi^{\mu_i-1}\left[\bar{q}_{i,\mu_i-1}^{\mathrm{T}}\right]b_j=\begin{cases}1, & i=j\\ 0, & i>j\\ X_{ij}, & i<j,\end{cases} \tag{B.4}$$

where X_{ij} is a uniquely specified function of time.

Proof. Noticing the expressions of $\bar{Q}(t)$ and $\bar{Q}^{-1}(t)$, we have

$$\begin{aligned}&\bar{Q}^{-1}(t)\bar{Q}(t)\\ &=\begin{bmatrix}\bar{q}_{10}^{\mathrm{T}}\\ \vdots\\ \bar{q}_{1,\mu_1-1}^{\mathrm{T}}\\ \vdots\\ \bar{q}_{r0}^{\mathrm{T}}\\ \vdots\\ \bar{q}_{r,\mu_r-1}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\bar{q}_1^1 & \cdots & \bar{q}_{\mu_1}^1 & \cdots & \bar{q}_1^r & \cdots & \bar{q}_{\mu_r}^r\end{bmatrix}\\ &=\begin{bmatrix}\bar{q}_{10}^{\mathrm{T}}\bar{q}_1^1 & \cdots & \bar{q}_{10}^{\mathrm{T}}\bar{q}_{\mu_1}^1 & \cdots & \bar{q}_{10}^{\mathrm{T}}\bar{q}_1^r & \cdots & \bar{q}_{10}^{\mathrm{T}}\bar{q}_{\mu_r}^r\\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots\\ \bar{q}_{1,\mu_1-1}^{\mathrm{T}}\bar{q}_1^1 & \cdots & \bar{q}_{1,\mu_1-1}^{\mathrm{T}}\bar{q}_{\mu_1}^1 & \cdots & \bar{q}_{1,\mu_1-1}^{\mathrm{T}}\bar{q}_1^r & \cdots & \bar{q}_{1,\mu_1-1}^{\mathrm{T}}\bar{q}_{\mu_r}^r\\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots\\ \bar{q}_{r0}^{\mathrm{T}}\bar{q}_1^1 & \cdots & \bar{q}_{r0}^{\mathrm{T}}\bar{q}_{\mu_1}^1 & \cdots & \bar{q}_{r0}^{\mathrm{T}}\bar{q}_1^r & \cdots & \bar{q}_{r0}^{\mathrm{T}}\bar{q}_{\mu_r}^r\\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots\\ \bar{q}_{r,\mu_r-1}^{\mathrm{T}}\bar{q}_1^1 & \cdots & \bar{q}_{r,\mu_r-1}^{\mathrm{T}}\bar{q}_{\mu_1}^1 & \cdots & \bar{q}_{r,\mu_r-1}^{\mathrm{T}}\bar{q}_1^r & \cdots & \bar{q}_{r,\mu_r-1}^{\mathrm{T}}\bar{q}_{\mu_r}^r\end{bmatrix}.\end{aligned}$$

Therefore,

$$\begin{cases}\bar{q}_{i,j-1}^{\mathrm{T}}\bar{q}_j^i=1,\ j=1,2,\cdots,\mu_i,\ i=1,2,\cdots,r\\ \bar{q}_{i,k}^{\mathrm{T}}\bar{q}_j^l=0,\ i\neq l,\text{or}\ k\neq j-1.\end{cases} \tag{B.5}$$

Further, recall

$$\begin{cases}\bar{q}_1^i=b_i\\ \bar{q}_2^i=A(t)\bar{q}_1^i-\dot{\bar{q}}_1^i=\Pi[\bar{q}_1^i]=\Pi[b_i]\\ \bar{q}_3^i=A(t)\bar{q}_2^i-\dot{\bar{q}}_2^i=\Pi[\bar{q}_2^i]=\Pi^2[b_i]\\ \quad\vdots\\ \bar{q}_{k+1}^i=\Pi^k[b_i].\end{cases}$$

Substituting these into (B.5), and using Proposition B.1, finally produce (B.3) and (B.4). ■

Based on the above, we now provide the detail proof of Theorem 2.3.

For each $i = 1, 2, \ldots, r$, we define

$$z_i = \bar{q}_{i,\mu_i-1}^{\mathrm{T}} x.$$

Taking derivative of the above equation and using the operators given by (B.1), yields

$$\begin{aligned} \dot{z}_i &= \left(\dot{\bar{q}}_{i,\mu_i-1}^{\mathrm{T}} + \bar{q}_{i,\mu_i-1}^{\mathrm{T}} A(t)\right) x + \bar{q}_{i,\mu_i-1}^{\mathrm{T}} B(t) u \\ &= \Xi\left[\bar{q}_{i,\mu_i-1}^{\mathrm{T}}\right] x + \bar{q}_{i,\mu_i-1}^{\mathrm{T}} \Pi^0 [B(t)] u. \end{aligned} \tag{B.6}$$

Further, substituting (B.3) into (B.6) gives

$$\dot{z}_i = \Xi\left[\bar{q}_{i,\mu_i-1}^{\mathrm{T}}\right] x. \tag{B.7}$$

Then, calculating higher order derivatives until the μ_i-th order gives

$$\left\{ \begin{array}{l} z_i^{(2)} = \Xi^2\left[\bar{q}_{i,\mu_i-1}^{\mathrm{T}}\right] x \\ z_i^{(3)} = \Xi^3\left[\bar{q}_{i,\mu_i-1}^{\mathrm{T}}\right] x \\ \quad \vdots \\ z_i^{(\mu_i)} = \Xi^{\mu_i}\left[\bar{q}_{i,\mu_i-1}^{\mathrm{T}}\right] x + u_i + \sum_{j=i+1}^{r} X_{ij} u_j. \end{array} \right. \tag{B.8}$$

Additionally, leveraging the definition of the controllability indices, it is not hard to see that the row vector $\Xi^{\mu_i}\left[\bar{q}_{i,\mu_i-1}^{\mathrm{T}}\right]$ is a linear combination of the rows of $P(t)$ given by (2.114). That is, there exist the linear coefficients $l_{jk}^{i}(t)$, $i, j = 1, 2, \ldots, r$, $k = 0, 1, \ldots, \mu_i - 1$, such that

$$\Xi^{\mu_i}\left[\bar{q}_{i,\mu_i-1}^{\mathrm{T}}\right] = \sum_{j=1}^{r} \sum_{k=0}^{\mu_j-1} l_{jk}^{i}(t)\, \Xi^k\left[\bar{q}_{i,\mu_i-1}^{\mathrm{T}}\right].$$

In the matrix form of (B.7)-(B.8), one can obtain

$$z_k^{(\mu_k)}\Big|_{k=1\sim r} = L(t)\, z_k^{(0\sim\mu_k-1)}\Big|_{k=1\sim r} + B_z(t) u, \tag{B.9}$$

where $z_k^{(0\sim\mu_k-1)}\Big|_{k=1\sim r} = P(t) x$, and

$$L(t) = \begin{bmatrix} l_{10}^1 & \cdots & l_{1,\mu_1-1}^1 & \cdots & l_{r0}^1 & \cdots & l_{r,\mu_r-1}^1 \\ l_{10}^2 & \cdots & l_{1,\mu_1-1}^2 & \cdots & l_{r0}^2 & \cdots & l_{r,\mu_r-1}^2 \\ \vdots & & \vdots & & \vdots & \cdots & \vdots \\ l_{10}^r & \cdots & l_{1,\mu_1-1}^r & \cdots & l_{r0}^r & \cdots & l_{r,\mu_r-1}^r \end{bmatrix}, \tag{B.10}$$

$$B_z(t) = \begin{bmatrix} 1 & X_{12} & \cdots & X_{1r} \\ & 1 & \cdots & X_{2r} \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix}. \tag{B.11}$$

Note that $B_z(t)$ is commonly an upper-triangular matrix with all elements on the diagonal being one and clearly nonsingular for $\forall t \geq t_0$. Obviously, there exists a number $\eta \leq r$ such that system (B.9) is in the form of (2.117) with $L(t) \in \mathbb{R}^{r\times n}$ and $B_z(t) \in \mathbb{R}^{r\times r}$ given by (B.10) and (B.11), respectively. The proof is then completed.

B.2 Theorems in Chapter 5

B.2.1 Proof of Theorem 5.1

Substituting control law (5.10) into system (5.2), gives the following closed-loop system:

$$x^{(n)} + A_{0\sim n-1}x^{(0\sim n-1)} = \phi(x^{(0\sim n-1)}), \tag{B.12}$$

where

$$\phi(x^{(0\sim n-1)}) = -\frac{1}{4\varepsilon}\rho^2(x^{(0\sim n-1)})P_L^{\mathrm{T}}(A_{0\sim n-1})x^{(0\sim n-1)} + \Delta f(x^{(0\sim n-1)}). \tag{B.13}$$

The closed-loop system (B.12) can be written in the following state-space form:

$$\dot{x}^{(0\sim n-1)} = \Phi(A_{0\sim n-1})x^{(0\sim n-1)} + \begin{bmatrix} 0_{(n-1)r} \\ \phi(x^{(0\sim n-1)}) \end{bmatrix}. \tag{B.14}$$

Since $A_i \in \mathbb{R}^{r\times r}$, $i = 0, 1, \ldots, n-1$, satisfy condition (5.5), there exists a positive definite matrix $P(A_{0\sim n-1})$ satisfying (5.6). Then the following Lyapunov function can be chosen for system (B.14):

$$V = \frac{1}{2}\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P(A_{0\sim n-1})x^{(0\sim n-1)}.$$

In view of (5.6), (5.8) and (B.14), we have

$$\begin{aligned}
\dot{V} &= \frac{1}{2}\left(\dot{x}^{(0\sim n-1)}\right)^{\mathrm{T}} Px^{(0\sim n-1)} + \frac{1}{2}\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P\dot{x}^{(0\sim n-1)} \\
&= \frac{1}{2}\left(\Phi x^{(0\sim n-1)} + \begin{bmatrix} 0_{(n-1)r} \\ \phi(x^{(0\sim n-1)}) \end{bmatrix}\right)^{\mathrm{T}} Px^{(0\sim n-1)} \\
&\quad + \frac{1}{2}\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P\left(\Phi x^{(0\sim n-1)} + \begin{bmatrix} 0_{(n-1)r} \\ \phi(x^{(0\sim n-1)}) \end{bmatrix}\right) \\
&= \frac{1}{2}\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} \left(\Phi^{\mathrm{T}}P + P\Phi\right)x^{(0\sim n-1)} + \left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P\begin{bmatrix} 0_{(n-1)r} \\ \phi(x^{(0\sim n-1)}) \end{bmatrix} \\
&\leq -\frac{\mu}{2}\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} Px^{(0\sim n-1)} + \left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L\phi(x^{(0\sim n-1)}) \\
&= -\mu V + \left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L\phi(x^{(0\sim n-1)}).
\end{aligned}$$

Next, let us consider the last term of the above equation. It follows from Assumption 5.2, Lemma 5.3 and equation (B.13) that

$$\begin{aligned}
& \left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L \phi(x^{(0\sim n-1)}) \\
= & -\frac{\rho^2(x^{(0\sim n-1)})}{4\varepsilon}\left[\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L P_L^{\mathrm{T}} x^{(0\sim n-1)}\right] \\
& +\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L \Delta f(x^{(0\sim n-1)}) \\
= & -\frac{\rho^2(x^{(0\sim n-1)})}{4\varepsilon}\left\| P_L^{\mathrm{T}} x^{(0\sim n-1)}\right\|^2 + \left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L \Delta f(x^{(0\sim n-1)}) \\
\leq & -\frac{\rho^2(x^{(0\sim n-1)})}{4\varepsilon}\left\| P_L^{\mathrm{T}} x^{(0\sim n-1)}\right\|^2 + \left\|\Delta f(x^{(0\sim n-1)})\right\| \left\| P_L^{\mathrm{T}} x^{(0\sim n-1)}\right\| \\
\leq & -\frac{\rho^2(x^{(0\sim n-1)})}{4\varepsilon}\left\| P_L^{\mathrm{T}} x^{(0\sim n-1)}\right\|^2 + \rho(x^{(0\sim n-1)}) \left\| P_L^{\mathrm{T}} x^{(0\sim n-1)}\right\| \\
\leq & \ \varepsilon.
\end{aligned}$$

Combining the above two equations, gives

$$\dot{V} \leq -\mu V + \varepsilon. \tag{B.15}$$

It thus follows from the Comparison Theorem that

$$V \leq V(0)e^{-\mu t} + \frac{\varepsilon}{\mu}\left(1 - e^{-\mu t}\right),$$

which gives

$$V \leq \left(V(0) - \frac{\varepsilon}{\mu}\right)e^{-\mu t} + \frac{\varepsilon}{\mu} \to \frac{\varepsilon}{\mu}, \quad t \to \infty.$$

Thus the state $x^{(0\sim n-1)}$ eventually converges into the ellipsoid $\Theta_{\mu,\varepsilon}(0)$. The proof is then completed.

B.2.2 Proof of Theorem 5.3

Substituting control law (5.31) into (5.27), gives a closed-loop system composed of

$$\dot{\hat{\theta}} = H(x^{(0\sim n-1)}) P_L^{\mathrm{T}}(A_{0\sim n-1}) x^{(0\sim n-1)},$$

and

$$x^{(n)} + A_{0\sim n-1} x^{(0\sim n-1)} = H^{\mathrm{T}}(x^{(0\sim n-1)})\tilde{\theta}, \tag{B.16}$$

where

$$\tilde{\theta} = \theta - \hat{\theta}. \tag{B.17}$$

The part of closed-loop system (B.16) can be expressed in the following state-space form:

$$\dot{x}^{(0\sim n-1)} = \Phi(A_{0\sim n-1}) x^{(0\sim n-1)} + \begin{bmatrix} 0_{(n-1)r} \\ H^{\mathrm{T}}(x^{(0\sim n-1)})\tilde{\theta} \end{bmatrix}. \tag{B.18}$$

Since $A_i \in \mathbb{R}^{r\times r}$, $i = 0, 1, \ldots, n-1$, are a set of matrices such that $\Phi(A^{0\sim n-1})$ is stable, there exists a positive definite matrix $P(A_{0\sim n-1})$ satisfying (5.28). Therefore, the following Lyapunov function can be selected for system (B.18):

$$V = \left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P(A_{0\sim n-1}) x^{(0\sim n-1)} + \tilde{\theta}^{\mathrm{T}}\tilde{\theta}.$$

In view of (5.28) and the first equation in (5.31), and also

$$\dot{\tilde{\theta}} = \dot{\theta} - \dot{\hat{\theta}} = -\dot{\hat{\theta}},$$

we have

$$\begin{aligned}
\dot{V} &= \left(\dot{x}^{(0\sim n-1)}\right)^{\mathrm{T}} P x^{(0\sim n-1)} + \left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P \dot{x}^{(0\sim n-1)} + \dot{\tilde{\theta}}^{\mathrm{T}}\tilde{\theta} + \tilde{\theta}^{\mathrm{T}}\dot{\tilde{\theta}} \\
&= \left(\Phi x^{(0\sim n-1)} + \begin{bmatrix} 0_{(n-1)r} \\ H^{\mathrm{T}}(x^{(0\sim n-1)})\tilde{\theta} \end{bmatrix}\right)^{\mathrm{T}} P x^{(0\sim n-1)} \\
&\quad + \left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P \left(\Phi x^{(0\sim n-1)} + \begin{bmatrix} 0_{(n-1)r} \\ H^{\mathrm{T}}(x^{(0\sim n-1)})\tilde{\theta} \end{bmatrix}\right) - 2\dot{\hat{\theta}}^{\mathrm{T}}\tilde{\theta} \\
&= \left(x^{(0\sim n-1)}\right)^{\mathrm{T}} \left(\Phi^{\mathrm{T}} P + P\Phi\right) x^{(0\sim n-1)} - 2\dot{\hat{\theta}}^{\mathrm{T}}\tilde{\theta} \\
&\quad + 2\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P(A^{0\sim n-1}) \begin{bmatrix} 0_{(n-1)r} \\ H^{\mathrm{T}}(x^{(0\sim n-1)})\tilde{\theta} \end{bmatrix} \\
&\leq -\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} x^{(0\sim n-1)} + 2\left[\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L(A^{0\sim n-1}) H^{\mathrm{T}}(x^{(0\sim n-1)}) - \dot{\hat{\theta}}^{\mathrm{T}}\right]\tilde{\theta} \\
&= -\left\|x^{(0\sim n-1)}\right\|^2.
\end{aligned}$$

Thus the conclusion of the theorem holds following from the Lyapunov Theorem.

B.2.3 Proof of Theorem 5.5

The proof is composed of four steps.

Step 1. The Lyapunov function

Substituting control law (5.54)-(5.57) into system (5.50), gives part of the closed-loop system as:

$$x^{(n)} + A_{0\sim n-1} x^{(0\sim n-1)} = \phi_r(x^{(0\sim n-1)}) + \phi_a(x^{(0\sim n-1)}), \tag{B.19}$$

where

$$\phi_r(x^{(0\sim n-1)}) = \Delta f(x^{(0\sim n-1)}) - \frac{1}{4\varepsilon}\rho^2(x^{(0\sim n-1)}) P_L^{\mathrm{T}}(A_{0\sim n-1}) x^{(0\sim n-1)}, \tag{B.20}$$

$$\phi_a(x^{(0\sim n-1)}) = H^{\mathrm{T}}(x^{(0\sim n-1)})\tilde{\theta}. \tag{B.21}$$

The closed-loop system (B.19) can be expressed in the following state-space form:

$$\dot{x}^{(0\sim n-1)} = \Phi(A_{0\sim n-1})x^{(0\sim n-1)} + \begin{bmatrix} 0_{(n-1)r} \\ \phi_r + \phi_a \end{bmatrix}. \tag{B.22}$$

When $A_i \in \mathbb{R}^{r\times r}$, $i = 0, 1, \ldots, n-1$, satisfy condition (5.52), we know that there exists a positive definite matrix $P = P(A_{0\sim n-1})$ satisfying equation (5.53). Thus, the following Lyapunov function can be chosen for system (B.22):

$$V = \frac{1}{2}(x^{(0\sim n-1)})^{\mathrm{T}} P x^{(0\sim n-1)} + \frac{1}{2}\tilde{\theta}^{\mathrm{T}}\tilde{\theta}. \tag{B.23}$$

It follows from (5.52) that equation (5.53) holds. Then, in view of (5.53) and the above equations (B.22)-(B.23), we have

$$\begin{aligned}
\dot{V} &= \frac{1}{2}\left(\dot{x}^{(0\sim n-1)}\right)^{\mathrm{T}} P x^{(0\sim n-1)} + \frac{1}{2}\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P \dot{x}^{(0\sim n-1)} + \frac{1}{2}\dot{\tilde{\theta}}^{\mathrm{T}}\tilde{\theta} + \frac{1}{2}\tilde{\theta}^{\mathrm{T}}\dot{\tilde{\theta}} \\
&= \frac{1}{2}\left(\Phi x^{(0\sim n-1)} + \begin{bmatrix} 0_{(n-1)r} \\ \phi_r + \phi_a \end{bmatrix}\right)^{\mathrm{T}} P x^{(0\sim n-1)} \\
&\quad + \frac{1}{2}\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P \left(\Phi x^{(0\sim n-1)} + \begin{bmatrix} 0_{(n-1)r} \\ \phi_r + \phi_a \end{bmatrix}\right) + \dot{\tilde{\theta}}^{\mathrm{T}}\tilde{\theta} \\
&= \frac{1}{2}\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} \left(\Phi^{\mathrm{T}} P + P\Phi\right) x^{(0\sim n-1)} \\
&\quad + \left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P \begin{bmatrix} 0_{(n-1)r} \\ \phi_r + \phi_a \end{bmatrix} + \dot{\tilde{\theta}}^{\mathrm{T}}\tilde{\theta} \\
&\leq -\frac{\mu}{2}\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P x^{(0\sim n-1)} + \left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L\left(\phi_r + \phi_a\right) + \dot{\tilde{\theta}}^{\mathrm{T}}\tilde{\theta} \\
&= -V_0 + V_r + V_a,
\end{aligned} \tag{B.24}$$

where

$$V_0 = \frac{\mu}{2}\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P x^{(0\sim n-1)}, \tag{B.25}$$

$$V_r = \left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L \phi_r, \tag{B.26}$$

and

$$V_a = \left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L \phi_a + \dot{\tilde{\theta}}^{\mathrm{T}}\tilde{\theta}. \tag{B.27}$$

Step 2. Dealing with V_0 and V_r

Adding and subtracting $\frac{\mu}{2}\tilde{\theta}^{\mathrm{T}}\tilde{\theta}$ in the last step of the above equation (B.24), yields

$$\begin{aligned}
\dot{V} &\leq -V_0 - \frac{\mu}{2}\tilde{\theta}^{\mathrm{T}}\tilde{\theta} + V_r + V_a + \frac{\mu}{2}\tilde{\theta}^{\mathrm{T}}\tilde{\theta} \\
&= -\mu V + V_r + V_a + \frac{\mu}{2}\tilde{\theta}^{\mathrm{T}}\tilde{\theta}.
\end{aligned} \tag{B.28}$$

Using Assumption 5.2 and Lemma 5.3, we have

$$\begin{aligned}
V_r &= \left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L \phi_r \\
&= \left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L \left[\Delta f(x^{(0\sim n-1)}) - \frac{1}{4\varepsilon}\rho^2(x^{(0\sim n-1)}) P_L^{\mathrm{T}}(A_{0\sim n-1}) x^{(0\sim n-1)}\right] \\
&= -\frac{\rho^2(x^{(0\sim n-1)})}{4\varepsilon}\left[\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L P_L^{\mathrm{T}} x^{(0\sim n-1)}\right] + \left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L \Delta f \\
&= -\frac{\rho^2(x^{(0\sim n-1)})}{4\varepsilon}\left\| P_L^{\mathrm{T}} x^{(0\sim n-1)}\right\|^2 + \left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L \Delta f \\
&\le -\frac{\rho^2(x^{(0\sim n-1)})}{4\varepsilon}\left\| P_L^{\mathrm{T}} x^{(0\sim n-1)}\right\|^2 + \|\Delta f\| \left\| P_L^{\mathrm{T}} x^{(0\sim n-1)}\right\| \\
&\le -\frac{\rho^2(x^{(0\sim n-1)})}{4\varepsilon}\left\| P_L^{\mathrm{T}} x^{(0\sim n-1)}\right\|^2 + \rho(x^{(0\sim n-1)}) \left\| P_L^{\mathrm{T}} x^{(0\sim n-1)}\right\| \\
&\le \varepsilon. \qquad \text{(B.29)}
\end{aligned}$$

Step 3. Dealing with V_a

It follows from (B.27) that

$$\begin{aligned}
V_a + \frac{\mu}{2}\tilde{\theta}^{\mathrm{T}}\tilde{\theta} &= \left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L \phi_a + \dot{\tilde{\theta}}^{\mathrm{T}}\tilde{\theta} + \frac{\mu}{2}\tilde{\theta}^{\mathrm{T}}\tilde{\theta} \\
&= \left[\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L H^{\mathrm{T}} + \dot{\tilde{\theta}}^{\mathrm{T}}\right]\tilde{\theta} + \frac{\mu}{2}\tilde{\theta}^{\mathrm{T}}\tilde{\theta}. \qquad \text{(B.30)}
\end{aligned}$$

Further using the first conclusion in Lemma 5.4, gives

$$\xi \triangleq \tilde{\theta}_{r0} - \tilde{\theta}_{e0} - \tilde{\theta} = 0,$$

and

$$\begin{aligned}
& V_a + \frac{\mu}{2}\tilde{\theta}^{\mathrm{T}}\tilde{\theta} \\
= & \left[\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L H^{\mathrm{T}} + \dot{\tilde{\theta}}^{\mathrm{T}}\right]\tilde{\theta} + \frac{\mu}{2}\tilde{\theta}^{\mathrm{T}}\tilde{\theta} + (\mu+1)\xi^{\mathrm{T}}\tilde{\theta} \\
= & \left[\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L H^{\mathrm{T}} - \dot{\tilde{\theta}}_{e0}^{\mathrm{T}} + \dot{\tilde{\theta}}_{r0}^{\mathrm{T}}\right]\tilde{\theta} \\
& + (\mu+1)\left(-\tilde{\theta}_{e0}^{\mathrm{T}} + \tilde{\theta}_{r0}^{\mathrm{T}} - \tilde{\theta}^{\mathrm{T}}\right)\tilde{\theta} + \frac{\mu}{2}\tilde{\theta}^{\mathrm{T}}\tilde{\theta} \\
= & \left[\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L H^{\mathrm{T}} - \dot{\tilde{\theta}}_{e0}^{\mathrm{T}}\right]\tilde{\theta} + \dot{\tilde{\theta}}_{r0}^{\mathrm{T}}\tilde{\theta} \\
& - (\mu+1)\tilde{\theta}_{e0}^{\mathrm{T}}\tilde{\theta} + (\mu+1)\left(\tilde{\theta}_{r0}^{\mathrm{T}}\tilde{\theta} - \tilde{\theta}^{\mathrm{T}}\tilde{\theta}\right) + \frac{\mu}{2}\tilde{\theta}^{\mathrm{T}}\tilde{\theta} \\
= & \left[\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L H^{\mathrm{T}} - \dot{\tilde{\theta}}_{e0}^{\mathrm{T}} - (\mu+1)\tilde{\theta}_{e0}^{\mathrm{T}}\right]\tilde{\theta} \\
& + \dot{\tilde{\theta}}_{r0}^{\mathrm{T}}\tilde{\theta} + (\mu+1)\tilde{\theta}_{r0}^{\mathrm{T}}\tilde{\theta} - \frac{\mu+2}{2}\tilde{\theta}^{\mathrm{T}}\tilde{\theta}. \qquad \text{(B.31)}
\end{aligned}$$

Then, using the adaptive control law (5.57), and in view of (5.58) and the last two conclusions in Lemma 5.4, we obtain

$$\begin{aligned} & V_a + \frac{\mu}{2}\tilde{\theta}^{\mathrm{T}}\tilde{\theta} \\ = \;& \dot{\tilde{\theta}}_{r0}^{\mathrm{T}}\tilde{\theta} + (\mu+1)\,\tilde{\theta}_{r0}^{\mathrm{T}}\tilde{\theta} - \frac{\mu+2}{2}\left\|\tilde{\theta}\right\|^2 \\ \leq \;& \frac{1}{2}\left(\delta_1^2 + \left\|\tilde{\theta}\right\|^2\right) + \frac{(\mu+1)}{2}\left(\delta_0^2 + \left\|\tilde{\theta}\right\|^2\right) - \frac{\mu+2}{2}\left\|\tilde{\theta}\right\|^2 \\ = \;& \frac{1}{2}\left(\delta_1^2 + (\mu+1)\,\delta_0^2\right) \\ = \;& \delta. \end{aligned} \tag{B.32}$$

Step 4. Applying the Comparison Theorem

Combining (B.28), (B.29) and (B.32), gives

$$\dot{V} \leq -\mu V + \varepsilon + \delta. \tag{B.33}$$

It thus follows from the Comparison Theorem that

$$V \leq V(0)\,\mathrm{e}^{-\mu t} + \frac{\varepsilon+\delta}{\mu}\left(1-\mathrm{e}^{-\mu t}\right),$$

which implies

$$\begin{aligned} V \;\leq\; & \left(V(0) - \frac{\varepsilon+\delta}{\mu}\right)\mathrm{e}^{-\mu t} + \frac{\varepsilon+\delta}{\mu} \\ \to\; & \frac{\varepsilon+\delta}{\mu}, \quad t \to \infty. \end{aligned}$$

Thus $\begin{bmatrix} x^{(0\sim n-1)} \\ \tilde{\theta} \end{bmatrix}$ eventually converges into the ellipsoid $\Theta_{(\mu,\varepsilon,\delta)}(0)$. The proof is completed.

B.2.4 Proof of Theorem 5.7

For clarity, we divide the proof into three steps.

Step 1. The Lyapunov function

It can be easily observed that this step is exactly the same as the Step 1 in the proof of Theorem 5.5. We thus need only to start from (B.24)-(B.27).

Step 2. Dealing with V_r

Similar to the Step 2 in the proof of Theorem 5.5, using Assumption 5.4, we have

$$\begin{aligned}
V_r &= \left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L \phi_r \\
&= -\frac{\rho(x^{(0\sim n-1)})}{\varepsilon}\left[\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L P_L^{\mathrm{T}} x^{(0\sim n-1)}\right] + \left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L \Delta f \\
&= -\frac{\rho(x^{(0\sim n-1)})}{\varepsilon}\left\| P_L^{\mathrm{T}} x^{(0\sim n-1)}\right\|^2 + \left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L \Delta f \\
&\le -\frac{\rho(x^{(0\sim n-1)})}{\varepsilon}\left\| P_L^{\mathrm{T}} x^{(0\sim n-1)}\right\|^2 + \|\Delta f\|\left\| P_L^{\mathrm{T}} x^{(0\sim n-1)}\right\| \\
&\le -\frac{\rho(x^{(0\sim n-1)})}{\varepsilon}\left\| P_L^{\mathrm{T}} x^{(0\sim n-1)}\right\|^2 + \rho(x^{(0\sim n-1)})\left\| P_L^{\mathrm{T}} x^{(0\sim n-1)}\right\|^2 \\
&= \left(-\frac{1}{\varepsilon}+1\right)\rho(x^{(0\sim n-1)})\left\| P_L^{\mathrm{T}} x^{(0\sim n-1)}\right\|^2 \\
&= -\frac{1-\varepsilon}{\varepsilon}\rho(x^{(0\sim n-1)})\left\| P_L^{\mathrm{T}} x^{(0\sim n-1)}\right\|^2. \qquad \text{(B.34)}
\end{aligned}$$

Step 3. Dealing with V_a and V

Recalling that the parameter vector θ is constant, we have

$$\dot{\tilde{\theta}} = \dot{\theta} - \dot{\hat{\theta}} = -\dot{\hat{\theta}}.$$

It thus follows from (B.21) and (5.76) that

$$\begin{aligned}
V_a &= \left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L \phi_a + \dot{\tilde{\theta}}^{\mathrm{T}} \tilde{\theta} \\
&= \left[\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L H^{\mathrm{T}} + \dot{\tilde{\theta}}^{\mathrm{T}}\right]\tilde{\theta} \\
&= \left[\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L H^{\mathrm{T}} - \dot{\hat{\theta}}^{\mathrm{T}}\right]\tilde{\theta} \\
&= 0. \qquad \text{(B.35)}
\end{aligned}$$

Using (B.24), (B.25) and the above (B.34) and (B.35), we obtain

$$\begin{aligned}
\dot{V} &\le -\frac{\mu}{2}\left\| x^{(0\sim n-1)}\right\|_P - \frac{1-\varepsilon}{\varepsilon}\rho(x^{(0\sim n-1)})\left\| P_L^{\mathrm{T}} x^{(0\sim n-1)}\right\|^2 \\
&\le 0,
\end{aligned}$$

when $\varepsilon \le 1$. Therefore, it follows from the Lyapunov Theorem that the conclusion holds.

B.3 Theorems in Chapter 6

B.3.1 Proof of Theorem 6.1

We use mathematical induction to prove this result.

Firstly, let us consider the case of $n = 2$. In this case, the system is

$$\begin{cases} \dot{x}_1 = f_1(x_1,\zeta,t) + G_1(x_1,\zeta,t)x_2 \\ \dot{x}_2 = f_2(x_1^{(0\sim 1)},x_2,\zeta,t) + G_2(x_1^{(0\sim 1)},x_2,\zeta,t)u. \end{cases} \tag{B.36}$$

It follows from the first equation in (B.36) that

$$x_2 = G_1^{-1}(x_1,\zeta,t)[\dot{x}_1 - f_1(x_1,\zeta,t)], \tag{B.37}$$

which gives the following transformation

$$\begin{cases} x_1 = z \\ x_2 = G_1^{-1}(z,\zeta,t)[\dot{z} - f_1(z,\zeta,t)]. \end{cases} \tag{B.38}$$

Next, taking the derivatives of both sides of the first equation in (B.36), and substituting the second one into the result, give

$$\begin{aligned} \ddot{x}_1 &= \dot{f}_1(x_1,\zeta,t) + \dot{G}_1(x_1,\zeta,t)x_2 + G_1(x_1,\zeta,t)\, f_2(x_1^{(0\sim 1)},x_2,\zeta,t) \\ &\quad + G_1(x_1,\zeta,t)G_2(x_1^{(0\sim 1)},x_2,\zeta,t)u. \end{aligned} \tag{B.39}$$

Further, substituting (B.38) into the above equation, produces the following second-order FAS:

$$\ddot{z} = h_2\left(z^{(0\sim 1)},\zeta,t\right) + L_2\left(z^{(0\sim 1)},\zeta,t\right)u, \tag{B.40}$$

where

$$L_2\left(z^{(0\sim 1)},\zeta,t\right) = G_1(z,\zeta,t)G_2(z^{(0\sim 1)},x_2,\zeta,t), \tag{B.41}$$

$$\begin{aligned} h_2\left(z^{(0\sim 1)},\zeta,t\right) &= \dot{f}_1(z,\zeta,t) + \dot{G}_1(z,\zeta,t)x_2 \\ &\quad + G_1(z,\zeta,t)\, f_2(z^{(0\sim 1)},x_2,\zeta,t), \end{aligned} \tag{B.42}$$

and the variable x_2 in the above two equations should be substituted by the second equation in (B.38). In view of $h_1(z,\zeta,t) = f_1(z,\zeta,t)$ and $L_1(z,\zeta,t) = G_1(z,\zeta,t)$, the above (B.42) gives the formula (6.6) for the case of $n = 2$.

Finally, recalling the nonsingularity of $G_1(z,\zeta,t)$ and $G_2(z^{(0\sim 1)},x_2,\zeta,t)$, and the definition of $L_2(z^{(0\sim 1)},\zeta,t)$, we know that system (B.40) is fully-actuated, thus the theorem holds in the case of $n = 2$.

Now let us assume that the theorem holds when $n = k$. For convenience, we still denote u as x_{k+1}. In this case, the system takes the form of

$$\begin{cases} \dot{x}_1 = f_1(x_1,\zeta,t) + G_1(x_1,\zeta,t)x_2 \\ \dot{x}_2 = f_2(x_1^{(0\sim 1)},x_2,\zeta,t) + G_2(x_1^{(0\sim 1)},x_2,\zeta,t)x_3 \\ \qquad \vdots \\ \dot{x}_k = f_k(x_{1\sim k-1}^{(0\sim 1)},x_k,\zeta,t) + G_k(x_{1\sim k-1}^{(0\sim 1)},x_k,\zeta,t)x_{k+1}, \end{cases} \tag{B.43}$$

which, under the following transformation

$$\begin{cases} x_1 = z \\ x_2 = G_1^{-1}(z,\zeta,t)[\dot{z} - f_1(z,\zeta,t)] \\ x_{i+1} = L_i^{-1}(z^{(0\sim i-1)},\zeta,t)[z^{(i)} - h_i(z^{(0\sim i-1)},\zeta,t)], \\ \qquad i = 2,3,\ldots,k-1, \end{cases} \tag{B.44}$$

can be expressed as the following high-order system form:

$$z^{(k)} = h_k(z^{(0\sim k-1)},\zeta,t) + L_k(z^{(0\sim k-1)},\zeta,t)x_{k+1}, \tag{B.45}$$

where

$$L_k(z^{(0\sim k-1)},\zeta,t) = G_1(z,\zeta,t)G_2(z^{(0\sim 1)},x_2,\zeta,t)\cdots G_k(x_{1\sim k-1}^{(0\sim 1)},x_k,\zeta,t). \tag{B.46}$$

Now let us prove the conclusion for the case of $n = k+1$.

When $n = k+1$, noting that the first k formulas of the system are exactly system (B.43), we know from the assumption that the system can be equivalently rewritten as

$$\begin{cases} z^{(k)} = h_k(z^{(0\sim k-1)},\zeta,t) + L_k(z^{(0\sim k-1)},\zeta,t)x_{k+1} \\ \dot{x}_{k+1} = f_{k+1}(x_{1\sim k}^{(0\sim 1)},x_{k+1},\zeta,t) + G_{k+1}(x_{1\sim k}^{(0\sim 1)},x_{k+1},\zeta,t)u. \end{cases} \tag{B.47}$$

It follows from the first equation in (B.47) that

$$x_{k+1} = L_k^{-1}(z^{(0\sim k-1)},\zeta,t)[x_1^{(k)} - h_k(z^{(0\sim k-1)},\zeta,t)]. \tag{B.48}$$

Combining (B.48) with (B.44), gives the following transformation

$$\begin{cases} x_1 = z \\ x_2 = G_1^{-1}(z,\zeta,t)[\dot{z} - f_1(z,\zeta,t)] \\ x_{i+1} = L_i^{-1}(z^{(0\sim i-1)},\zeta,t)[z^{(i)} - h_i(z^{(0\sim i-1)},\zeta,t)], \\ \qquad i = 2,3,\ldots,k. \end{cases} \tag{B.49}$$

Further, taking the derivatives of both sides of the first equation in (B.47), and then substituting the second one, give

$$\begin{aligned} z^{(k+1)} \;=\; & \dot{h}_k(z^{(0\sim k-1)},\zeta,t) + \dot{L}_k(z^{(0\sim k-1)},\zeta,t)x_{k+1} \\ & + L_k(z^{(0\sim k-1)},\zeta,t)\, f_{k+1}(x_{1\sim k}^{(0\sim 1)},x_{k+1},\zeta,t) \\ & + L_k(z^{(0\sim k-1)},\zeta,t)G_{k+1}(x_{1\sim k}^{(0\sim 1)},x_{k+1},\zeta,t)u. \end{aligned} \tag{B.50}$$

Then, substituting (B.49) and its derivative into (B.50), yields the following high-order system:

$$z^{(k+1)} = h_{k+1}(z^{(0\sim k)},\zeta,t) + L_{k+1}(z^{(0\sim k)},\zeta,t)u, \tag{B.51}$$

where

$$\begin{aligned} & L_{k+1}(z^{(0\sim k)},\zeta,t) \\ = \;& L_k(z^{(0\sim k-1)},\zeta,t)G_{k+1}(x_{1\sim k}^{(0\sim 1)},x_{k+1},\zeta,t) \\ = \;& G_1(x_1,\zeta,t)G_2(x_1,\dot{x}_1,x_2,\zeta,t)\cdots G_{k+1}(x_{1\sim k}^{(0\sim 1)},x_{k+1},\zeta,t), \end{aligned} \tag{B.52}$$

with $h_{k+1}(z^{(0\sim k)},\zeta,t)$ being given by (6.6).

Finally, it is known from the nonsingularity of $G_k(x_{1\sim k-1}^{(0\sim 1)},x_k,\zeta,t),\ k=1,2,\ldots,n$, and the definitions of the above $L_{k+1}(z^{(0\sim k)},\zeta,t)$ that system (B.51) is fully-actuated. Thus the theorem also holds in the case of $n=k+1$. Therefore, the whole proof is completed.

B.3.2 Proof of Theorem 6.3

Again we prove this result using mathematical induction.

Let us firstly consider the case of $n=2$. For convenience, we still use the notation x_3 in the position of u, in this case the system is

$$\begin{cases} \ddot{x}_1 = f_1(x_1^{(0\sim 1)},\zeta,t)+G_1(x_1^{(0\sim 1)},\zeta,t)x_2 \\ \ddot{x}_2 = f_2(x_1^{(0\sim 2)},x_2^{(0\sim 1)},\zeta,t)+G_2(x_1^{(0\sim 2)},x_2^{(0\sim 1)},\zeta,t)x_3. \end{cases} \tag{B.53}$$

It follows from the first equation in (B.53) that

$$x_2 = G_1^{-1}(x_1^{(0\sim 1)},\zeta,t)[\ddot{x}_1 - f_1(x_1^{(0\sim 1)},\zeta,t)]. \tag{B.54}$$

Thus the following transformation can be obtained

$$\begin{cases} x_1 = z \\ x_2 = G_1^{-1}(z^{(0\sim 1)},\zeta,t)[\ddot{z} - f_1(z^{(0\sim 1)},\zeta,t)], \end{cases} \tag{B.55}$$

which gives

$$\begin{cases} \dot{x}_1 = \dot{z} \\ \dot{x}_2 \triangleq d_1^{x2}(z^{(0\sim 3)},\zeta,t), \end{cases} \tag{B.56}$$

with

$$\begin{aligned} d_1^{x2}(z^{(0\sim 3)},\zeta,t) &= \frac{\mathrm{d}}{\mathrm{d}t}G_1^{-1}(z^{(0\sim 1)},\zeta,t)[\ddot{z} - f_1(z^{(0\sim 1)},\zeta,t)] \\ &\quad + G_1^{-1}(z^{(0\sim 1)},\zeta,t)[\dddot{z} - \dot{f}_1(z^{(0\sim 1)},\zeta,t)]. \end{aligned}$$

Next, taking the derivatives of both sides of the first equation in (B.53), gives

$$\dddot{z} = \dddot{x}_1 = \dot{f}_1(z^{(0\sim 1)},\zeta,t)+\dot{G}_1(z^{(0\sim 1)},\zeta,t)x_2+G_1(z^{(0\sim 1)},\zeta,t)\dot{x}_2.$$

Hence,

$$\begin{aligned} z^{(4)} &= \ddot{f}_1(z^{(0\sim 1)},\zeta,t)+\ddot{G}_1(z^{(0\sim 1)},\zeta,t)x_2+\dot{G}_1(z^{(0\sim 1)},\zeta,t)\dot{x}_2 \\ &\quad +\dot{G}_1(z^{(0\sim 1)},\zeta,t)\dot{x}_2+G_1(z^{(0\sim 1)},\zeta,t)\ddot{x}_2. \end{aligned}$$

By using the second equation in (B.53), the above equation can be written as

$$\begin{aligned} z^{(4)} &= \ddot{f}_1(z^{(0\sim 1)},\zeta,t)+\ddot{G}_1(z^{(0\sim 1)},\zeta,t)x_2+2\dot{G}_1(z^{(0\sim 1)},\zeta,t)\dot{x}_2 \\ &\quad +G_1(z^{(0\sim 1)},\zeta,t)\,f_2(z^{(0\sim 2)},x_2^{(0\sim 1)},\zeta,t) \\ &\quad +G_1(z^{(0\sim 1)},\zeta,t)G_2(z^{(0\sim 2)},x_2^{(0\sim 1)},\zeta,t)x_3. \end{aligned} \tag{B.57}$$

Further, substituting (B.55) and (B.56) into the (B.57), yields the following fourth-order system:

$$z^{(4)} = h_2(z^{(0\sim 3)}, \zeta, t) + L_2(z^{(0\sim 3)}, \zeta, t)x_3, \tag{B.58}$$

where

$$L_2(z^{(0\sim 3)}, \zeta, t) = G_1(x_1^{(0\sim 1)}, \zeta, t)G_2(x_1^{(0\sim 2)}, x_2^{(0\sim 1)}, \zeta, t), \tag{B.59}$$

and

$$\begin{aligned} h_2(z^{(0\sim 3)}, \zeta, t) \quad = \quad & \ddot{f}_1(z^{(0\sim 1)}, \zeta, t) + \ddot{G}_1(z^{(0\sim 1)}, \zeta, t)x_2 + 2\dot{G}_1(z^{(0\sim 1)}, \zeta, t)\dot{x}_2 \\ & + G_1(z^{(0\sim 1)}, \zeta, t)\, f_2(x_1^{(0\sim 2)}, x_2^{(0\sim 1)}, \zeta, t), \end{aligned} \tag{B.60}$$

and the variable x_2 and its derivative in the above equation are substituted with (B.55) and (B.56). Recalling that $h_1(z^{(0\sim 1)}, \zeta, t) = f_1(z^{(0\sim 1)}, \zeta, t)$ and $L_1(z^{(0\sim 1)}, \zeta, t) = G_1(z^{(0\sim 1)}, \zeta, t)$, we know that the above (B.60) gives the formula (6.22) for the case of $n = 2$.

Finally, in view of the nonsingularity of $G_1(z^{(0\sim 1)}, \zeta, t)$ and $G_2(z^{(0\sim 2)}, x_2^{(0\sim 1)}, \zeta, t)$, and the definition of $L_2(z^{(0\sim 3)}, \zeta, t)$ above, system (B.58) is easily seen to be fully-actuated. Thus the theorem holds when $n = 2$.

Now assume that the theorem holds when $n = k$. For convenience, we still denote u as x_{k+1}, in this case the system is

$$\begin{cases} \ddot{x}_1 = f_1(x_1^{(0\sim 1)}, \zeta, t) + G_1(x_1^{(0\sim 1)}, \zeta, t)x_2 \\ \ddot{x}_2 = f_2(x_1^{(0\sim 2)}, x_2^{(0\sim 1)}, \zeta, t) + G_2(x_1^{(0\sim 2)}, x_2^{(0\sim 1)}, \zeta, t)x_3 \\ \qquad \vdots \\ \ddot{x}_k = f_k(x_{1\sim k-1}^{(0\sim 2)}, x_k^{(0\sim 1)}, \zeta, t) + G_k(x_{1\sim k-1}^{(0\sim 2)}, x_k^{(0\sim 1)}, \zeta, t)x_{k+1}, \end{cases} \tag{B.61}$$

which, under the following transformation,

$$\begin{cases} x_1 = z \\ x_2 = G_1^{-1}(z^{(0\sim 1)}, \zeta, t)[\ddot{z} - f_1(z^{(0\sim 1)}, \zeta, t)] \\ x_{i+1} = L_i^{-1}(z^{(0\sim 2i-1)}, \zeta, t)[z^{(2i)} - h_i(z^{(0\sim 2i-1)}, \zeta, t)], \\ \qquad\qquad i = 2, 3, \ldots, k-1, \end{cases} \tag{B.62}$$

can be rewritten into

$$z^{(2k)} = h_k(z^{(0\sim 2k-1)}, \zeta, t) + L_k(z^{(0\sim 2k-1)}, \zeta, t)x_{k+1}, \tag{B.63}$$

where

$$L_k(z^{(0\sim 2k-1)}, \zeta, t) = L_{k-1}(z^{(0\sim 2k-2)}, \zeta, t)G_k(x_{1\sim k-1}^{(0\sim 2)}, x_k^{(0\sim 1)}, \zeta, t). \tag{B.64}$$

Let us now prove the conclusion for the case of $n = k+1$.

When $n = k+1$, it is easy to know that the first k equations of the system are exactly system (B.61). Thus it is known from the assumptions that in this case the system can be expressed as

$$\begin{cases} z^{(2k)} = h_k(z^{(0\sim 2k-1)}, \zeta, t) + L_k(z^{(0\sim 2k-1)}, \zeta, t)x_{k+1} \\ \ddot{x}_{k+1} = f_{k+1}(x_{1\sim k}^{(0\sim 2)}, x_{k+1}^{(0\sim 1)}, \zeta, t) + G_{k+1}(x_{1\sim k}^{(0\sim 2)}, x_{k+1}^{(0\sim 1)}, \zeta, t)u. \end{cases} \tag{B.65}$$

It follows from the first equation in (B.65) that

$$x_{k+1} = L_k^{-1}(z^{(0\sim 2k-1)}, \zeta, t)[z^{(2k)} - h_k(z^{(0\sim 2k-1)}, \zeta, t)]. \tag{B.66}$$

Combining the above equation with (B.62), yields

$$\begin{cases} x_1 = z \\ x_2 = G_1^{-1}(z^{(0\sim 1)}, \zeta, t)[\ddot{z} - f_1(z^{(0\sim 1)}, \zeta, t)] \\ x_{i+1} = L_i^{-1}(z^{(0\sim 2i-1)}, \zeta, t)[z^{(2i)} - h_i(z^{(0\sim 2i-1)}, \zeta, t)], \\ \qquad\qquad i = 2, 3, \ldots, k. \end{cases} \tag{B.67}$$

Taking the derivatives of both sides of the first equation in (B.65), gives

$$z^{(2k+1)} = \dot{h}_k(z^{(0\sim 2k-1)}, \zeta, t) + \dot{L}_k(z^{(0\sim 2k-1)}, \zeta, t)x_{k+1} + L_k(z^{(0\sim 2k-1)}, \zeta, t)\dot{x}_{k+1}.$$

Hence,

$$\begin{aligned} z^{(2k+2)} &= \ddot{h}_k(z^{(0\sim 2k-1)}, \zeta, t) + \ddot{L}_k(z^{(0\sim 2k-1)}, \zeta, t)x_{k+1} + \dot{L}_k(z^{(0\sim 2k-1)}, \zeta, t)\dot{x}_{k+1} \\ &\quad + \dot{L}_k(z^{(0\sim 2k-1)}, \zeta, t)\dot{x}_{k+1} + L_k(z^{(0\sim 2k-1)}, \zeta, t)\ddot{x}_{k+1}. \end{aligned}$$

Substituting the second equation in (B.65) into the above equation, gives

$$\begin{aligned} z^{(2k+2)} &= \ddot{h}_k(z^{(0\sim 2k-1)}, \zeta, t) + \ddot{L}_k(z^{(0\sim 2k-1)}, \zeta, t)x_{k+1} + 2\dot{L}_k(z^{(0\sim 2k-1)}, \zeta, t)\dot{x}_{k+1} \\ &\quad + L_k(z^{(0\sim 2k-1)}, \zeta, t)\, f_{k+1}(x_{1\sim k}^{(0\sim 2)}, x_{k+1}^{(0\sim 1)}, \zeta, t) \\ &\quad + L_k(z^{(0\sim 2k-1)}, \zeta, t)G_{k+1}(x_{1\sim k}^{(0\sim 2)}, x_{k+1}^{(0\sim 1)}, \zeta, t)u, \end{aligned} \tag{B.68}$$

where $x_i,\ i = 2, 3, \ldots, k$, and their derivatives are given by (B.67). Taking

$$L_{k+1}(z^{(0\sim 2k+1)}, \zeta, t) = L_k(z^{(0\sim 2k-1)}, \zeta, t)G_{k+1}(x_{1\sim k}^{(0\sim 2)}, x_{k+1}^{(0\sim 1)}, \zeta, t), \tag{B.69}$$

and $h_{k+1}(z^{(0\sim 2k+1)}, \zeta, t)$ as in (6.22), the high-order system can be obtained as

$$z^{(2k+2)} = h_{k+1}(z^{(0\sim 2k+1)}, \zeta, t) + L_{k+1}(z^{(0\sim 2k+1)}, \zeta, t)u. \tag{B.70}$$

Finally, due to the nonsingularity of $G_k(x_{1\sim k-1}^{(0\sim 2)}, x_k^{(0\sim 1)}, \zeta, t),\ k = 1, 2, \ldots, n$, and the above definitions of $L_{k+1}(z^{(0\sim 2k+1)}, \zeta, t)$, system (B.70) is clearly fully-actuated. Thus the theorem is also true in the case of $n = k+1$. Therefore, the whole proof is completed.

B.3.3 Proof of Theorem 6.5

We again use mathematical induction to prove the result.

Firstly, let us consider the case of $n = 2$. Similarly, using the notation x_3 at the position of u, in this case the system is

$$\begin{cases} x_1^{(m_1)} = f_1(x_1^{(0\sim m_1-1)}, \zeta, t) + G_1(x_1^{(0\sim m_1-1)}, \zeta, t)x_2 \\ x_2^{(m_2)} = f_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2}, \zeta, t) + G_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2}, \zeta, t)x_3. \end{cases} \tag{B.71}$$

It follows from the first equation in (B.71) that

$$x_2 = G_1^{-1}(x_1^{(0\sim m_1-1)}, \zeta, t)[x_1^{(m_1)} - f_1(x_1^{(0\sim m_1-1)}, \zeta, t)]. \tag{B.72}$$

Thus the following transformation can be obtained:

$$\begin{cases} x_1 = z \\ x_2 = G_1^{-1}(z^{(0\sim m_1-1)}, \zeta, t)[z^{(m_1)} - f_1(z^{(0\sim m_1-1)}, \zeta, t)], \end{cases} \tag{B.73}$$

which further gives

$$\begin{cases} x_1^{(i)} = z^{(i)},\ i = 1 \sim m_1 - 1 \\ x_2^{(i)} \triangleq d_i^{x2}(z^{(0\sim m_1+i)}, \zeta, t),\ i = 1 \sim m_2 - 1, \end{cases} \tag{B.74}$$

with

$$\begin{aligned} d_i^{x2}(z^{(0\sim m_1+i)}, \zeta, t) &= \frac{\mathrm{d}}{\mathrm{d}t} G_1^{-1}(z^{(0\sim m_1-1)}, \zeta, t)[z^{(m_1)} - f_1(z^{(0\sim m_1-1)}, \zeta, t)] \\ &\quad + G_1^{-1}(z^{(0\sim m_1-1)}, \zeta, t)[z^{(m_1+1)} - \dot{f}_1(z^{(0\sim m_1-1)}, \zeta, t)]. \end{aligned}$$

Taking the m_2-order derivatives of both sides of the first equation in (B.71), and then substituting the second one into the result, give

$$\begin{aligned} x_1^{(p_2)} &= f_1^{(m_2)}(x_1^{(0\sim m_1-1)}, \zeta, t) + \frac{\mathrm{d}^{m_2}}{\mathrm{d}t^{m_2}}[G_1(x_1^{(0\sim m_1-1)}, \zeta, t)x_2] \\ &= f_1^{(m_2)}(x_1^{(0\sim m_1-1)}, \zeta, t) + \sum_{k=0}^{m_2} C(k, m_2) G_1^{(m_2-k)}(x_1^{(0\sim m_1-1)}, \zeta, t) x_2^{(k)} \\ &= f_1^{(m_2)}(x_1^{(0\sim m_1-1)}, \zeta, t) + \sum_{k=0}^{m_2-1} C(k, m_2) G_1^{(m_2-k)}(x_1^{(0\sim m_1-1)}, \zeta, t) x_2^{(k)} \\ &\quad + G_1(x_1^{(0\sim m_1-1)}, \zeta, t)\, f_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2}, \zeta, t) \\ &\quad + G_1(x_1^{(0\sim m_1-1)}, \zeta, t) G_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2}, \zeta, t) u. \end{aligned} \tag{B.75}$$

Further, substituting (B.73) and (B.74) into the above equation, produces the following high-order system:

$$z^{(p_2)} = h_2(z^{(0\sim p_2-1)}, \zeta, t) + L_2(z^{(0\sim p_2-1)}, \zeta, t) u, \tag{B.76}$$

where

$$L_2(z^{(0\sim p_2-1)}, \zeta, t) = G_1(x_1^{(0\sim m_1-1)}, \zeta, t) G_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2}, \zeta, t), \tag{B.77}$$

$$\begin{aligned} h_2(z^{(0\sim p_2-1)}, \zeta, t) &= f_1^{(m_2)}(x_1^{(0\sim m_1-1)}, \zeta, t) \\ &\quad + \sum_{k=0}^{m_2-1} C(k, m_2) G_1^{(m_2-k)}(x_1^{(0\sim m_1-1)}, \zeta, t) x_2^{(k)} \\ &\quad + G_1(x_1^{(0\sim m_1-1)}, \zeta, t)\, f_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2}, \zeta, t). \end{aligned} \tag{B.78}$$

Further, in view of

$$\begin{aligned} h_1(z^{(0\sim m_1-1)},\zeta,t) &= f_1(z^{(0\sim m_1-1)},\zeta,t), \\ L_1(z^{(0\sim m_1-1)},\zeta,t) &= G_1(z^{(0\sim m_1-1)},\zeta,t), \end{aligned}$$

the above (B.78) turns out to be (6.38) for the case of $n=2$.

Finally, it follows from the nonsingularity of $G_1(x_1^{(0\sim m_1-1)},\zeta,t)$ and $G_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2},\zeta,t)$, and the definitions of the above $L_2(z^{(0\sim p_2-1)},\zeta,t)$ that system (B.76) is fully-actuated. Thus the theorem holds when $n=2$.

Now assume that the theorem holds when $n=k$. Again, for convenience, we still denote u as x_{k+1}, in this case the system is

$$\left\{\begin{array}{l} x_1^{(m_1)} = f_1(x_1^{(0\sim m_1-1)},\zeta,t)+G_1(x_1^{(0\sim m_1-1)},\zeta,t)x_2 \\ x_2^{(m_2)} = f_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2},\zeta,t)+G_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2},\zeta,t)x_3 \\ \qquad\qquad \vdots \\ x_k^{(m_k)} = f_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k},\zeta,t)+G_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k},\zeta,t)x_{k+1}, \end{array}\right. \tag{B.79}$$

which, under the following transformation

$$\left\{\begin{array}{l} x_1 = z \\ x_2 = G_1^{-1}(z^{(0\sim m_1-1)},\zeta,t)[z^{(m_1)} - f_1(z^{(0\sim m_1-1)},\zeta,t)] \\ x_{i+1} = L_i^{-1}(z^{(0\sim p_i-1)},\zeta,t)[z^{(p_i)} - h_i(z^{(0\sim p_i-1)},\zeta,t)], \\ \qquad\qquad i=2,3,\ldots,k-1, \end{array}\right. \tag{B.80}$$

can be expressed as the following FAS:

$$z^{(p_k)} = h_k(z^{(0\sim p_k-1)},\zeta,t)+L_k(z^{(0\sim p_k-1)},\zeta,t)x_{k+1}, \tag{B.81}$$

where

$$L_k(z^{(0\sim p_k-1)},\zeta,t) = L_{k-1}(z^{(0\sim p_k-2)},\zeta,t)G_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k},\zeta,t). \tag{B.82}$$

Now let us prove that the conclusion is also true when $n=k+1$.

When $n=k+1$, the first k formulas of the system are exactly system (B.79). Thus it is known from the assumptions that the system is equivalent to

$$\left\{\begin{array}{l} z^{(p_k)} = h_k(z^{(0\sim p_k-1)},\zeta,t)+L_k(z^{(0\sim p_k-1)},\zeta,t)x_{k+1} \\ x_{k+1}^{(m_{k+1})} = f_{k+1}(x_i^{(0\sim m_i-1)}|_{i=1\sim k+1},\zeta,t)+G_{k+1}(x_i^{(0\sim m_i-1)}|_{i=1\sim k+1},\zeta,t)u. \end{array}\right. \tag{B.83}$$

From the first equation in (B.83), we can obtain

$$x_{k+1} = L_k^{-1}(z^{(0\sim p_k-1)},\zeta,t)[z^{(p_k)} - h_k(z^{(0\sim p_k-1)},\zeta,t)]. \tag{B.84}$$

Combining the above equation with (B.80), gives the following transformation

$$\left\{\begin{array}{l} x_1 = z \\ x_2 = G_1^{-1}(z^{(0\sim m_1-1)},\zeta,t)[z^{(m_1)} - f_1(z^{(0\sim m_1-1)},\zeta,t)] \\ x_{i+1} = L_i^{-1}(z^{(0\sim p_i-1)},\zeta,t)[z^{(p_i)} - h_i(z^{(0\sim p_i-1)},\zeta,t)], \\ \qquad\qquad i=2,3,\ldots,k, \end{array}\right. \tag{B.85}$$

which further gives

$$\begin{cases} x_1^{(i)} = z^{(i)},\ i = 1 \sim m_1 - 1 \\ x_j^{(i)} \triangleq d_i^{xj}(z^{(0\sim p_{j-1}+i)}),\ i = 1 \sim m_j - 1,\ j = 2 \sim k. \end{cases} \tag{B.86}$$

Taking the m_{k+1}-order derivatives of both sides of the first equation in (B.83), and then substituting the second one into the result, give

$$\begin{aligned} z^{(p_{k+1})} &= h_k^{m_{k+1}}(z^{(0\sim p_k-1)},\zeta,t) + \left[L_k(z^{(0\sim p_k-1)},\zeta,t)x_{k+1}\right]^{(m_{k+1})} \\ &= h_k^{m_{k+1}}(z^{(0\sim p_k-1)},\zeta,t) + \sum_{j=0}^{m_{k+1}} C(j,m_{k+1})L_k^{(m_{k+1}-j)}(z^{(0\sim p_k-1)},\zeta,t)x_{k+1}^{(j)} \\ &= h_k^{m_{k+1}}(z^{(0\sim p_k-1)},\zeta,t) + \sum_{j=0}^{m_{k+1}-1} C(j,m_{k+1})L_k^{(m_{k+1}-j)}(z^{(0\sim p_k-1)},\zeta,t)x_{k+1}^{(j)} \\ &\quad + L_k(x_1^{(0\sim m_{k+1}-1)},\zeta,t)\, f_{k+1}(x_i^{(0\sim m_i-1)}|_{i=1\sim k+1},\zeta,t) \\ &\quad + L_k(x_1^{(0\sim m_{k+1}-1)},\zeta,t)G_{k+1}(x_i^{(0\sim m_i-1)}|_{i=1\sim k+1},\zeta,t)u. \end{aligned} \tag{B.87}$$

Further, substituting (B.85) and (B.86) into the above equation, we can obtain the following high-order FAS:

$$z^{(p_{k+1})} = h_k(z^{(0\sim p_{k+1}-1)},\zeta,t) + L_k(z^{(0\sim p_{k+1}-1)},\zeta,t)u, \tag{B.88}$$

where

$$\begin{aligned} & L_{k+1}(z^{(0\sim p_{k+1}-1)},\zeta,t) \\ =\ & L_k(x_1^{(0\sim m_{k+1}-1)},\zeta,t)G_{k+1}(x_i^{(0\sim m_i-1)}|_{i=1\sim k+1},\zeta,t) \\ =\ & G_1(x_1^{(0\sim m_1-1)},\zeta,t) \\ & \times G_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2},\zeta,t)\cdots G_{k+1}(x_i^{(0\sim m_i-1)}|_{i=1\sim k+1},\zeta,t), \end{aligned} \tag{B.89}$$

and $h_{k+1}(z^{(0\sim p_{k+1}-1)},\zeta,t)$ is given by (6.38).

Finally, it follows from the nonsingularity of $G_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k},\zeta,t), k = 1,2,\ldots,n$, and the definitions of the above $L_{k+1}(z^{(0\sim p_{k+1}-1)},\zeta,t)$ that system (B.88) is fully-actuated. Thus the theorem holds for the case of $n = k+1$. Therefore, the whole proof is complete.

B.4 Proofs of Theorems in Chapter 7

B.4.1 Proof of Theorem 7.2

Again, we can prove the theorem via the n-step induction.

Step 1: For the first subsystem, we already have

$$\begin{aligned} \dot{\bar{x}}_{12} &= \bar{f}_1(\bar{x}_{12}) + \bar{G}_1(\bar{x}_{12})\bar{x}_{21} \\ &= L_1(\bar{x}_{12}) + \bar{G}_{1\sim 1}(\bar{x}_{12})\bar{x}_{21}. \end{aligned} \tag{B.90}$$

Step i ($2 \le i \le n-1$)**:** Assume that

$$\begin{aligned}\bar{x}_{p2}^{(i-p)}\Big|_{p=1\sim i-1} &= L_{i-1}\left(\bar{x}_{p2}^{(0\sim i-1-p)}\Big|_{p=1\sim i-1}\right)\\ &\quad +\bar{G}_{1\sim i-1}\left(\bar{x}_{p2}^{(0\sim i-1-p)}\Big|_{p=1\sim i-1}\right)\bar{x}_{i1}, \end{aligned} \tag{B.91}$$

then we prove that

$$\bar{x}_{p2}^{(i+1-p)}\Big|_{p=1\sim i} = L_i\left(\bar{x}_{p2}^{(0\sim i-p)}\Big|_{p=1\sim i}\right)+\bar{G}_{1\sim i}\left(\bar{x}_{p2}^{(0\sim i-p)}\Big|_{p=1\sim i}\right)\bar{x}_{i+1,1}. \tag{B.92}$$

It follows from (B.91) that

$$\bar{x}_{i1} = \bar{G}_{1\sim i-1}^{-1}(\cdot)\left(\bar{x}_{p2}^{(i-p)}\Big|_{p=1\sim i-1} - L_{i-1}\left(\bar{x}_{p2}^{(0\sim i-1-p)}\Big|_{p=1\sim i-1}\right)\right). \tag{B.93}$$

Also, recalling system (7.34), we have the following equations:

$$\begin{cases} \dot{\bar{x}}_{i1} = \bar{f}_{i1}(\bar{x}_{2\sim i,1},\bar{x}_{1\sim i,2})+\begin{bmatrix} I_{r_{i-1}} & 0\end{bmatrix}\bar{Q}_{i-1}^{-1}\bar{G}_i(\bar{x}_{2\sim i,1},\bar{x}_{1\sim i,2})\bar{x}_{i+1,1}\\ \dot{\bar{x}}_{i2} = \bar{f}_{i2}(\bar{x}_{2\sim i,1},\bar{x}_{1\sim i,2})+\begin{bmatrix} 0 & I_{r_i-r_{i-1}}\end{bmatrix}\bar{Q}_{i-1}^{-1}\bar{G}_i(\bar{x}_{2\sim i,1},\bar{x}_{1\sim i,2})\bar{x}_{i+1,1}. \end{cases} \tag{B.94}$$

Next, taking the derivatives of both sides of equation (B.91) and substituting first equation of (B.94) into the result, one has that

$$\begin{aligned} &\dot{\bar{x}}_{p2}^{(i+1-p)}\Big|_{p=1\sim i-1}\\ =\;& \dot{L}_{i-1}(\cdot)+\dot{\bar{G}}_{1\sim i-1}(\cdot)\bar{x}_{i1}+\bar{G}_{1\sim i-1}(\cdot)\dot{\bar{x}}_{i1}\\ =\;& \dot{L}_{i-1}(\cdot)+\dot{\bar{G}}_{1\sim i-1}(\cdot)\bar{x}_{i1}\\ &+\bar{G}_{1\sim i-1}(\cdot)\left(\bar{f}_{i1}(\cdot)+\begin{bmatrix} I_{r_{i-1}} & 0\end{bmatrix}\bar{Q}_{i-1}^{-1}(\cdot)\bar{G}_i(\cdot)\bar{x}_{i+1,1}\right)\\ =\;& \dot{L}_{i-1}(\cdot)+\dot{\bar{G}}_{1\sim i-1}(\cdot)\bar{x}_{i1}+\bar{G}_{1\sim i-1}(\cdot)\,\bar{f}_{i1}(\cdot)\\ &+\begin{bmatrix} \bar{G}_{1\sim i-1}(\cdot) & 0\end{bmatrix}\bar{Q}_{i-1}^{-1}(\cdot)\bar{G}_i(\cdot)\bar{x}_{i+1,1}. \end{aligned} \tag{B.95}$$

By combining (B.95) with the last equation of (B.94), it can be obtained that

$$\bar{x}_{p2}^{(i+1-p)}\Big|_{p=1\sim i} = L_i\left(\bar{x}_{p2}^{(0\sim i-p)}\Big|_{p=1\sim i}\right)+\bar{G}_{1\sim i}\left(\bar{x}_{p2}^{(0\sim i-p)}\Big|_{p=1\sim i}\right)\bar{x}_{i+1,1}, \tag{B.96}$$

where

$$\begin{aligned} L_i(\cdot) &= \begin{bmatrix} \dot{L}_{i-1}(\cdot)+\dot{\bar{G}}_{1\sim i-1}(\cdot)\bar{x}_{i1}+\bar{G}_{1\sim i-1}(\cdot)\,\bar{f}_{i1}(\cdot)\\ \bar{f}_{i2}(\cdot)\end{bmatrix},\\ \bar{G}_{1\sim i}(\cdot) &= \begin{bmatrix} \bar{G}_{1\sim i-1}(\cdot) & 0\\ 0 & I_{r_i-r_{i-1}}\end{bmatrix}\bar{Q}_{i-1}^{-1}(\cdot)\bar{G}_i(\cdot). \end{aligned} \tag{B.97}$$

Step n: Based on the previous $n-1$ steps, we can obtain that

$$\bar{x}_{i1} = \bar{G}_{1\sim i-1}^{-1}\left(\bar{x}_{p2}^{(i-p)}\Big|_{p=1\sim i-1} - L_{i-1}\left(\bar{x}_{p2}^{(0\sim i-1-p)}\Big|_{p=1\sim i-1}\right)\right), \quad i=2,3,\cdots,n-1, \tag{B.98}$$

and

$$\begin{aligned}\bar{x}_{p2}^{(n-p)}\Big|_{p=1\sim n-1} &= L_{n-1}\left(\bar{x}_{p2}^{(0\sim n-1-p)}\Big|_{p=1\sim n-1}\right)\\ &\quad +\bar{G}_{1\sim n-1}\left(\bar{x}_{p2}^{(0\sim n-1-p)}\Big|_{p=1\sim n-1}\right)\bar{x}_{n1},\end{aligned} \tag{B.99}$$

where $L_i(\cdot)$ and $\bar{G}_{1\sim i}$ are given iteratively by

$$\begin{aligned}&L_{i+1}\left(\bar{x}_{p2}^{(0\sim i+1-p)}\Big|_{p=1\sim i+1}\right)\\ =&\begin{bmatrix}\dot{L}_i(\cdot)+\dot{\bar{G}}_{1\sim i}(\cdot)\bar{x}_{i+1,1}+\bar{G}_{1\sim i}(\cdot)\bar{f}_{i+1,1}(\cdot)\\ \bar{f}_{i+1,2}(\cdot)\end{bmatrix},\end{aligned} \tag{B.100}$$

$$\begin{aligned}&\bar{G}_{1\sim i+1}\left(\bar{x}_{p2}^{(0\sim i+1-p)}\Big|_{p=1\sim i+1}\right)\\ =&\begin{bmatrix}\bar{G}_{1\sim i} & 0\\ 0 & I_{r_{i+1}-r_i}\end{bmatrix}\bar{Q}_i^{-1}(\cdot)\bar{G}_{i+1}(\cdot), \quad i=1,2,\cdots,n-1.\end{aligned} \tag{B.101}$$

Meanwhile, recalling (7.34), we have the following equations:

$$\begin{cases}\dot{\bar{x}}_{n1}=\bar{f}_{n1}(\bar{x}_{2\sim n,1},\bar{x}_{1\sim n,2})+\begin{bmatrix}I_{r_{n-1}} & 0\end{bmatrix}\bar{Q}_{n-1}^{-1}\bar{G}_n(\bar{x}_{2\sim n,1},\bar{x}_{1\sim n,2})\bar{u}_1\\ \dot{\bar{x}}_{n2}=\bar{f}_{n2}(\bar{x}_{2\sim n,1},\bar{x}_{1\sim n,2})+\begin{bmatrix}0 & I_{r_n-r_{n-1}}\end{bmatrix}\bar{Q}_{n-1}^{-1}\bar{G}_n(\bar{x}_{2\sim n,1},\bar{x}_{1\sim n,2})\bar{u}_1.\end{cases} \tag{B.102}$$

Similar to the discussion in the previous steps, taking the derivative of both sides of (B.99) and substituting the first equation of (B.102) into the result, we obtain

$$\begin{aligned}\bar{x}_{p2}^{(n+1-p)}\Big|_{p=1\sim n-1} &= \dot{L}_{n-1}+\dot{\bar{G}}_{1\sim n-1}\bar{x}_{n1}+\bar{G}_{1\sim n-1}\dot{\bar{x}}_{n1}\\ &= \dot{L}_{n-1}+\dot{\bar{G}}_{1\sim n-1}\bar{x}_{n1}+\bar{G}_{1\sim n-1}\bar{f}_{n1}\\ &\quad +\begin{bmatrix}\bar{G}_{1\sim n-1} & 0\end{bmatrix}\bar{Q}_{n-1}^{-1}\bar{G}_n\bar{u}_1.\end{aligned} \tag{B.103}$$

Combining (B.103) with the last equation of (B.102), gives

$$\bar{x}_{p2}^{(0\sim n+1-p)}\Big|_{p=1\sim n}=L_n\left(\bar{x}_{p2}^{(0\sim n-p)}\Big|_{p=1\sim n}\right)+\bar{G}_{1\sim n}\left(\bar{x}_{p2}^{(0\sim n-p)}\Big|_{p=1\sim n}\right)\bar{u}_1, \tag{B.104}$$

where

$$L_n(\cdot)=\begin{bmatrix}\dot{L}_{n-1}(\cdot)+\dot{\bar{G}}_{1\sim n-1}(\cdot)\bar{x}_{n1}+\bar{G}_{1\sim n-1}(\cdot)\bar{f}_{n1}(\cdot)\\ f_{n2}(\cdot)\end{bmatrix}, \tag{B.105}$$

$$\bar{G}_{1\sim n}=\begin{bmatrix}\bar{G}_{1\sim n-1}(\cdot) & 0\\ 0 & I_{r_n-r_{n-1}}\end{bmatrix}\bar{Q}_{n-1}^{-1}(\cdot)\bar{G}_n(\cdot). \tag{B.106}$$

It follows from the definition of $\bar{G}_{1\sim n}(\cdot)$ that $\bar{G}_{1\sim n}(\cdot)$ is nonsingular for any $\left(\bar{x}_{p2}^{(0\sim n-p)}\Big|_{p=1\sim n}\right) \in \mathbb{R}^{\varpi_0}$, with $\varpi_0 = \sum_{i=1}^{n}(n-i)(r_i - r_{i-1})$. Therefore, system (B.104), which is identical with system (7.36), is a FAS.

The above process proves the theorem.

B.4.2 Proof of Theorem 7.3

We prove the theorem via the n-step induction.

Step 1: When $i = 1$, we have

$$\begin{aligned}\ddot{x}_1 &= f_1\left(x_1^{(0\sim 1)}\right) + \bar{G}_1 x_{21}\\ &= L_1\left(x_1^{(0\sim 1)}\right) + \bar{G}_{1\sim 1} x_{21}.\end{aligned} \tag{B.107}$$

Step i ($2 \le i \le n-1$): Assume that

$$\begin{bmatrix} x_1^{(2i-2)} \\ x_{22}^{(2i-4)} \\ \vdots \\ x_{i-1,2}^{(2)} \end{bmatrix} = L_{i-1}\left(x_1^{(0\sim 2i-3)}, x_{p2}^{(0\sim 2i-1-2p)}\Big|_{p=2\sim i-1}\right) + \bar{G}_{1\sim i-1} x_{i1}, \tag{B.108}$$

then we prove that

$$\begin{bmatrix} x_1^{(2i)} \\ x_{22}^{(2i-2)} \\ \vdots \\ x_{i2}^{(2)} \end{bmatrix} = L_i\left(x_1^{(0\sim 2i-1)}, x_{p2}^{(0\sim 2i+1-2p)}\Big|_{p=2\sim i}\right) + \bar{G}_{1\sim i} x_{i+1,1}.$$

It follows from (B.108) that

$$x_{i1} = \bar{G}_{1\sim i-1}^{-1}\left(x_{p2}^{(2i-2p)}\Big|_{p=1\sim i-1} - L_{i-1}\left(x_1^{(0\sim 2i-3)}, x_{p2}^{(0\sim 2i-1-2p)}\Big|_{p=2\sim i-1}\right)\right).$$

We also recall from system (7.39) the following equations:

$$\begin{cases} \ddot{x}_{i1} = f_{i1} + \begin{bmatrix} I_{r_{i-1}} & 0 \end{bmatrix} Q_{i-1}^{-1} \bar{G}_i x_{i+1,1} \\ \ddot{x}_{i2} = f_{i2} + \begin{bmatrix} 0 & I_{r_i - r_{i-1}} \end{bmatrix} Q_{i-1}^{-1} \bar{G}_i x_{i+1,1}. \end{cases} \tag{B.109}$$

Next, taking the second-order derivatives of both sides of equation (B.108) and substi-

tuting the first equation of (B.109) into the result, one has that

$$\begin{aligned}\begin{bmatrix} x_1^{(2i)} \\ x_{22}^{(2i-2)} \\ \vdots \\ x_{i-1,2}^{(4)} \end{bmatrix} &= \ddot{L}_{i-1} + \bar{G}_{1\sim i-1}\ddot{x}_{i1} \\ &= \ddot{L}_{i-1} + \bar{G}_{1\sim i-1}\left(f_{i1} + \begin{bmatrix} I_{r_{i-1}} & 0 \end{bmatrix} Q_{i-1}^{-1}\bar{G}_i x_{i+1,1}\right) \\ &= \ddot{L}_{i-1} + \bar{G}_{1\sim i-1} f_{i1} + \begin{bmatrix} \bar{G}_{1\sim i-1} & 0 \end{bmatrix} Q_{i-1}^{-1}\bar{G}_i x_{i+1,1}. \end{aligned} \tag{B.110}$$

By combining (B.110) with the last equation of (B.109), it can be obtained that

$$\begin{bmatrix} x_1^{(2i)} \\ x_{22}^{(2i-2)} \\ \vdots \\ x_{i2}^{(2)} \end{bmatrix} = L_i\left(x_1^{(0\sim 2i-1)}, x_{p2}^{(0\sim 2i-2p+1)}\Big|_{p=2\sim i}\right) + \bar{G}_{1\sim i}x_{i+1,1},$$

where

$$L_i(\cdot) = \begin{bmatrix} \ddot{L}_{i-1}(\cdot) + \bar{G}_{1\sim i-1}(\cdot)\, f_{i1}(\cdot) \\ f_{i2}(\cdot) \end{bmatrix}, \tag{B.111}$$

$$\bar{G}_{1\sim i}(\cdot) = \begin{bmatrix} \bar{G}_{1\sim i-1}(\cdot) & 0 \\ 0 & I_{r_i - r_{i-1}} \end{bmatrix} Q_{i-1}^{-1}\bar{G}_i(\cdot). \tag{B.112}$$

Step n: Based on the previous $n-1$ steps, we can obtain that

$$\begin{aligned} x_{i1} &= \bar{G}_{1\sim i-1}^{-1}\left(x_{p2}^{(2i-2p)}\Big|_{p=1\sim i-1} - L_{i-1}\left(x_1^{(0\sim 2i-3)}, x_{p2}^{(0\sim 2i-1-2p)}\Big|_{p=2\sim i-1}\right)\right), \\ &\qquad i = 2, 3, \cdots, n-1, \end{aligned}$$

and

$$\begin{bmatrix} x_1^{(2n-2)} \\ x_{22}^{(2n-4)} \\ \vdots \\ x_{n-1,2}^{(2)} \end{bmatrix} = L_{n-1}\left(x_1^{(0\sim 2n-3)}, x_{p2}^{(0\sim 2n-1-2p)}\Big|_{p=2\sim n-1}\right) + \bar{G}_{1\sim n-1}x_{n1}, \tag{B.113}$$

where, for $i = 1, 2, \cdots, n-1$,

$$L_{i+1}(\cdot) = \begin{bmatrix} \ddot{L}_i(\cdot) + \bar{G}_{1\sim i}\, f_{i+1,1}(\cdot) \\ f_{i+1,2}(\cdot) \end{bmatrix}, \tag{B.114}$$

$$\bar{G}_{1\sim i+1} = \begin{bmatrix} \bar{G}_{1\sim i} & 0 \\ 0 & I_{r_{i+1} - r_i} \end{bmatrix} Q_i^{-1}\bar{G}_{i+1}. \tag{B.115}$$

We recall from (7.39) the following equations:

$$\begin{cases} \ddot{x}_{n1} = f_{n1} + \begin{bmatrix} I_{r_{n-1}} & 0 \end{bmatrix} Q_{n-1}^{-1} \bar{G}_n u_1 \\ \ddot{x}_{n2} = f_{n2} + \begin{bmatrix} 0 & I_{r_n - r_{n-1}} \end{bmatrix} Q_{n-1}^{-1} \bar{G}_n u_1. \end{cases} \tag{B.116}$$

Similar to the discussion in the previous steps, taking the second-order derivative of both sides of (B.113) and substituting the first equation of (B.116) into the result, we obtain

$$\begin{aligned} & x_{p2}^{(2n+2-2p)}\Big|_{p=1\sim n-1} \\ = \;& \ddot{L}_{n-1} + \bar{G}_{1\sim n-1} \ddot{x}_{n1} \\ = \;& \ddot{L}_{n-1} + \bar{G}_{1\sim n-1} \left(f_{n1} + \begin{bmatrix} I_{k_{n-1}} & 0 \end{bmatrix} Q_{n-1}^{-1} \bar{G}_n u_1 \right) \\ = \;& \ddot{L}_{n-1} + \bar{G}_{1\sim n-1} f_{n1} + \begin{bmatrix} \bar{G}_{1\sim n-1} & 0 \end{bmatrix} Q_{n-1}^{-1} \bar{G}_n u_1. \end{aligned} \tag{B.117}$$

Combining (B.117) with the last equation of (B.116), gives

$$\begin{bmatrix} x_1^{(2n)} \\ x_{22}^{(2n-2)} \\ \vdots \\ x_{n2}^{(2)} \end{bmatrix} = L_n \left(x_1^{(0\sim 2n-1)}, x_{p2}^{(0\sim 2n-2p+1)}\Big|_{p=2\sim n} \right) + \bar{G}_{1\sim n} u_1, \tag{B.118}$$

where

$$L_n = \begin{bmatrix} \ddot{L}_{n-1} + \bar{G}_{1\sim n-1} f_{n1} \\ f_{n2} \end{bmatrix}, \tag{B.119}$$

$$\bar{G}_{1\sim n} = \begin{bmatrix} \bar{G}_{1\sim n-1} & 0 \\ 0 & I_{r_n - r_{n-1}} \end{bmatrix} Q_{n-1}^{-1} \bar{G}_n. \tag{B.120}$$

It follows from the definition of $\bar{G}_{1\sim n}$ that $\bar{G}_{1\sim n}$ is invertible. Therefore, system (B.118), which is identical with system (7.41), is fully-actuated.

Due to the above process, the conclusion of the theorem holds.

B.4.3 Proof of Theorem 7.5

We prove the theorem via the n-step induction.

Step 1: For the case of $i = 1$, we have

$$\begin{aligned} x_1^{(m_1)} &= f_1 \left(x_1^{(0\sim m_1 - 1)} \right) + \bar{G}_1 x_{21} \\ &= L_1 (x_1^{(0\sim q_{1\sim 1} - 1)}) + \bar{G}_{1\sim 1} x_{21}. \end{aligned} \tag{B.121}$$

Step i ($2 \le i \le n-1$)**:** Assume that

$$\begin{bmatrix} x_1^{(q_{1\sim i-1})} \\ x_{22}^{(q_{2\sim i-1})} \\ \vdots \\ x_{i-1,2}^{(q_{i-1\sim i-1})} \end{bmatrix} = L_{i-1}\left(x_1^{(0\sim q_{1\sim i-1}-1)}, x_{p2}^{(0\sim q_{p\sim i-1}-1)}\Big|_{p=2\sim i-1}\right) + \bar{G}_{1\sim i-1}x_{i1}, \tag{B.122}$$

then we prove that

$$\begin{bmatrix} x_1^{(q_{1\sim i})} \\ x_{22}^{(q_{2\sim i})} \\ \vdots \\ x_{i2}^{(q_{i\sim i})} \end{bmatrix} = L_i\left(x_1^{(0\sim q_{1\sim i}-1)}, x_{p2}^{(0\sim q_{p\sim i}-1)}\Big|_{p=2\sim i}\right) + \bar{G}_{1\sim i}x_{i+1,1}.$$

It follows from (B.122) that

$$x_{i1} = \bar{G}_{1\sim i-1}^{-1}\left(x_{p2}^{(q_{p\sim i-1})}\Big|_{p=1\sim i-1} - L_{i-1}\left(x_1^{(0\sim q_{1\sim i-1}-1)}, x_{p2}^{(0\sim q_{p\sim i-1}-1)}\Big|_{p=2\sim i-1}\right)\right).$$

We also recall the following equations from system (7.57):

$$\begin{cases} x_{i1}^{(m_i)} = f_{i1} + \begin{bmatrix} I_{r_{i-1}} & 0 \end{bmatrix} Q_{i-1}^{-1}\bar{G}_i x_{i+1,1} \\ x_{i2}^{(m_i)} = f_{i2} + \begin{bmatrix} 0 & I_{r_i - r_{i-1}} \end{bmatrix} Q_{i-1}^{-1}\bar{G}_i x_{i+1,1}. \end{cases} \tag{B.123}$$

Next, taking the m_i-th order derivatives of both sides of equation (B.122) and substituting first equation of (B.123) into the result, one has that

$$\begin{aligned} \begin{bmatrix} x_1^{(q_{1\sim i})} \\ x_{22}^{(q_{2\sim i})} \\ \vdots \\ x_{i-1,2}^{(q_{i-1\sim i})} \end{bmatrix} &= L_{i-1}^{(m_i)} + \bar{G}_{1\sim i-1}x_{i1}^{(m_i)} \\ &= L_{i-1}^{(m_i)} + \bar{G}_{1\sim i-1}\left(f_{i1} + \begin{bmatrix} I_{r_{i-1}} & 0 \end{bmatrix} Q_{i-1}^{-1}\bar{G}_i x_{i+1,1}\right) \\ &= L_{i-1}^{(m_i)} + \bar{G}_{1\sim i-1}f_{i1} + \begin{bmatrix} \bar{G}_{1\sim i-1} & 0 \end{bmatrix} Q_{i-1}^{-1}\bar{G}_i x_{i+1,1}. \end{aligned} \tag{B.124}$$

By combining (B.124) with the last equation of (B.123), it can be obtained that

$$\begin{bmatrix} x_1^{(q_{1\sim i})} \\ x_{22}^{(q_{2\sim i})} \\ \vdots \\ x_{i2}^{(q_{i\sim i})} \end{bmatrix} = L_i\left(x_1^{(0\sim q_{1\sim i}-1)}, x_{p2}^{(0\sim q_{p\sim i}-1)}\Big|_{p=2\sim i}\right) + \bar{G}_{1\sim i}x_{i+1,1},$$

where

$$L_i(\cdot) = \begin{bmatrix} L_{i-1}^{(m_i)}(\cdot) + \bar{G}_{1\sim i-1} f_{i1}(\cdot) \\ f_{i2}(\cdot) \end{bmatrix}, \tag{B.125}$$

$$\bar{G}_{1\sim i} = \begin{bmatrix} \bar{G}_{1\sim i-1} & 0 \\ 0 & I_{r_i - r_{i-1}} \end{bmatrix} Q_{i-1}^{-1} \bar{G}_i. \tag{B.126}$$

Step n: Based on the previous $n-1$ steps, we can obtain that

$$\begin{aligned} & x_{i1} \\ = \; & \bar{G}_{1\sim i-1}^{-1} \left(x_{p2}^{(q_{p\sim i-1})}\Big|_{p=1\sim i-1} - L_{i-1}\left(x_1^{(0\sim q_{1\sim i-1}-1)}, x_{p2}^{(0\sim q_{p\sim i-1}-1)}\Big|_{p=2\sim i-1} \right) \right), \\ & i = 2, 3, \cdots, n-1, \end{aligned}$$

and

$$\begin{bmatrix} x_1^{(q_{1\sim n-1})} \\ x_{22}^{(q_{2\sim n-1})} \\ \vdots \\ x_{i2}^{(q_{n-1\sim n-1})} \end{bmatrix} = L_{n-1}\left(x_1^{(0\sim q_{1\sim n-1}-1)}, x_{p2}^{(0\sim q_{p\sim n-1}-1)}\Big|_{p=2\sim n-1} \right) + \bar{G}_{1\sim n-1} x_{n1}, \tag{B.127}$$

where, for $i = 1, 2, \cdots, n-1$,

$$L_{i+1}(\cdot) = \begin{bmatrix} L_i^{(m_{i+1})}(\cdot) + \bar{G}_{1\sim i} f_{i+1,1}(\cdot) \\ f_{i+1,2}(\cdot) \end{bmatrix}, \tag{B.128}$$

$$\bar{G}_{1\sim i+1} = \begin{bmatrix} \bar{G}_{1\sim i} & 0 \\ 0 & I_{r_{i+1} - r_i} \end{bmatrix} Q_i^{-1} \bar{G}_{i+1}. \tag{B.129}$$

We recall from (7.57) the following equations:

$$\begin{cases} x_{n1}^{(m_n)} = f_{n1} + \begin{bmatrix} I_{r_{n-1}} & 0 \end{bmatrix} Q_{n-1}^{-1} \bar{G}_n u_1 \\ x_{n2}^{(m_n)} = f_{n2} + \begin{bmatrix} 0 & I_{r_n - r_{n-1}} \end{bmatrix} Q_{n-1}^{-1} \bar{G}_n u_1. \end{cases} \tag{B.130}$$

Similar to the discussion in the previous steps, taking the m_i-th order derivative of both sides of (B.127) and substituting the first equation of (B.130) into the result, we obtain

$$\begin{aligned} & x_{p2}^{(q_{p\sim n})}\Big|_{p=1\sim n-1} \\ = \; & L_{n-1}^{(m_n)} + \bar{G}_{1\sim n-1} x_{n1}^{(m_n)} \\ = \; & L_{n-1}^{(m_n)} + \bar{G}_{1\sim n-1} f_{n1} + \begin{bmatrix} \bar{G}_{1\sim n-1} & 0 \end{bmatrix} Q_{n-1}^{-1} \bar{G}_n u_1. \end{aligned} \tag{B.131}$$

Combining (B.131) with the last equation of (B.130), gives

$$\begin{bmatrix} x_1^{(q_{1\sim n})} \\ x_{22}^{(q_{2\sim n})} \\ \vdots \\ x_{n2}^{(q_{n\sim n})} \end{bmatrix} = L_n\left(x_1^{(0\sim q-1)}, x_{p2}^{(0\sim q_{p\sim n}-1)}\Big|_{p=2\sim n} \right) + \bar{G}_{1\sim n} u_1, \tag{B.132}$$

where

$$L_n(\cdot) = \begin{bmatrix} L_{n-1}^{m_n}(\cdot) + \bar{G}_{1\sim i-1} f_{n1}(\cdot) \\ f_{i2}(\cdot) \end{bmatrix}, \tag{B.133}$$

$$\bar{G}_{1\sim n} = \begin{bmatrix} \bar{G}_{1\sim i-1} & 0 \\ 0 & I_{r_n - r_{n-1}} \end{bmatrix} Q_{n-1}^{-1} \bar{G}_n. \tag{B.134}$$

It follows from the definition of $\bar{G}_{1\sim n}$ that $\bar{G}_{1\sim n}$ is invertible. Therefore, system (B.132), which is identical with system (7.59), is fully-actuated.

Due to the above process, the conclusion of the theorem clearly holds.

B.5 Proofs of Theorems in Chapter 8

B.5.1 Proof of Theorem 8.1

We use mathematical induction to prove this result.

Step 1 Case of $n = 2$ and 3

Firstly, let us consider the case of $n = 2$. In this case, the system is

$$\begin{cases} \dot{x}_1 = f_1(x_1, \zeta, t) + g_1(x_{1\sim 2}, \zeta, t) \\ \dot{x}_2 = f_2(x_{1\sim 2}, \zeta, t) + g_2(x_{1\sim 2}, u, \zeta, t). \end{cases} \tag{B.135}$$

From the first equation in (B.135), we can solve x_2 as follows:

$$x_2 = g_1^{-1}(x_1, y_1, \zeta, t),\ y_1 = \dot{x}_1 - f_1(x_1, \zeta, t), \tag{B.136}$$

which gives the following transformation:

$$\begin{cases} x_1 = z \\ x_2 = g_1^{-1}(z, y_1, \zeta, t),\ y_1 = \dot{z} - f_1(z, \zeta, t). \end{cases} \tag{B.137}$$

Next, taking the derivative of both sides of the first equation in (B.135), and substituting the second one into the result, give

$$\begin{aligned} \ddot{x}_1 &= \dot{f}_1(x_1, \zeta, t) + \dot{g}_1(x_{1\sim 2}, \zeta, t) \\ &= \dot{f}_1(x_1, \zeta, t) + \frac{\partial g_1(x_{1\sim 2}, \zeta, t)}{\partial x_1}\dot{x}_1 + \frac{\partial g_1(x_{1\sim 2}, \zeta, t)}{\partial x_2}\dot{x}_2 \\ &\quad + \frac{\partial g_1(x_{1\sim 2}, \zeta, t)}{\partial \zeta}\dot{\zeta} + \frac{\partial g_1(x_{1\sim 2}, \zeta, t)}{\partial t} \\ &= \dot{f}_1(x_1, \zeta, t) + \frac{\partial g_1(x_{1\sim 2}, \zeta, t)}{\partial x_1}(f_1(x_1, \zeta, t) + g_1(x_{1\sim 2}, \zeta, t)) \\ &\quad + \frac{\partial g_1(x_{1\sim 2}, \zeta, t)}{\partial x_2}(f_2(x_{1\sim 2}, \zeta, t) + g_2(x_{1\sim 2}, u, \zeta, t)) \\ &\quad + \frac{\partial g_1(x_{1\sim 2}, \zeta, t)}{\partial \zeta}\dot{\zeta} + \frac{\partial g_1(x_{1\sim 2}, \zeta, t)}{\partial t}. \end{aligned} \tag{B.138}$$

Further, substituting (B.137) into the above equation, produces the following second-order fully-actuated system:

$$\ddot{z} = h_2\left(z^{(0\sim 1)}, \zeta^{(0\sim 1)}, t\right) + l_2\left(z^{(0\sim 1)}, u, \zeta, t\right), \tag{B.139}$$

where

$$\begin{aligned} h_2\left(z^{(0\sim 1)}, \zeta^{(0\sim 1)}, t\right) &= \dot{f}_1(z,\zeta,t) + \frac{\partial g_1(x_{1\sim 2},\zeta,t)}{\partial x_1}\left(f_1(x_1,\zeta,t) + g_1(x_{1\sim 2},\zeta,t)\right) \\ &\quad + \frac{\partial g_1(x_{1\sim 2},\zeta,t)}{\partial x_2} f_2(x_{1\sim 2},\zeta,t) \\ &\quad + \frac{\partial g_1(x_{1\sim 2},\zeta,t)}{\partial \zeta}\dot{\zeta} + \frac{\partial g_1(x_{1\sim 2},\zeta,t)}{\partial t}, \end{aligned} \tag{B.140}$$

$$l_2\left(z^{(0\sim 1)}, u, \zeta, t\right) = \frac{\partial g_1(x_{1\sim 2},\zeta,t)}{\partial x_2} g_2(x_{1\sim 2}, u, \zeta, t), \tag{B.141}$$

and the variable x_2 in the above two equations should be substituted by the second equation in (B.137). In view of $h_1(z,\zeta,t) = f_1(x_1,\zeta,t)$ and $l_1(z,x_2,\zeta,t) = g_1(x_{1\sim 2},\zeta,t)$, the above (B.140) gives the formula (8.9) for the case of $n = 2$.

For clearness, let us also consider the case of $n = 3$. In this case, the system is

$$\begin{cases} \ddot{z} = h_2\left(z^{(0\sim 1)}, \zeta^{(0\sim 1)}, t\right) + l_2\left(z^{(0\sim 1)}, u, \zeta, t\right) \\ \dot{x}_3 = f_3(x_{1\sim 3},\zeta,t) + g_3(x_{1\sim 3}, u, \zeta, t). \end{cases} \tag{B.142}$$

By replacing the u with the variable x_3 in the first equation of (B.142), it follows that

$$x_3 = l_2^{-1}(z^{(0\sim 1)}, y_2, \zeta, t),\ y_2 = \ddot{z} - h_2\left(z^{(0\sim 1)}, \zeta^{(0\sim 1)}, t\right), \tag{B.143}$$

which gives the following transformation:

$$\begin{cases} x_1 = z \\ x_2 = g_1^{-1}(z, y_1, \zeta, t),\ y_1 = \dot{z} - f_1(z,\zeta,t) \\ x_3 = l_2^{-1}(z^{(0\sim 1)}, y_2, \zeta, t),\ y_2 = \ddot{z} - h_2\left(z^{(0\sim 1)}, \zeta^{(0\sim 1)}, t\right). \end{cases} \tag{B.144}$$

Next, taking the derivative of (B.139), by replacing u into x_3, and substituting the second one into the result, give

$$\begin{aligned} \dddot{z} &= \dot{h}_2\left(z^{(0\sim 1)}, \zeta^{(0\sim 1)}, t\right) + \frac{\partial l_2\left(z^{(0\sim 1)}, x_3, \zeta, t\right)}{\partial z^{(0\sim 1)}}\dot{z}^{(0\sim 1)} \\ &\quad + \frac{\partial l_2\left(z^{(0\sim 1)}, x_3, \zeta, t\right)}{\partial \zeta}\dot{\zeta} + \frac{\partial l_2\left(z^{(0\sim 1)}, x_3, \zeta, t\right)}{\partial t} \\ &\quad + \frac{\partial l_2\left(z^{(0\sim 1)}, x_3, \zeta, t\right)}{\partial x_3}\left(f_3(x_{1\sim 3},\zeta,t) + g_3(x_{1\sim 3}, u, \zeta, t)\right) \\ &= h_3\left(z^{(0\sim 2)}, \zeta^{(0\sim 2)}, t\right) + l_3\left(z^{(0\sim 2)}, u, \zeta^{(0\sim 1)}, t\right), \end{aligned} \tag{B.145}$$

where

$$h_3\left(z^{(0\sim 2)},\zeta^{(0\sim 2)},t\right)=\dot{h}_2\left(z^{(0\sim 1)},\zeta^{(0\sim 1)},t\right)+\frac{\partial l_2\left(z^{(0\sim 1)},x_3,\zeta,t\right)}{\partial z^{(0\sim 1)}}\dot{z}^{(0\sim 1)}$$
$$+\frac{\partial l_2\left(z^{(0\sim 1)},x_3,\zeta,t\right)}{\partial \zeta}\dot{\zeta}+\frac{\partial l_2\left(z^{(0\sim 1)},x_3,\zeta,t\right)}{\partial t}$$
$$+\frac{\partial l_2\left(z^{(0\sim 1)},x_3,\zeta,t\right)}{\partial x_3}f_3(x_{1\sim 3},\zeta,t), \tag{B.146}$$

$$l_3\left(z^{(0\sim 2)},u,\zeta^{(0\sim 1)},t\right)=\frac{\partial l_2\left(z^{(0\sim 1)},x_3,\zeta,t\right)}{\partial x_3}g_3(x_{1\sim 3},u,\zeta,t), \tag{B.147}$$

and the variable x_3 in the above three equations should be substituted by the third equation in (B.144) , and then the above (B.146) gives the formula (8.9) for the case of $n=3$.

Finally, according to Assumption 8.1, we can obtain the nonsingularity of $\frac{\partial}{\partial x_2}g_1(x_{1\sim 2},\zeta,t)$, and $\frac{\partial}{\partial x_3}g_2(x_{1\sim 3},\zeta,t)$ and the definition of $l_3(z^{(0\sim 2)},u,\zeta^{(0\sim 1)},t)$, we know that system (B.145) is fully-actuated, thus the theorem holds in the case of $n=3$.

Step 2 Case of $n=k$

Now let us assume that the theorem holds when $n=k$. For convenience, we still denote u as x_{k+1}. In this case, the system takes the form of

$$\begin{cases}\dot{x}_1=f_1(x_1,\zeta,t)+g_1(x_{1\sim 2},\zeta,t)\\ \dot{x}_2=f_2(x_{1\sim 2},\zeta,t)+g_2(x_{1\sim 3},\zeta,t)\\ \quad\vdots\\ \dot{x}_k=f_k(x_{1\sim k},\zeta,t)+g_k(x_{1\sim k},x_{k+1},\zeta,t),\end{cases} \tag{B.148}$$

which, under the following transformation

$$\begin{cases}x_1=z\\ x_2=g_1^{-1}(z,y_1,\zeta,t),\ y_1=\dot{z}-f_1(z,\zeta,t)\\ x_{i+1}=l_i^{-1}(z^{(0\sim i-1)},y_i,\zeta^{(0\sim i-2)},t),\ y_i=z^{(i)}-h_i(z^{(0\sim i-1)},\zeta^{(0\sim i-1)},t),\\ \qquad i=2,3,\ldots,k-1,\end{cases} \tag{B.149}$$

can be expressed as the following high-order system form:

$$z^{(k)}=h_k(z^{(0\sim k-1)},\zeta^{(0\sim k-1)},t)+l_k(z^{(0\sim k-1)},x_{k+1},\zeta^{(0\sim k-2)},t), \tag{B.150}$$

where

$$l_k(z^{(0\sim k-1)},x_{k+1},\zeta^{(0\sim k-2},t)=\frac{\partial l_{k-1}\left(z^{(0\sim k-2)},x_k,\zeta^{(0\sim k-3)},t\right)}{\partial x_k}g_k(x_{1\sim k},x_{k+1},\zeta,t), \tag{B.151}$$

with $h_k(z^{(0\sim k-1)},\zeta^{(0\sim k-1)},t)$ given by (8.9). Now let us prove the conclusion for the case of $n=k+1$.

Step 3 Case of $n = k+1$

When $n = k+1$, noting that the first k formulas of the system are exactly system (B.148), system (8.1) can be equivalently rewritten as

$$\begin{cases} z^{(k)} = h_k(z^{(0\sim k-1)}, \zeta^{(0\sim k-1)}, t) + l_k(z^{(0\sim k-1)}, x_{k+1}, \zeta^{(0\sim k-2)}, t) \\ \dot{x}_{k+1} = f_{k+1}(x_{1\sim k+1}, \zeta, t) + g_{k+1}(x_{1\sim k+1}, u, \zeta, t). \end{cases} \tag{B.152}$$

It follows from the first equation in (B.152) that

$$x_{k+1} = l_k^{-1}(z^{(0\sim k-1)}, y_k, \zeta^{(0\sim k-2)}, t),\ y_k = z^{(k)} - h_k(z^{(0\sim k-1)}, \zeta^{(0\sim k-1)}, t). \tag{B.153}$$

Combining (B.153) with (B.149), gives the following transformation:

$$\begin{cases} x_1 = z \\ x_2 = g_1^{-1}(z, y_1, \zeta, t),\ y_1 = \dot{z} - f_1(z, \zeta, t) \\ x_{i+1} = l_i^{-1}(z^{(0\sim i-1)}, y_i, \zeta^{(0\sim i-2)}, t),\ y_i = z^{(i)} - h_i(z^{(0\sim i-1)}, \zeta^{(0\sim i-1)}, t), \\ \qquad i = 2, 3, \ldots, k. \end{cases} \tag{B.154}$$

Further, taking the derivative of both sides of the first equation in (B.152), and then substituting the second one into the result, give

$$\begin{aligned} z^{(k+1)} = {} & \dot{h}_k(z^{(0\sim k-1)}, \zeta^{(0\sim k-1)}, t) + \frac{\partial l_k(z^{(0\sim k-1)}, x_{k+1}, \zeta^{(0\sim k-2)}, t)}{\partial z^{(0\sim k-1)}} \dot{z}^{(0\sim k-1)} \\ & + \frac{\partial l_k(z^{(0\sim k-1)}, x_{k+1}, \zeta^{(0\sim k-2)}, t)}{\partial \zeta^{(0\sim k-2)}} \dot{\zeta}^{(0\sim k-2)} + \frac{\partial l_k(z^{(0\sim k-1)}, x_{k+1}, \zeta^{(0\sim k-2)}, t)}{\partial t} \\ & + \frac{\partial l_k(z^{(0\sim k-1)}, x_{k+1}, \zeta^{(0\sim k-2)}, t)}{\partial x_{k+1}} f_{k+1}(x_{1\sim k+1}, \zeta, t) \\ & + \frac{\partial l_k(z^{(0\sim k-1)}, x_{k+1}, \zeta^{(0\sim k-2)}, t)}{\partial x_{k+1}} g_{k+1}(x_{1\sim k+1}, u, \zeta, t). \end{aligned} \tag{B.155}$$

Then, substituting (B.154) and its derivative into (B.155), yields the following high-order system:

$$z^{(k+1)} = h_{k+1}(z^{(0\sim k)}, \zeta^{(0\sim k)}, t) + l_{k+1}(z^{(0\sim k)}, u, \zeta^{(0\sim k-1)}, t), \tag{B.156}$$

where

$$\begin{aligned} h_{k+1}(z^{(0\sim k)}, \zeta^{(0\sim k)}, t) = {} & \dot{h}_k(z^{(0\sim k-1)}, \zeta^{(0\sim k-1)}, t) \\ & + \frac{\partial l_k(z^{(0\sim k-1)}, x_{k+1}, \zeta^{(0\sim k-2)}, t)}{\partial z^{(0\sim k-1)}} \dot{z}^{(0\sim k-1)} \\ & + \frac{\partial l_k(z^{(0\sim k-1)}, x_{k+1}, \zeta^{(0\sim k-2)}, t)}{\partial \zeta^{(0\sim k-2)}} \dot{\zeta}^{(0\sim k-2)} \\ & + \frac{\partial l_k(z^{(0\sim k-1)}, x_{k+1}, \zeta^{(0\sim k-2)}, t)}{\partial t} \end{aligned}$$

$$+\frac{\partial l_k(z^{(0\sim k-1)},x_{k+1},\zeta^{(0\sim k-2)},t)}{\partial x_{k+1}}f_{k+1}(x_{1\sim k+1},\zeta,t), \tag{B.157}$$

$$l_{k+1}(z^{(0\sim k)},u,\zeta^{(0\sim k-1)},t)=\frac{\partial l_k(z^{(0\sim k-1)},x_{k+1},\zeta^{(0\sim k-2)},t)}{\partial x_{k+1}}g_{k+1}(x_{1\sim k+1},u,\zeta,t). \tag{B.158}$$

Finally, it is known from the nonsingularity of $\frac{\partial}{\partial x_{k+1}}g_k(x_{1\sim k+1},u,\zeta,t)$ and the definitions of the above $l_{k+1}(z^{(0\sim k)},u,\zeta^{(0\sim k-1)},t)$ that system (B.156) is fully-actuated. Thus the theorem also holds in the case of $n=k+1$. Therefore, the whole proof is completed.

B.5.2 Proof of Theorem 8.2

We use mathematical induction to prove this result.

Step 1 Case of $n=2$ and 3

Firstly, let us consider the case of $n=2$. For convenience, we still use the notation x_3 in the position of u. In this case, the system is

$$\begin{cases}\ddot{x}_1=f_1(x_1^{(0\sim 1)},\zeta,t)+g_1(x_1^{(0\sim 1)},x_2,\zeta,t)\\ \ddot{x}_2=f_2(x_{1\sim 2}^{(0\sim 1)},\zeta,t)+g_2(x_{1\sim 2}^{(0\sim 1)},x_3,\zeta,t).\end{cases} \tag{B.159}$$

From the first equation in (B.159), we can solve x_2 as follows:

$$x_2=g_1^{-1}(x_1^{(0\sim 1)},y_1,\zeta,t),\ y_1=\ddot{x}_1-f_1(x_1^{(0\sim 1)},\zeta,t). \tag{B.160}$$

Thus the following transformation can be obtained:

$$\begin{cases}x_1=z\\ x_2=g_1^{-1}(z^{(0\sim 1)},y_1,\zeta,t),\ y_1=\ddot{z}-\ f_1(z^{(0\sim 1)},\zeta,t),\end{cases} \tag{B.161}$$

which gives

$$\begin{cases}\dot{x}_1=\dot{z}\\ \dot{x}_2\triangleq d_1^{x_2}(z^{(0\sim 3)},\zeta^{(0\sim 1)},t),\end{cases} \tag{B.162}$$

with

$$d_1^{x_2}(z^{(0\sim 3)},\zeta^{(0\sim 1)},t)=\frac{dg_1^{-1}(z^{(0\sim 1)},y_1,\zeta,t)}{dt}.$$

Next, taking the derivative of both sides of the first equation in (B.159), gives

$$\begin{aligned}\dddot{z}=\dddot{x}_1&=\dot{f}_1(z^{(0\sim 1)},\zeta,t)+\dot{g}_1(z^{(0\sim 1)},x_2,\zeta,t)\\ &=\dot{f}_1(z^{(0\sim 1)},\zeta,t)+\frac{\partial g_1(z^{(0\sim 1)},x_2,\zeta,t)}{\partial z^{(0\sim 1)}}\dot{z}^{(0\sim 1)}+\frac{\partial g_1(z^{(0\sim 1)},x_2,\zeta,t)}{\partial\zeta}\dot{\zeta}\\ &\quad+\frac{\partial g_1(z^{(0\sim 1)},x_2,\zeta,t)}{\partial t}+\frac{\partial g_1(z^{(0\sim 1)},x_2,\zeta,t)}{\partial x_2}\dot{x}_2\\ &\triangleq\tilde{f}_1(z^{(0\sim 2)},\zeta^{(0\sim 1)},t)+\tilde{g}_1(z^{(0\sim 2)},\zeta,t)\dot{x}_2,\end{aligned}$$

where

$$\begin{aligned}\tilde{f}_1(z^{(0\sim 2)},\zeta^{(0\sim 1)},t) &= \dot{f}_1(z^{(0\sim 1)},\zeta,t)+\frac{\partial g_1(z^{(0\sim 1)},x_2,\zeta,t)}{\partial z^{(0\sim 1)}}\dot{z}^{(0\sim 1)}\\ &\quad+\frac{\partial g_1(z^{(0\sim 1)},x_2,\zeta,t)}{\partial \zeta}\dot{\zeta}+\frac{\partial g_1(z^{(0\sim 1)},x_2,\zeta,t)}{\partial t},\\ \tilde{g}_1(z^{(0\sim 2)},\zeta,t) &= \frac{\partial g_1(z^{(0\sim 1)},x_2,\zeta,t)}{\partial x_2}.\end{aligned}$$

Hence,

$$z^{(4)}=\dot{\tilde{f}}_1(z^{(0\sim 2)},\zeta^{(0\sim 1)},t)+\dot{\tilde{g}}_1(z^{(0\sim 2)},\zeta,t)\dot{x}_2+\tilde{g}_1(z^{(0\sim 2)},\zeta,t)\ddot{x}_2.$$

By using the second equation in (B.159), the above equation can be written as

$$\begin{aligned}z^{(4)} &= \dot{\tilde{f}}_1(z^{(0\sim 2)},\zeta^{(0\sim 1)},t)+\dot{\tilde{g}}_1(z^{(0\sim 2)},\zeta,t)\dot{x}_2\\ &\quad+\tilde{g}_1(z^{(0\sim 2)},\zeta,t)\,f_2(x_{1\sim 2}^{(0\sim 1)},\zeta,t)\\ &\quad+\tilde{g}_1(z^{(0\sim 2)},\zeta,t)g_2(x_{1\sim 2}^{(0\sim 1)},x_3,\zeta,t).\end{aligned}\tag{B.163}$$

Further, substituting (B.160) and (B.161) into equation (B.163), yields the following fourth-order fully-actuated system:

$$z^{(4)}=h_2(z^{(0\sim 3)},\zeta^{(0\sim 2)},t)+l_2(z^{(0\sim 3)},x_3,\zeta,t),\tag{B.164}$$

where

$$\begin{aligned}h_2(z^{(0\sim 3)},\zeta^{(0\sim 2)},t) &= \dot{\tilde{f}}_1(z^{(0\sim 2)},\zeta^{(0\sim 1)},t)+\dot{\tilde{g}}_1(z^{(0\sim 2)},\zeta,t)\dot{x}_2\\ &\quad+\tilde{g}_1(z^{(0\sim 2)},\zeta,t)\,f_2(x_{1\sim 2}^{(0\sim 1)},\zeta,t),\end{aligned}\tag{B.165}$$

and

$$\begin{aligned}l_2(z^{(0\sim 3)},x_3,\zeta,t) &= \tilde{g}_1(z^{(0\sim 2)},\zeta,t)g_2(x_{1\sim 2}^{(0\sim 1)},x_3,\zeta,t)\\ &= \frac{\partial g_1(z^{(0\sim 1)},x_2,\zeta,t)}{\partial x_2}g_2(x_{1\sim 2}^{(0\sim 1)},x_3,\zeta,t),\end{aligned}\tag{B.166}$$

and the variable x_2 and its derivative in the above equation are substituted with (B.161) and (B.162). Recalling that $h_1(z^{(0\sim 1)},\zeta,t)=f_1(x_1^{(0\sim 1)},\zeta,t)$ and $l_1(z^{(0\sim 1)},x_2,\zeta,t)=g_1(x_1^{(0\sim 1)},x_2,\zeta,t)$, we know that the above (B.165) gives the formula (8.38) for the case of $n=2$.

For clearness, let us also consider the case of $n=3$. For convenience, we still use the notation x_4 in the position of u. In this case, the system is

$$\begin{cases}\ddot{x}_1 = f_1(x_1^{(0\sim 1)},\zeta,t)+g_1(x_1^{(0\sim 1)},x_2,\zeta,t)\\ \ddot{x}_2 = f_2(x_{1\sim 2}^{(0\sim 1)},\zeta,t)+g_2(x_{1\sim 2}^{(0\sim 1)},x_3,\zeta,t)\\ \ddot{x}_3 = f_3(x_{1\sim 3}^{(0\sim 1)},\zeta,t)+g_3(x_{1\sim 3}^{(0\sim 1)},x_4,\zeta,t).\end{cases}\tag{B.167}$$

Then, according to the above process of $n = 2$, it is easy to know that the first two equations of system (B.167) are exactly system (B.164). Thus it is known from the assumptions that in this case system (B.167) can be expressed as

$$\begin{cases} z^{(4)} = h_2(z^{(0\sim 3)}, \zeta^{(0\sim 2)}, t) + l_2(z^{(0\sim 3)}, x_3, \zeta, t) \\ \ddot{x}_3 = f_3(x_{1\sim 3}^{(0\sim 1)}, \zeta, t) + g_3(x_{1\sim 3}^{(0\sim 1)}, x_4, \zeta, t). \end{cases} \tag{B.168}$$

It follows from the first equation in (B.168) that

$$x_3 = l_2^{-1}(z^{(0\sim 3)}, y_2, \zeta, t),\ y_2 = z^{(4)} - h_2(z^{(0\sim 3)}, \zeta^{(0\sim 2)}, t). \tag{B.169}$$

Hence the following transformation can be obtained:

$$\begin{cases} x_1 = z \\ x_2 = g_1^{-1}(z^{(0\sim 1)}, y_1, \zeta, t),\ y_1 = \ddot{z} - f_1(z^{(0\sim 1)}, \zeta, t) \\ x_3 = l_2^{-1}(z^{(0\sim 3)}, y_2, \zeta, t),\ y_2 = z^{(4)} - h_2(z^{(0\sim 3)}, \zeta^{(0\sim 2)}, t), \end{cases} \tag{B.170}$$

which further derives

$$\dot{x}_3 \triangleq d_1^{x_3}(z^{(0\sim 5)}, \zeta^{(0\sim 3)}, t), \tag{B.171}$$

with

$$d_1^{x_3}(z^{(0\sim 5)}, \zeta^{(0\sim 3)}, t) = \frac{dl_2^{-1}(z^{(0\sim 3)}, y_2, \zeta, t)}{dt}.$$

Next, taking the derivative of both sides of the first equation in (B.168) gives

$$\begin{aligned} z^{(5)} &= \dot{h}_2\left(z^{(0\sim 3)}, \zeta^{(0\sim 2)}, t\right) + \frac{\partial l_2\left(z^{(0\sim 3)}, x_3, \zeta, t\right)}{\partial z^{(0\sim 3)}}\dot{z}^{(0\sim 3)} + \frac{\partial l_2\left(z^{(0\sim 3)}, x_3, \zeta, t\right)}{\partial t} \\ &\quad + \frac{\partial l_2\left(z^{(0\sim 3)}, x_3, \zeta, t\right)}{\partial \zeta}\dot{\zeta} + \frac{\partial l_2\left(z^{(0\sim 3)}, x_3, \zeta, t\right)}{\partial x_3}\dot{x}_3 \\ &\triangleq \tilde{f}_2(z^{(0\sim 4)}, \zeta^{(0\sim 3)}, t) + \tilde{g}_2(z^{(0\sim 4)}, \zeta^{(0\sim 2)}, t)\dot{x}_3, \end{aligned} \tag{B.172}$$

where

$$\begin{aligned} \tilde{f}_2(z^{(0\sim 4)}, \zeta^{(0\sim 3)}, t) &= \dot{h}_2\left(z^{(0\sim 3)}, \zeta^{(0\sim 2)}, t\right) + \frac{\partial l_2\left(z^{(0\sim 3)}, x_3, \zeta, t\right)}{\partial z^{(0\sim 3)}}\dot{z}^{(0\sim 3)} \\ &\quad + \frac{\partial l_2\left(z^{(0\sim 3)}, x_3, \zeta, t\right)}{\partial t} + \frac{\partial l_2\left(z^{(0\sim 3)}, x_3, \zeta, t\right)}{\partial \zeta}\dot{\zeta}, \end{aligned} \tag{B.173}$$

$$\tilde{g}_2(z^{(0\sim 4)}, \zeta^{(0\sim 2)}, t) = \frac{\partial l_2\left(z^{(0\sim 3)}, x_3, \zeta, t\right)}{\partial x_3}. \tag{B.174}$$

Hence,

$$z^{(6)} = \dot{\tilde{f}}_2(z^{(0\sim 4)}, \zeta^{(0\sim 3)}, t) + \dot{\tilde{g}}_2(z^{(0\sim 4)}, \zeta^{(0\sim 2)}, t)\dot{x}_3 + \tilde{g}_2(z^{(0\sim 4)}, \zeta^{(0\sim 2)}, t)\ddot{x}_3.$$

Substituting the second equation in (B.168) into the above equation, gives

$$\begin{aligned} z^{(6)} = {} & \dot{\tilde{f}}_2(z^{(0\sim 4)}, \zeta^{(0\sim 3)}, t) + \dot{\tilde{g}}_2(z^{(0\sim 4)}, \zeta^{(0\sim 2)}, t)\dot{x}_3 \\ & + \tilde{g}_2(z^{(0\sim 4)}, \zeta^{(0\sim 2)}, t)\, f_3(x_{1\sim 3}^{(0\sim 1)}, \zeta, t) \\ & + \tilde{g}_2(z^{(0\sim 4)}, \zeta^{(0\sim 2)}, t) g_3(x_{1\sim 3}^{(0\sim 1)}, x_4, \zeta, t), \end{aligned} \tag{B.175}$$

where $x_i,\ i = 1,2,3$, and their derivatives are given by (B.170) and (B.171). Using equation (B.174) and taking

$$l_3(z^{(0\sim5)}, x_4, \zeta^{(0\sim2)}, t) = \tilde{g}_2(z^{(0\sim4)}, \zeta^{(0\sim2)}, t) g_3(x_{1\sim3}^{(0\sim1)}, x_4, \zeta, t),$$

and $h_3(z^{(0\sim5)}, \zeta^{(0\sim4)}, t)$ as in (8.38) with $k = 3$, the high-order system can be obtained as

$$z^{(6)} = h_3(z^{(0\sim5)}, \zeta^{(0\sim4)}, t) + l_3(z^{(0\sim5)}, x_4, \zeta^{(0\sim2)}, t). \tag{B.176}$$

Finally, recalling the nonsingularity of $g_3(x_{1\sim3}^{(0\sim1)}, x_4, \zeta, t)$ and $\tilde{g}_2(z^{(0\sim4)}, \zeta^{(0\sim2)}, t)$, and the definition of $l_3(z^{(0\sim5)}, x_4, \zeta^{(0\sim2)}, t)$, we know that system (B.176) is fully-actuated, thus the theorem holds in the case of $n = 3$.

Step 2 Case of $n = k$

Now let us assume that the theorem holds when $n = k$. For convenience, we still denote u as x_{k+1}, in this case the system takes the form of

$$\begin{cases} \ddot{x}_1 = f_1(x_1^{(0\sim1)}, \zeta, t) + g_1(x_1^{(0\sim1)}, x_2, \zeta, t) \\ \ddot{x}_2 = f_2(x_{1\sim2}^{(0\sim1)}, \zeta, t) + g_2(x_{1\sim2}^{(0\sim1)}, x_3, \zeta, t) \\ \qquad\vdots \\ \ddot{x}_k = f_k(x_{1\sim k}^{(0\sim1)}, \zeta, t) + g_k(x_{1\sim k}^{(0\sim1)}, x_{k+1}, \zeta, t), \end{cases} \tag{B.177}$$

which, under the following transformation:

$$\begin{cases} x_1 = z \\ x_2 = g_1^{-1}(z^{(0\sim1)}, y_1, \zeta, t),\ y_1 = \ddot{z} - f_1(z^{(0\sim1)}, \zeta, t) \\ x_{i+1} = l_i^{-1}(z^{(0\sim2i-1)}, y_i, \zeta^{(0\sim2i-4)}, t),\ y_i = z^{(2i)} - h_i(z^{(0\sim2i-1)}, \zeta^{(0\sim2i-2)}, t), \\ \qquad\qquad i = 2,3,\ldots,k-1, \end{cases} \tag{B.178}$$

can be rewritten into

$$z^{(2k)} = h_k(z^{(0\sim2k-1)}, \zeta^{(0\sim2k-2)}, t) + l_k(z^{(0\sim2k-1)}, x_{k+1}, \zeta^{(0\sim2k-4)}, t), \tag{B.179}$$

where

$$\begin{aligned} & h_k(z^{(0\sim2k-1)}, \zeta^{(0\sim2k-2)}, t) \\ = \ & \dot{\tilde{f}}_{k-1}(z^{(0\sim2k-2)}, \zeta^{(0\sim2k-3)}, t) \\ & + \dot{\tilde{g}}_{k-1}(z^{(0\sim2k-2)}, \zeta^{(0\sim2k-4)}, t)\dot{x}_k \\ & + \tilde{g}_{k-1}(z^{(0\sim2k-2)}, \zeta^{(0\sim2k-4)}, t)\, f_k(x_{1\sim k}^{(0\sim1)}, \zeta, t), \end{aligned} \tag{B.180}$$

$$\begin{aligned} & l_k(z^{(0\sim2k-1)}, x_{k+1}, \zeta^{(0\sim2k-4)}, t) \\ = \ & \tilde{g}_{k-1}(z^{(0\sim2k-2)}, \zeta^{(0\sim2k-4)}, t) g_k(x_{1\sim k}^{(0\sim1)}, x_{k+1}, \zeta, t), \end{aligned} \tag{B.181}$$

with

$$\begin{aligned}
&\tilde{f}_{k-1}(z^{(0\sim 2k-2)},\zeta^{(0\sim 2k-3)},t)\\
=\;&\dot{h}_{k-1}(z^{(0\sim 2k-3)},\zeta^{(0\sim 2k-4)},t) \qquad\text{(B.182)}\\
&+\frac{\partial l_{k-1}(z^{(0\sim 2k-3)},x_k,\zeta^{(0\sim 2k-6)},t)}{\partial t}\\
&+\frac{\partial l_{k-1}(z^{(0\sim 2k-3)},x_k,\zeta^{(0\sim 2k-6)},t)}{\partial z^{(0\sim 2k-3)}}\dot{z}^{(0\sim 2k-3)}\\
&+\frac{\partial l_{k-1}(z^{(0\sim 2k-3)},x_k,\zeta^{(0\sim 2k-6)},t)}{\partial \zeta^{(0\sim 2k-6)}}\dot{\zeta}^{(0\sim 2k-6)}, \qquad\text{(B.183)}\\
&\tilde{g}_{k-1}(z^{(0\sim 2k-2)},\zeta^{(0\sim 2k-4)},t)\\
=\;&\frac{\partial l_{k-1}(z^{(0\sim 2k-3)},x_k,\zeta^{(0\sim 2k-6)},t)}{\partial x_k}. \qquad\text{(B.184)}
\end{aligned}$$

Let us now prove the conclusion for the case of $n=k+1$.

Step 3 Case of $n=k+1$

When $n=k+1$, it is easy to know that the first k equations of the system are exactly system (B.179). Thus it is known from the assumptions that in this case the system can be expressed as

$$\begin{cases} z^{(2k)}=h_k(z^{(0\sim 2k-1)},\zeta^{(0\sim 2k-2)},t)+l_k(z^{(0\sim 2k-1)},x_{k+1},\zeta^{(0\sim 2k-4)},t)\\ \ddot{x}_{k+1}=f_{k+1}(x_{1\sim k+1}^{(0\sim 1)},\zeta,t)+g_{k+1}(x_{1\sim k+1}^{(0\sim 1)},u,\zeta,t). \end{cases} \qquad\text{(B.185)}$$

It follows from the first equation in (B.185) that

$$x_{k+1}=l_k^{-1}(z^{(0\sim 2k-1)},y_k,\zeta^{(0\sim 2k-4)},t),\ y_k=z^{(2k)}-h_k(z^{(0\sim 2k-1)},\zeta^{(0\sim 2k-2)},t). \qquad\text{(B.186)}$$

Combining the above equation with (B.178), yields

$$\begin{cases} x_1=z\\ x_2=g_1^{-1}(z^{(0\sim 1)},y_1,\zeta,t),\ y_1=\ddot{z}-f_1(z^{(0\sim 1)},\zeta,t)\\ x_{i+1}=l_i^{-1}(z^{(0\sim 2i-1)},y_i,\zeta^{(0\sim 2i-4)},t),\ y_i=z^{(2i)}-h_i(z^{(0\sim 2i-1)},\zeta^{(0\sim 2i-2)},t),\\ \qquad\qquad i=2,3,\ldots,k. \end{cases} \qquad\text{(B.187)}$$

Further, taking the derivative of both sides of the first equation in (B.185), gives

$$\begin{aligned}
z^{(2k+1)}&=\dot{h}_k(z^{(0\sim 2k-1)},\zeta^{(0\sim 2k-2)},t)+\dot{l}_k(z^{(0\sim 2k-1)},x_{k+1},\zeta^{(0\sim 2k-4)},t)\\
&=\dot{h}_k(z^{(0\sim 2k-1)},\zeta^{(0\sim 2k-2)},t)+\frac{\partial l_k(z^{(0\sim 2k-1)},x_{k+1},\zeta^{(0\sim 2k-4)},t)}{\partial z^{(0\sim 2k-1)}}\dot{z}^{(0\sim 2k-1)}\\
&\quad+\frac{\partial l_k(z^{(0\sim 2k-1)},x_{k+1},\zeta^{(0\sim 2k-4)},t)}{\partial \zeta^{(0\sim 2k-4)}}\dot{\zeta}^{(0\sim 2k-4)}\\
&\quad+\frac{\partial l_k(z^{(0\sim 2k-1)},x_{k+1},\zeta^{(0\sim 2k-4)},t)}{\partial t}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\partial l_k(z^{(0\sim 2k-1)}, x_{k+1}, \zeta^{(0\sim 2k-4)}, t)}{\partial x_{k+1}}\dot{x}_{k+1}\\
&\triangleq \tilde{f}_k(z^{(0\sim 2k)}, \zeta^{(0\sim 2k-1)}, t)+\tilde{g}_k(z^{(0\sim 2k)}, \zeta^{(0\sim 2k-2)}, t)\dot{x}_{k+1},
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{f}_k(z^{(0\sim 2k)}, \zeta^{(0\sim 2k-1)}, t) &= \dot{h}_k(z^{(0\sim 2k-1)}, \zeta^{(0\sim 2k-2)}, t)+\frac{\partial l_k(z^{(0\sim 2k-1)}, x_{k+1}, \zeta^{(0\sim 2k-4)}, t)}{\partial t}\\
&\quad+\frac{\partial l_k(z^{(0\sim 2k-1)}, x_{k+1}, \zeta^{(0\sim 2k-4)}, t)}{\partial z^{(0\sim 2k-1)}}\dot{z}^{(0\sim 2k-1)}\\
&\quad+\frac{\partial l_k(z^{(0\sim 2k-1)}, x_{k+1}, \zeta^{(0\sim 2k-4)}, t)}{\partial \zeta^{(0\sim 2k-4)}}\dot{\zeta}^{(0\sim 2k-4)},\\
\tilde{g}_k(z^{(0\sim 2k)}, \zeta^{(0\sim 2k-2)}, t) &= \frac{\partial l_k(z^{(0\sim 2k-1)}, x_{k+1}, \zeta^{(0\sim 2k-4)}, t)}{\partial x_{k+1}}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
z^{(2k+2)} \;=\;& \dot{\tilde{f}}_k(z^{(0\sim 2k)}, \zeta^{(0\sim 2k-1)}, t)+\dot{\tilde{g}}_k(z^{(0\sim 2k)}, \zeta^{(0\sim 2k-2)}, t)\dot{x}_{k+1}\\
&+\tilde{g}_k(z^{(0\sim 2k)}, \zeta^{(0\sim 2k-2)}, t)\ddot{x}_{k+1}.
\end{aligned}
$$

Substituting the second equation in (B.185) into the above equation, gives

$$
\begin{aligned}
z^{(2k+2)} =\;& \dot{\tilde{f}}_k(z^{(0\sim 2k)}, \zeta^{(0\sim 2k-1)}, t)+\dot{\tilde{g}}_k(z^{(0\sim 2k)}, \zeta^{(0\sim 2k-2)}, t)\dot{x}_{k+1}\\
&+\tilde{g}_k(z^{(0\sim 2k)}, \zeta^{(0\sim 2k-2)}, t)\, f_{k+1}(x_{1\sim k+1}^{(0\sim 1)}, \zeta, t)\\
&+\tilde{g}_k(z^{(0\sim 2k)}, \zeta^{(0\sim 2k-2)}, t) g_{k+1}(x_{1\sim k+1}^{(0\sim 1)}, u, \zeta, t), \qquad \text{(B.188)}
\end{aligned}
$$

where $x_i,\ i=2,3,\ldots,k$, and their derivatives are given or determined by (B.187). Taking

$$
\begin{aligned}
&h_{k+1}(z^{(0\sim 2k+1)}, \zeta^{(0\sim 2k)}, t)\\
=\;& \dot{\tilde{f}}_k(z^{(0\sim 2k)}, \zeta^{(0\sim 2k-1)}, t)+\dot{\tilde{g}}_k(z^{(0\sim 2k)}, \zeta^{(0\sim 2k-2)}, t)\dot{x}_{k+1}\\
&+\tilde{g}_k(z^{(0\sim 2k)}, \zeta^{(0\sim 2k-2)}, t)\, f_{k+1}(x_{1\sim k+1}^{(0\sim 1)}, \zeta, t), \qquad \text{(B.189)}\\
&l_{k+1}(z^{(0\sim 2k+1)}, u, \zeta^{(0\sim 2k-2)}, t)\\
=\;& \tilde{g}_k(z^{(0\sim 2k)}, \zeta^{(0\sim 2k-2)}, t) g_{k+1}(x_{1\sim k+1}^{(0\sim 1)}, u, \zeta, t), \qquad \text{(B.190)}
\end{aligned}
$$

the high-order system can be obtained as

$$
z^{(2k+2)} = h_{k+1}(z^{(0\sim 2k+1)}, \zeta^{(0\sim 2k)}, t)+l_{k+1}(z^{(0\sim 2k+1)}, u, \zeta^{(0\sim 2k-2)}, t). \qquad \text{(B.191)}
$$

Finally, due to the nonsingularity of $\tilde{g}_k(z^{(0\sim 2k)}, \zeta^{(0\sim 2k-2)}, t)$, and $g_{k+1}(x_{1\sim k+1}^{(0\sim 1)}, u, \zeta, t),\ k=1,2,\ldots,n$, and the above definitions of $l_{k+1}(z^{(0\sim 2k+1)}, u, \zeta^{(0\sim 2k-2)}, t)$, system (B.191) is clearly fully-actuated. Thus the theorem is also true in the case of $n=k+1$. Therefore, the whole proof is completed.

B.5.3 Proof of Theorem 8.3

We again use mathematical induction to prove the result.

Step 1 Case of $n = 2$ and 3

Firstly, let us consider the case of $n = 2$. In this case, system (8.69) is

$$\begin{cases} x_1^{(m_1)} = f_1(x_1^{(0\sim m_1-1)}, \zeta, t) + g_1(x_1^{(0\sim m_1-1)}, x_2, \zeta, t) \\ x_2^{(m_2)} = f_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2}, \zeta, t) + g_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2}, u, \zeta, t). \end{cases} \tag{B.192}$$

Based on the Assumption 8.5, we can solve x_2 from the first equation in (B.192) as

$$x_2 = g_1^{-1}(x_1^{(0\sim m_1-1)}, y_1, \zeta, t),\ y_1 = z^{(m_1)} - f_1(z^{(0\sim m_1-1)}, \zeta, t), \tag{B.193}$$

Thus, the following state transformation can be obtained:

$$\begin{cases} x_1 = z \\ x_2 = l_1^{-1}(z^{(0\sim m_1-1)}, y_1, \zeta, t),\ y_1 = z^{(m_1)} - h_1(z^{(0\sim m_1-1)}, \zeta, t), \end{cases} \tag{B.194}$$

which further gives

$$\begin{cases} x_1^{(i)} = z^{(i)}, \quad i = 1, 2, \ldots, m_1 - 1 \\ x_2^{(i)} = \frac{\mathrm{d}^i}{\mathrm{d}t^i} l_1^{-1}(z^{(0\sim m_1-1)}, y_1, \zeta, t), \quad i = 1, 2, \ldots, m_2 - 1, \end{cases} \tag{B.195}$$

Taking the first-order derivative of the first equation in (B.192), yields

$$\begin{aligned} x_1^{(m_1+1)} &= \dot{f}_1(x_1^{(0\sim m_1-1)}, \zeta, t) + \dot{g}_1(x_1^{(0\sim m_1-1)}, x_2, \zeta, t) \\ &= \dot{f}_1(x_1^{(0\sim m_1-1)}, \zeta, t) + \sum_{i=0}^{m_1-1} \frac{\partial g_1(x_1^{(0\sim m_1-1)}, x_2, \zeta, t)}{\partial x_1^{(i)}} x_1^{(i+1)} \\ &\quad + \frac{\partial g_1(x_1^{(0\sim m_1-1)}, x_2, \zeta, t)}{\partial x_2} \dot{x}_2 + \frac{\partial g_1(x_1^{(0\sim m_1-1)}, x_2, \zeta, t)}{\partial t} \\ &= F_1(x_1^{(0\sim m_1)}, x_2, \zeta, t) + \frac{\partial g_1(x_1^{(0\sim m_1-1)}, x_2, \zeta, t)}{\partial x_2} \dot{x}_2, \end{aligned} \tag{B.196}$$

where

$$\begin{aligned} F_1(x_1^{(0\sim m_1)}, x_2, \zeta, t) = \dot{f}_1(x_1^{(0\sim m_1-1)}, \zeta, t) &+ \frac{\partial g_1(x_1^{(0\sim m_1-1)}, x_2, \zeta, t)}{\partial t} \\ &+ \sum_{i=0}^{m_1-1} \frac{\partial g_1(x_1^{(0\sim m_1-1)}, x_2, \zeta, t)}{\partial x_1^{(i)}} x_1^{(i+1)}. \end{aligned}$$

Taking the (m_2-1)-order derivatives of both sides of (B.196), and then substituting the second one of (B.192) into the result, give

$$\begin{aligned}
x_1^{(p_2)} &= F_1^{(m_2-1)}(x_1^{(0\sim m_1)}, x_2, \zeta, t) + \sum_{j=0}^{m_2-1} C(j, m_2-1) \\
&\quad \times \left(\frac{\partial g_1(x_1^{(0\sim m_1-1)}, x_2, \zeta, t)}{\partial x_2} \right)^{(m_2-1-j)} x_2^{(j+1)} \\
&= F_1^{(m_2-1)}(x_1^{(0\sim m_1)}, x_2, \zeta, t) + \sum_{j=0}^{m_2-2} C(j, m_2-1) \\
&\quad \times \left(\frac{\partial g_1(x_1^{(0\sim m_1-1)}, x_2, \zeta, t)}{\partial x_2} \right)^{(m_2-1-j)} x_2^{(j+1)} \\
&\quad + \frac{\partial g_1(x_1^{(0\sim m_1-1)}, x_2, \zeta, t)}{\partial x_2} \Big(f_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2}, \zeta, t) \\
&\quad + g_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2}, u, \zeta, t) \Big),
\end{aligned} \tag{B.197}$$

which produces the following high-order FAS:

$$z^{(p_2)} = h_2(z^{(0\sim p_2-1)}, \zeta, t) + l_2(z^{(0\sim p_2-1)}, u, \zeta, t), \tag{B.198}$$

where

$$\begin{aligned}
h_2(z^{(0\sim p_2-1)}, \zeta, t) &= F_1^{(m_2-1)}(x_1^{(0\sim m_1)}, x_2, \zeta, t) + \sum_{j=0}^{m_2-2} C(j, m_2-1) \\
&\quad \times \left(\frac{\partial}{\partial x_2} g_1(x_1^{(0\sim m_1-1)}, x_2, \zeta, t) \right)^{(m_2-1-j)} x_2^{(j+1)} \\
&\quad + \frac{\partial g_1(x_1^{(0\sim m_1-1)}, x_2, \zeta, t)}{\partial x_2} \\
&\quad \times f_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2}, \zeta, t),
\end{aligned} \tag{B.199}$$

$$\begin{aligned}
l_2(z^{(0\sim p_2-1)}, \zeta, u, t) &= \frac{\partial g_1(x_1^{(0\sim m_1-1)}, x_2, \zeta, t)}{\partial x_2} \\
&\quad \times g_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2}, u, \zeta, t).
\end{aligned} \tag{B.200}$$

Further, in view of

$$\begin{cases}
h_1(z^{(0\sim m_1-1)}, \zeta, t) = f_1(x_1^{(0\sim m_1-1)}, \zeta, t) \\
l_1(z^{(0\sim m_1-1)}, x_2, \zeta, t) = g_1(x_1^{(0\sim m_1-1)}, x_2, \zeta, t),
\end{cases}$$

the above (B.198) turns out to be (8.78) for the case of $n = 2$.

For clearness, let us also consider the case of $n = 3$. Similarly, in this case system (8.69) is

$$\begin{cases} x_1^{(m_1)} = f_1(x_1^{(0\sim m_1-1)}, \zeta, t) + g_1(x_1^{(0\sim m_1-1)}, x_2, \zeta, t) \\ x_2^{(m_2)} = f_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2}, \zeta, t) + g_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2}, x_3, \zeta, t) \\ x_3^{(m_3)} = f_3(x_i^{(0\sim m_i-1)}|_{i=1\sim 3}, \zeta, t) + g_3(x_i^{(0\sim m_i-1)}|_{i=1\sim 3}, u, \zeta, t). \end{cases} \tag{B.201}$$

By replacing the u with the variable x_3 in equation (B.198), it follows that

$$\begin{cases} z^{(p_2)} = h_2(z^{(0\sim p_2-1)}, \zeta, t) + l_2(z^{(0\sim p_2-1)}, x_3, \zeta, t) \\ x_3^{(m_3)} = f_3(x_i^{(0\sim m_i-1)}|_{i=1\sim 3}, \zeta, t) + g_3(x_i^{(0\sim m_i-1)}|_{i=1\sim 3}, u, \zeta, t). \end{cases} \tag{B.202}$$

From the first equation in (B.202), we can solve x_3 as

$$x_3 = l_2^{-1}(z^{(0\sim p_2-1)}, y_2, \zeta, t),\ y_2 = z^{(p_2)} - h_2(z^{(0\sim p_2-1)}, \zeta, t). \tag{B.203}$$

Thus the following transformation can be obtained:

$$\begin{cases} x_1 = z \\ x_2 = l_1^{-1}(z^{(0\sim m_1-1)}, y_1, \zeta, t),\ y_1 = z^{(m_1)} - h_1(z^{(0\sim m_1-1)}, \zeta, t) \\ x_3 = l_2^{-1}(z^{(0\sim p_2-1)}, y_2, \zeta, t),\ y_2 = z^{(p_2)} - h_2(z^{(0\sim p_2-1)}, \zeta, t), \end{cases} \tag{B.204}$$

which further gives

$$\begin{cases} x_1^{(i)} = z^{(i)}, \quad i = 1, 2, \ldots, m_1 - 1 \\ x_2^{(i)} = \frac{\mathrm{d}^i}{\mathrm{d}t^i} l_1^{-1}(z^{(0\sim m_1-1)}, y_1, \zeta, t), \quad i = 1, 2, \ldots, m_2 - 1 \\ x_3^{(i)} = \frac{\mathrm{d}^i}{\mathrm{d}t^i} l_2^{-1}(z^{(0\sim p_2-1)}, y_2, \zeta, t), \quad i = 1, 2, \ldots, m_3 - 1, \end{cases} \tag{B.205}$$

Taking the first-order derivative of the first equation in (B.202), yields

$$\begin{aligned} z^{(p_2+1)} &= \dot{h}_2(z^{(0\sim p_2-1)}, \zeta, t) + \dot{l}_2(z^{(0\sim p_2-1)}, x_3, \zeta, t) \\ &= \dot{h}_2(z^{(0\sim p_2-1)}, \zeta, t) + \frac{\partial l_2(z^{(0\sim p_2-1)}, x_3, \zeta, t)}{\partial z^{(0\sim p_2-1)}} z^{(1\sim p_2)} \\ &\quad + \frac{\partial l_2(z^{(0\sim p_2-1)}, x_3, \zeta, t)}{\partial x_3} \dot{x}_3 + \frac{\partial l_2(z^{(0\sim p_2-1)}, x_3, \zeta, t)}{\partial t} \\ &= F_2(z^{(0\sim p_2)}, x_3, \zeta, t) + \frac{\partial l_2(z^{(0\sim p_2-1)}, x_3, \zeta, t)}{\partial x_3} \dot{x}_3, \end{aligned} \tag{B.206}$$

where

$$\begin{aligned} F_2(z^{(0\sim p_2)}, x_3, \zeta, t) = \dot{h}_2(z^{(0\sim p_2-1)}, \zeta, t) &+ \frac{\partial l_2(z^{(0\sim p_2-1)}, x_3, \zeta, t)}{\partial t} \\ &+ \frac{\partial l_2(z^{(0\sim p_2-1)}, x_3, \zeta, t)}{\partial z^{(0\sim p_2-1)}} z^{(1\sim p_2)}. \end{aligned} \tag{B.207}$$

Taking the (m_3-1)-order derivatives of both sides of (B.206), gives

$$\begin{aligned} z^{(p_3)} &= F_2^{(m_3-1)}(z^{(0\sim p_2)}, x_3, \zeta, t) + \sum_{j=0}^{m_3-1} C(j, m_3-1) \\ &\quad \times \left(\frac{\partial l_2(z^{(0\sim p_2-1)}, x_3, \zeta, t)}{\partial x_3}\right)^{(m_3-1-j)} x_3^{(j+1)} \\ &= F_2^{(m_3-1)}(z^{(0\sim p_2)}, x_3, \zeta, t) + \sum_{j=0}^{m_3-2} C(j, m_3-1) \\ &\quad \times \left(\frac{\partial l_2(z^{(0\sim p_2-1)}, x_3, \zeta, t)}{\partial x_3}\right)^{(m_3-1-j)} x_3^{(j+1)} \\ &\quad + \frac{\partial l_2(z^{(0\sim p_2-1)}, x_3, \zeta, t)}{\partial x_3} x_3^{(m_3)}. \end{aligned} \tag{B.208}$$

Substituting the second one of (B.202) into the above result, produces

$$z^{(p_3)} = h_3(z^{(0\sim p_3-1)}, \zeta, t) + l_3(z^{(0\sim p_3-1)}, u, \zeta, t), \tag{B.209}$$

where

$$\begin{aligned} h_3(z^{(0\sim p_3-1)}, \zeta, t) &= F_2^{(m_3-1)}(z^{(0\sim p_2)}, x_3, \zeta, t) + \sum_{j=0}^{m_3-2} C(j, m_3-1) \\ &\quad \times \left(\frac{\partial l_2(z^{(0\sim p_2-1)}, x_3, \zeta, t)}{\partial x_3}\right)^{(m_3-1-j)} x_3^{(j+1)} \\ &\quad + \frac{\partial l_2(z^{(0\sim p_2-1)}, x_3, \zeta, t)}{\partial x_3} f_3(x_i^{(0\sim m_i)}|_{i=1\sim 3}, \zeta, t), \end{aligned}$$

$$l_3(z^{(0\sim p_3-1)}, u, \zeta, t) = \frac{\partial l_2(z^{(0\sim p_2-1)}, x_3, \zeta, t)}{\partial x_3} g_3(x_i^{(0\sim m_i-1)}|_{i=1\sim 3}, u, \zeta, t).$$

Finally, recalling the nonsingularity of $\frac{\partial}{\partial x_3} g_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2}, x_3, \zeta, t)$ and the definition of $l_3(z^{(0\sim p_3-1)}, u, \zeta, t)$, system (B.209) is fully-actuated. Thus the theorem holds when $n = 3$.

Step 2 Case of $n = k$

Now assume that the theorem holds when $n = k$. Again, for convenience, we still denote u as x_{k+1}, in this case the system is

$$\begin{cases} x_1^{(m_1)} = f_1(x_1^{(0\sim m_1-1)}, \zeta, t) + g_1(x_1^{(0\sim m_1-1)}, x_2, \zeta, t) \\ x_2^{(m_2)} = f_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2}, \zeta, t) + g_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2}, x_3, \zeta, t) \\ \quad \vdots \\ x_k^{(m_k)} = f_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k}, \zeta, t) + g_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k}, x_{k+1}, \zeta, t), \end{cases} \tag{B.210}$$

which, under the following transformation:

$$\begin{cases} x_1 = z \\ x_2 = l_1^{-1}(z^{(0\sim m_1-1)}, \zeta, t),\ y_1 = z^{(m_1)} - h_1(z^{(0\sim m_1-1)}, \zeta, t) \\ x_{i+1} = l_i^{-1}(z^{(0\sim p_i-1)}, \zeta, t),\ y_i = z^{(p_i)} - h_i(z^{(0\sim p_i-1)}, \zeta, t), \\ \qquad i = 2, 3, \ldots, k-1, \end{cases} \tag{B.211}$$

can be expressed as the following FAS:

$$z^{(p_k)} = h_k(z^{(0\sim p_k-1)}, \zeta, t) + l_k(z^{(0\sim p_k-1)}, x_{k+1}, \zeta, t), \tag{B.212}$$

where

$$\begin{aligned} h_k(z^{(0\sim p_k-1)}, \zeta, t) = F_{k-1}^{(m_k-1)}(z^{(0\sim p_{k-1})}, x_k, \zeta, t) + \sum_{j=0}^{m_k-2} C(j, m_k-1) \\ \times \left(\frac{\partial l_{k-1}(z^{(0\sim p_{k-1}-1)}, x_k, \zeta, t)}{\partial x_k} \right)^{(m_k-1-j)} x_k^{(j+1)} \\ + \frac{\partial l_{k-1}(z^{(0\sim p_{k-1}-1)}, x_k, \zeta, t)}{\partial x_k} f_k(x_i^{(0\sim m_i)}|_{i=1\sim k}, \zeta, t), \end{aligned}$$

and

$$\begin{aligned} & l_k(z^{(0\sim p_{k-1}-1)}, x_{k+1}, \zeta, t) \\ = & \frac{\partial l_{k-1}(z^{(0\sim p_{k-1}-1)}, x_k, \zeta, t)}{\partial x_k} g_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k}, x_{k+1}, \zeta, t), \end{aligned}$$

with

$$\begin{aligned} & F_{k-1}(z^{(0\sim p_{k-1})}, x_k, \zeta, t) \\ = & \dot{h}_{k-1}(z^{(0\sim p_{k-1}-1)}, \zeta, t) + \frac{\partial l_{k-1}(z^{(0\sim p_{k-1}-1)}, x_k, \zeta, t)}{\partial t} \\ & + \frac{\partial l_{k-1}(z^{(0\sim p_{k-1}-1)}, x_k, \zeta, t)}{\partial z^{(0\sim p_{k-1}-1)}} z^{(1\sim p_{k-1})}. \end{aligned}$$

Now let us prove that the conclusion is also true when $n = k+1$.

Step 3 Case of $n = k+1$

When $n = k+1$, the first k formulas of the system are exactly system (B.212). Thus it is known from the assumptions that the system is equivalent to

$$\begin{cases} z^{(p_k)} = h_k(z^{(0\sim p_k-1)}, \zeta, t) + l_k(z^{(0\sim p_k-1)}, x_{k+1}, \zeta, t) \\ x_{k+1}^{(m_{k+1})} = f_{k+1}(x_i^{(0\sim m_i-1)}|_{i=1\sim k+1}, \zeta, t) \\ \qquad\qquad + g_{k+1}(x_i^{(0\sim m_i-1)}|_{i=1\sim k+1}, u, \zeta, t). \end{cases} \tag{B.213}$$

From the first equation in (B.213), we can obtain

$$x_{k+1} = l_k^{-1}(z^{(0\sim p_k-1)}, y_k, \zeta, t),\ y_k = z^{(p_k)} - h_k(z^{(0\sim p_k-1)}, \zeta, t). \tag{B.214}$$

Combining the above equation with (B.211), gives the following transformation:

$$\begin{cases} x_1 = z \\ x_2 = l_1^{-1}(z^{(0\sim m_1-1)}, y_1, \zeta, t),\ y_1 = z^{(m_1)} - f_1(z^{(0\sim m_1-1)}, \zeta, t) \\ x_{j+1} = l_j^{-1}(z^{(0\sim p_j-1)}, \zeta, t),\ y_j = z^{(p_j)} - h_j(z^{(0\sim p_j-1)}, \zeta, t), \\ \qquad j = 2, 3, \ldots, k, \end{cases} \tag{B.215}$$

which further gives

$$\begin{cases} x_1^{(i)} = z^{(i)}, \quad i = 1, 2, \ldots, m_1 - 1 \\ x_{j+1}^{(i)} = \frac{\mathrm{d}^i}{\mathrm{d}t^i} l_j^{-1}(z^{(0\sim p_j-1)}, \zeta, t), \quad i = 1, 2, \ldots, m_j - 1,\ j = 1, 2, \ldots, k. \end{cases} \tag{B.216}$$

Taking the first-order derivative of both sides of the first equation in (B.213), gives

$$\begin{aligned} z^{(p_k+1)} &= \dot{h}_k(z^{(0\sim p_k-1)}, \zeta, t) + \dot{l}_k(z^{(0\sim p_k-1)}, x_{k+1}, \zeta, t) \\ &= \dot{h}_k(z^{(0\sim p_k-1)}, \zeta, t) + \frac{\partial l_k(z^{(0\sim p_k-1)}, x_{k+1}, \zeta, t)}{\partial z^{(0\sim p_k-1)}} z^{(1\sim p_k)} \\ &\quad + \frac{\partial l_k(z^{(0\sim p_k-1)}, x_{k+1}, \zeta, t)}{\partial x_{k+1}} \dot{x}_{k+1} + \frac{\partial l_k(z^{(0\sim p_k-1)}, x_{k+1}, \zeta, t)}{\partial t} \\ &= F_k(z^{(0\sim p_k)}, x_{k+1}, \zeta, t) + \frac{\partial l_k(z^{(0\sim p_k-1)}, x_{k+1}, \zeta, t)}{\partial x_{k+1}} \dot{x}_{k+1}, \end{aligned} \tag{B.217}$$

where

$$\begin{aligned} F_k(z^{(0\sim p_k)}, x_{k+1}, \zeta, t) &= \dot{h}_k(z^{(0\sim p_k-1)}, \zeta, t) + \frac{\partial l_k(z^{(0\sim p_k-1)}, x_{k+1}, \zeta, t)}{\partial t} \\ &\quad + \frac{\partial l_k(z^{(0\sim p_k-1)}, x_{k+1}, \zeta, t)}{\partial z^{(0\sim p_k-1)}} z^{(1\sim p_k)}. \end{aligned} \tag{B.218}$$

Taking the $(m_{k+1}-1)$-order derivatives of both sides of (B.217), and then substituting the second one of (B.213) into the result, give

$$\begin{aligned} z^{(p_{k+1})} &= F_k^{(m_{k+1}-1)}(z^{(0\sim p_k)}, x_{k+1}, \zeta, t) + \sum_{j=0}^{m_{k+1}-2} C(j, m_{k+1}-1) \\ &\quad \times \left(\frac{\partial l_k(z^{(0\sim p_k-1)}, x_{k+1}, \zeta, t)}{\partial z^{(0\sim p_k-1)}} \right)^{(m_{k+1}-1-j)} x_{k+1}^{(j+1)} \\ &\quad + \frac{\partial l_k(z^{(0\sim p_k-1)}, x_{k+1}, \zeta, t)}{\partial x_{k+1}} x_{k+1}^{(m_{k+1})} \\ &= h_{k+1}(z^{(0\sim p_{k+1}-1)}, \zeta, t) + l_{k+1}(z^{(0\sim p_{k+1}-1)}, u, \zeta, t), \end{aligned} \tag{B.219}$$

where

$$\begin{aligned} & h_{k+1}(z^{(0\sim p_{k+1}-1)},\zeta,t) \\ = \; & F_k^{(m_{k+1}-1)}(z^{(0\sim p_k)},x_{k+1},\zeta,t) \\ & + \sum_{j=0}^{m_{k+1}-1} C(j,m_{k+1}-1)\left(\frac{\partial l_k(z^{(0\sim p_k-1)},x_{k+1},\zeta,t)}{\partial z^{(0\sim p_k-1)}}\right)^{(m_{k+1}-1-j)} x_{k+1}^{(j+1)} \\ & + \frac{\partial l_k(z^{(0\sim p_k-1)},x_{k+1},\zeta,t)}{\partial x_{k+1}} f_{k+1}(x_i^{(0\sim m_i)}|_{i=1\sim k+1},\zeta,t), \end{aligned}$$

and

$$\begin{aligned} & l_{k+1}(z^{(0\sim p_{k+1}-1)},u,\zeta,t) \\ = \; & \frac{\partial l_k(z^{(0\sim p_k-1)},x_{k+1},\zeta,t)}{\partial x_{k+1}} g_{k+1}(x_i^{(0\sim m_i-1)}|_{i=1\sim k},u,\zeta,t). \end{aligned}$$

Finally, it follows from Assumption 8.5, and the definitions of the above $l_{k+1}(z^{(0\sim p_{k+1}-1)},x_{k+1},\zeta,t)$ that system (B.219) is fully-actuated. Thus the theorem holds for the case of $n=k+1$. Therefore, the whole proof is completed.

B.6 Proofs of Theorems in Chapter 11

B.6.1 Proof of Theorem 11.1

Since $A_{0\sim n-1}\in\mathbb{R}^{r\times nr}$ and $L\in\mathbb{R}^{nr\times m}$ respectively satisfy (11.11) and (11.13), there exist two positive definite matrices P^{c} and P^{o} satisfying (11.12) and (11.14), respectively. Then the following Lyapunov function can be chosen for system (11.23):

$$V=V_1+V_2, \tag{B.220}$$

where

$$V_1=\frac{1}{2}\left(\hat{x}^{(0\sim n-1)}\right)^{\mathrm{T}} P^{\mathrm{c}}\hat{x}^{(0\sim n-1)}, \tag{B.221}$$

and

$$V_2=\frac{1}{2}\left(\tilde{x}^{(0\sim n-1)}\right)^{\mathrm{T}} P^{\mathrm{o}}\tilde{x}^{(0\sim n-1)}. \tag{B.222}$$

In view of (11.12) and (11.17), we have

$$\begin{aligned} \dot{V}_1 &= \frac{1}{2}\left(\hat{x}^{(0\sim n-1)}\right)^{\mathrm{T}}\left(\Phi^{\mathrm{T}}(A_{0\sim n-1})\,P^{\mathrm{c}}+P^{\mathrm{c}}\Phi(A_{0\sim n-1})\right)\hat{x}^{(0\sim n-1)} \\ &\quad -\left(\hat{x}^{(0\sim n-1)}\right)^{\mathrm{T}} P^{\mathrm{c}}LC_{0\sim n-1}\tilde{x}^{(0\sim n-1)} \\ &\le -\frac{\mu_{\mathrm{c}}}{2}\left(\hat{x}^{(0\sim n-1)}\right)^{\mathrm{T}} P^{\mathrm{c}}\hat{x}^{(0\sim n-1)}+W, \end{aligned} \tag{B.223}$$

where

$$W=-\left(\hat{x}^{(0\sim n-1)}\right)^{\mathrm{T}} P^{\mathrm{c}}LC_{0\sim n-1}\tilde{x}^{(0\sim n-1)}.$$

Using Young's inequality, and the following relations

$$\left(\hat{x}^{(0\sim n-1)}\right)^{\mathrm{T}} P^{\mathrm{c}} \left(P^{\mathrm{c}}\right)^{\mathrm{T}} \hat{x}^{(0\sim n-1)} \leq \lambda_{\max}\left(P^{\mathrm{c}}\right) \left(\hat{x}^{(0\sim n-1)}\right)^{\mathrm{T}} P^{\mathrm{c}} \hat{x}^{(0\sim n-1)}, \tag{B.224}$$

$$\lambda_{\min}\left(P^{\mathrm{o}}\right) \left(\tilde{x}^{(0\sim n-1)}\right)^{\mathrm{T}} \tilde{x}^{(0\sim n-1)} \leq \left(\tilde{x}^{(0\sim n-1)}\right)^{\mathrm{T}} P^{\mathrm{o}} \tilde{x}^{(0\sim n-1)}, \tag{B.225}$$

we can obtain

$$\begin{aligned} W &\leq \frac{1}{2}\left(\hat{x}^{(0\sim n-1)}\right)^{\mathrm{T}} P^{\mathrm{c}} \left(P^{\mathrm{c}}\right)^{\mathrm{T}} \hat{x}^{(0\sim n-1)} + \frac{1}{2}\left\| LC_{0\sim n-1}\tilde{x}^{(0\sim n-1)}\right\|^2 \\ &\leq \frac{1}{2}\lambda_{\max}\left(P^{\mathrm{c}}\right)\left(\hat{x}^{(0\sim n-1)}\right)^{\mathrm{T}} P^{\mathrm{c}} \hat{x}^{(0\sim n-1)} \\ &\quad + \frac{\|LC_{0\sim n-1}\|^2}{2\lambda_{\min}\left(P^{\mathrm{o}}\right)}\left(\tilde{x}^{(0\sim n-1)}\right)^{\mathrm{T}} P^{\mathrm{o}} \tilde{x}^{(0\sim n-1)}. \end{aligned} \tag{B.226}$$

Then it follows from (B.223) that

$$\begin{aligned} \dot{V}_1 &\leq -\frac{1}{2}\left(\mu_{\mathrm{c}} - \lambda_{\max}\left(P^{\mathrm{c}}\right)\right)\left(\hat{x}^{(0\sim n-1)}\right)^{\mathrm{T}} P^{\mathrm{c}} \hat{x}^{(0\sim n-1)} \\ &\quad + \frac{1}{2}\frac{\|LC_{0\sim n-1}\|^2}{\lambda_{\min}\left(P^{\mathrm{o}}\right)}\left(\tilde{x}^{(0\sim n-1)}\right)^{\mathrm{T}} P^{\mathrm{o}} \tilde{x}^{(0\sim n-1)}. \end{aligned} \tag{B.227}$$

In view of (11.14), (11.16), (11.20), and (B.225), we have

$$\begin{aligned} \dot{V}_2 &= \frac{1}{2}\left(\tilde{x}^{(0\sim n-1)}\right)^{\mathrm{T}} \left(\left(\Phi\left(0_{0\sim n-1}\right) + LC_{0\sim n-1}\right)^{\mathrm{T}} P^{\mathrm{o}} + \right. \\ &\quad \left. + P^{\mathrm{o}}\left(\Phi\left(0_{0\sim n-1}\right) + LC_{0\sim n-1}\right)\right)\tilde{x}^{(0\sim n-1)} + \left(\tilde{x}^{(0\sim n-1)}\right)^{\mathrm{T}} P^{\mathrm{o}} B_c \Delta f(\cdot) \\ &\leq -\frac{\mu_{\mathrm{o}}}{2}\left(\tilde{x}^{(0\sim n-1)}\right)^{\mathrm{T}} P^{\mathrm{o}} \tilde{x}^{(0\sim n-1)} + \rho\left\|P^{\mathrm{o}} B_c\right\|\left\|\tilde{x}^{(0\sim n-1)}\right\|^2 \\ &\leq -\frac{\mu_{\mathrm{o}}}{2}\left(\tilde{x}^{(0\sim n-1)}\right)^{\mathrm{T}} P^{\mathrm{o}} \tilde{x}^{(0\sim n-1)} \\ &\quad + \frac{\rho\left\|P^{\mathrm{o}} B_c\right\|}{\lambda_{\min}\left(P^{\mathrm{o}}\right)}\left(\tilde{x}^{(0\sim n-1)}\right)^{\mathrm{T}} P^{\mathrm{o}} \tilde{x}^{(0\sim n-1)}. \end{aligned} \tag{B.228}$$

Combining (B.227) and (B.228), gives

$$\begin{aligned} \dot{V} &\leq -\frac{1}{2}\left(\mu_{\mathrm{c}} - \lambda_{\max}\left(P^{\mathrm{c}}\right)\right)\left(\hat{x}^{(0\sim n-1)}\right)^{\mathrm{T}} P^{\mathrm{c}} \hat{x}^{(0\sim n-1)} \\ &\quad + \frac{1}{2}\frac{\|LC_{0\sim n-1}\|^2}{\lambda_{\min}\left(P^{\mathrm{o}}\right)}\left(\tilde{x}^{(0\sim n-1)}\right)^{\mathrm{T}} P^{\mathrm{o}} \tilde{x}^{(0\sim n-1)} \\ &\quad - \frac{\mu_{\mathrm{o}}}{2}\left(\tilde{x}^{(0\sim n-1)}\right)^{\mathrm{T}} P^{\mathrm{o}} \tilde{x}^{(0\sim n-1)} + \frac{\rho\left\|P^{\mathrm{o}} B_c\right\|}{\lambda_{\min}\left(P^{\mathrm{o}}\right)}\left(\tilde{x}^{(0\sim n-1)}\right)^{\mathrm{T}} P^{\mathrm{o}} \tilde{x}^{(0\sim n-1)} \\ &= -\frac{1}{2}\left(\mu_{\mathrm{c}} - \lambda_{\max}\left(P^{\mathrm{c}}\right)\right)\left(\hat{x}^{(0\sim n-1)}\right)^{\mathrm{T}} P^{\mathrm{c}} \hat{x}^{(0\sim n-1)} \\ &\quad - \frac{1}{2}\varpi\left(\tilde{x}^{(0\sim n-1)}\right)^{\mathrm{T}} P^{\mathrm{o}} \tilde{x}^{(0\sim n-1)}, \end{aligned} \tag{B.229}$$

where

$$\varpi = \mu_{\mathrm{o}} - \frac{2\rho \left\| P^{\mathrm{o}} B_c \right\|}{\lambda_{\min}\left(P^{\mathrm{o}}\right)} - \frac{\left\| L C_{0\sim n-1} \right\|^2}{\lambda_{\min}\left(P^{\mathrm{o}}\right)}.$$

If (11.24) holds, then we have

$$\dot{V} \leq -kV, \tag{B.230}$$

where

$$k = \min\left\{\mu_{\mathrm{c}} - \lambda_{\max}\left(P^{\mathrm{c}}\right), \varpi\right\} > 0. \tag{B.231}$$

Based on the Comparison Theorem, we have from (B.230) that

$$V \leq V(0)\,\mathrm{e}^{-kt},$$

thus the conclusion is easily obtained. Then the proof is completed.

B.6.2 Proof of Theorem 11.7

Firstly, let us show that under the given condition, subsystem (11.65) is globally exponentially stable. To do this, let us define a candidate Lyapunov function as

$$V(e) = e^{\mathrm{T}} P e,$$

whose time derivative along the trajectory of system (11.54) is

$$\dot{V}(e) = e^{\mathrm{T}}\left(\Phi_{33}^{\mathrm{T}} P + P\Phi_{33}\right) e + 2e^{\mathrm{T}} P B_L \Delta f_0.$$

By Assumption 11.5, we have

$$\left\| \Delta f_0 \right\| \leq \alpha \left\| x_2^{(0\sim n_2-1)} - \hat{x}_2^{(0\sim n_2-1)} \right\| = \alpha \left\| e \right\|,$$

which is equivalent to

$$(\Delta f_0)^{\mathrm{T}} \Delta f_0 - \alpha^2 e^{\mathrm{T}} e \leq 0. \tag{B.232}$$

Therefore, we have, using (11.55) and (11.56), the following:

$$\begin{aligned}
\dot{V}(e) &\leq e^{\mathrm{T}}\left(\Phi_{33}^{\mathrm{T}} P + P\Phi_{33}\right) e + 2e^{\mathrm{T}} P B_L \Delta f_0 - k\left((\Delta f_0)^{\mathrm{T}} \Delta f_0 - \alpha^2 e^{\mathrm{T}} e\right) \\
&= \begin{bmatrix} e \\ \Delta f_0 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Phi_{33}^{\mathrm{T}} P + P\Phi_{33} + k\alpha^2 I_n & P B_L \\ B_L^{\mathrm{T}} P & -kI_r \end{bmatrix} \begin{bmatrix} e \\ \Delta f_0 \end{bmatrix} \\
&\leq -\varepsilon \begin{bmatrix} e \\ \Delta f_0 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} e \\ \Delta f_0 \end{bmatrix} \\
&= -\varepsilon\left(\left\| e \right\|^2 + \left\| \Delta f_0 \right\|^2\right) \\
&\leq -\varepsilon \left\| e \right\|^2.
\end{aligned}$$

This immediately implies that subsystem (11.65) is globally exponentially stable.

Secondly, denote

$$X = \begin{bmatrix} x_1^{(0\sim n_1-1)} \\ x_2^{(0\sim n_2-1)} \end{bmatrix}.$$

Then, recalling (11.36), we have

$$\begin{aligned} \Delta f_0(\cdot) &= \Delta f_0\left(x_1^{(0\sim n_1-1)}, x_2^{(0\sim n_2-1)}, \hat{x}_2^{(0\sim n_2-1)}\right) \\ &= \Delta f_0\left(x_1^{(0\sim n_1-1)}, x_2^{(0\sim n_2-1)}, x_2^{(0\sim n_2-1)} - e\right) \\ &\triangleq \Delta \tilde{f}_0(X, e), \end{aligned}$$

and the following relation still holds:

$$\left\| \Delta \tilde{f}_0(X, e) \right\| \leq \alpha \|e\|. \tag{B.233}$$

Thus system (11.64) can be written as

$$\dot{X} = \Phi_F X + B_E \Delta \tilde{f}_0(X, e). \tag{B.234}$$

Finally, it suffices to show that the above system (B.234) is input-to-state stable with e being viewed as the input.

Since Φ_F is Hurwitz, there exists a positive definite matrix Q such that

$$\Phi_F^{\mathrm{T}} Q + Q\Phi_F = -2I. \tag{B.235}$$

Define a candidate Lyapunov function for system (B.234) as

$$V(X, t) = X^{\mathrm{T}} Q X.$$

Then, using (B.233) and (B.235), we can derive

$$\begin{aligned} \dot{V}(X) &= \dot{X}^{\mathrm{T}} P_1 X + X^{\mathrm{T}} P_1 \dot{X} \\ &= X^{\mathrm{T}}\left(\Phi_F^{\mathrm{T}} Q + Q\Phi_F\right) X + 2X^{\mathrm{T}} Q B_E \Delta \tilde{f}_0(X, e) \\ &\leq -2\|X\|^2 + \left\| 2X^{\mathrm{T}} Q B_E \Delta \tilde{f}_0(X, e) \right\| \\ &\leq -2\|X\|^2 + 2\left\|X^{\mathrm{T}}\right\| \left\| Q B_E \Delta \tilde{f}_0(X, e) \right\| \\ &\leq -2\|X\|^2 + \left\|X^{\mathrm{T}}\right\|^2 + \left\| Q B_E \Delta \tilde{f}_0(X, e) \right\|^2 \\ &\leq -\|X\|^2 + \|Q B_E\|^2 \left\| \Delta \tilde{f}_0(X, e) \right\|^2 \\ &\leq -\|X\|^2 + \alpha^2 \|Q B_E\|^2 \|e\|^2. \end{aligned}$$

Therefore, system (B.234) is input-to-state stable with e being viewed as the input. Combining this with the global exponential stability of subsystem (11.65), gives the conclusion of the theorem.

B.7 Proofs of Lemmas in Chapter 12

B.7.1 Proof of Lemma 12.3

Rewrite the closed-loop system (12.39) or (12.40) as

$$x^{(n)} + A_{0\sim n-1}x^{(0\sim n-1)} = \phi(\cdot) + h\left(z^{(0\sim m-1)}\right), \tag{B.236}$$

or, equivalently,

$$\dot{x}^{(0\sim n-1)} = \Phi(A_{0\sim n-1})\, x^{(0\sim n-1)} + \begin{bmatrix} 0_{(n-1)r} \\ \phi(\cdot) + h\left(z^{(0\sim m-1)}\right) \end{bmatrix}, \tag{B.237}$$

with

$$\phi(\cdot) = -\Theta x^{(0\sim n-1)} + \Delta f(\cdot). \tag{B.238}$$

Since $A_{0\sim n-1}$ satisfies condition (12.33), there exists a positive definite matrix $P(A_{0\sim n-1})$ satisfying the Lyapunov inequality (12.34). Then the following Lyapunov function can be chosen for system (B.237):

$$V = \frac{1}{2}\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P(A_{0\sim n-1})x^{(0\sim n-1)}, \tag{B.239}$$

and, in view of (12.34), (12.36) and (B.237), we have

$$\begin{aligned}
\dot{V} &= \frac{1}{2}\left(\dot{x}^{(0\sim n-1)}\right)^{\mathrm{T}} P x^{(0\sim n-1)} + \frac{1}{2}\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P\dot{x}^{(0\sim n-1)} \\
&= \frac{1}{2}\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} \left(\Phi^{\mathrm{T}}P + P\Phi\right) x^{(0\sim n-1)} + \left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P \begin{bmatrix} 0_{(n-1)r} \\ \phi + h\left(z^{(0\sim m-1)}\right) \end{bmatrix} \\
&\leq -\frac{\mu}{2}\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P x^{(0\sim n-1)} + \left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L\left(\phi + h\left(z^{(0\sim m-1)}\right)\right) \\
&= W_0 + W_1 + W_2,
\end{aligned} \tag{B.240}$$

where

$$W_0 = -\frac{\mu}{2}\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P x^{(0\sim n-1)} + \left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L h\left(z^{(0\sim m-1)}\right), \tag{B.241}$$

$$W_1 = -\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L \Theta x^{(0\sim n-1)}, \tag{B.242}$$

$$W_2 = \left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L \Delta f(\cdot). \tag{B.243}$$

Firstly, let us treat W_0. Using Assumption 12.10, we can show

$$W_0 \leq \frac{1}{2}\frac{\|P_L\|^2}{\mu\sigma_{\min}(P)}\kappa_h^2\left(\left\|z^{(0\sim m-1)}\right\|\right). \tag{B.244}$$

Next, using the expression for Θ, we can get

$$\begin{aligned} W_1 &= -\frac{1}{\varepsilon\sigma^2_{\min}(P_L)}\vartheta\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L P_L^{\mathrm{T}} x^{(0\sim n-1)} \qquad \text{(B.245)} \\ &\leq -\frac{1}{\varepsilon\sigma^2_{\min}(P_L)}\vartheta\sigma^2_{\min}(P_L)\left\|x^{(0\sim n-1)}\right\|^2 \\ &\leq -\frac{1}{\varepsilon}\vartheta\left\|x^{(0\sim n-1)}\right\|^2. \qquad \text{(B.246)} \end{aligned}$$

Further, using Assumption 12.9, we can prove

$$W_2 = \vartheta\left\|x^{(0\sim n-1)}\right\|^2 + \frac{1}{2}\left\|P_L^{\mathrm{T}}\right\|^2\kappa^2\left(\left\|z^{(0\sim m-1)}\right\|\right). \qquad \text{(B.247)}$$

Finally, using the above two obtained relations (B.246) and (B.247), together with (B.244), produces

$$\begin{aligned} \dot{V} \leq & -\frac{1-\varepsilon}{\varepsilon}\vartheta\left\|x^{(0\sim n-1)}\right\|^2 + \frac{1}{2}\left\|P_L^{\mathrm{T}}\right\|^2\kappa^2\left(\left\|z^{(0\sim m-1)}\right\|\right) \\ & +\frac{1}{2}\frac{\|P_L\|^2}{\mu\sigma_{\min}(P)}\kappa_h^2\left(\left\|z^{(0\sim m-1)}\right\|\right). \end{aligned}$$

Therefore the conclusion follows from Lemma 12.1.

B.7.2 Proof of Lemma 12.4

Obviously, system (12.54) can be written in the following state-space form:

$$\dot{x}^{(0\sim n-1)} = \Phi(A_{0\sim n-1})\,x^{(0\sim n-1)} + \begin{bmatrix} 0_{(n-1)r} \\ \phi'(\cdot) + h\left(z^{(0\sim m-1)}\right) \end{bmatrix}, \qquad \text{(B.248)}$$

with

$$\phi'(\cdot) = -\Theta x^{(0\sim n-1)} + \Delta f(\cdot) + \Delta f_b(\cdot). \qquad \text{(B.249)}$$

Again, since $A_{0\sim n-1}$ satisfies condition (12.33), there exists a positive definite matrix $P(A_{0\sim n-1})$ satisfying the Lyapunov inequality (12.34). Then the following Lyapunov function can be chosen for system (B.248):

$$V = \frac{1}{2}\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P(A_{0\sim n-1})x^{(0\sim n-1)}, \qquad \text{(B.250)}$$

and, in view of (12.34), (12.36) and (B.248), we can derive, parallel to (B.240), the following relation:

$$\begin{aligned} \dot{V} &= \frac{1}{2}\left(\dot{x}^{(0\sim n-1)}\right)^{\mathrm{T}} P x^{(0\sim n-1)} + \frac{1}{2}\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P\dot{x}^{(0\sim n-1)} \\ &\leq -\frac{\mu}{2}\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P x^{(0\sim n-1)} + \left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L\left(\phi' + h\left(z^{(0\sim m-1)}\right)\right) \\ &= W_0 + W_1 + W_2', \qquad \text{(B.251)} \end{aligned}$$

where W_0 is still given by (B.241), and also satisfies (B.244), W_1 is given by (B.242), and

$$W_2' = \left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L \left(\Delta f\left(\cdot\right) + \Delta f_b\left(\cdot\right)\right).$$

Using Assumption B2, and (12.56)-(12.58), we can prove

$$\begin{aligned} & \left\|\Delta f\left(\cdot\right) + \Delta f_b\left(\cdot\right)\right\| \\ \leq\ & \hat{\rho}\left(\cdot\right)\left\|x^{(0\sim n-1)}\right\| + \hat{\sigma}\left(\cdot\right)\hat{\kappa}\left(\left\|z^{(0\sim m-1)}\right\|\right) + \frac{c}{\varepsilon}\frac{1}{\sigma_{\min}^2\left(P_L\right)}\vartheta\left\|P_L^{\mathrm{T}}x^{(0\sim n-1)}\right\|. \end{aligned}$$

Therefore,

$$\begin{aligned} W_2' &= \left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L\left(\Delta f + \Delta f_b\right) \\ &\leq \left\|P_L^{\mathrm{T}}x^{(0\sim n-1)}\right\|\left\|\Delta f\left(\cdot\right) + \Delta f_b\left(\cdot\right)\right\| \\ &= W_{21} + W_{22}, \end{aligned}$$

with

$$W_{21} = \frac{c}{\varepsilon}\frac{1}{\sigma_{\min}^2\left(P_L\right)}\vartheta\left\|P_L^{\mathrm{T}}x^{(0\sim n-1)}\right\|^2, \tag{B.252}$$

and

$$W_{22} = \hat{\rho}\left(\cdot\right)\left\|P_L^{\mathrm{T}}\right\|\left\|x^{(0\sim n-1)}\right\|^2 + \hat{\sigma}\left(\cdot\right)\left\|P_L^{\mathrm{T}}x^{(0\sim n-1)}\right\|\hat{\kappa}\left(\left\|z^{(0\sim m-1)}\right\|\right),$$

which can be shown to satisfy

$$\begin{aligned} W_{22} &= \hat{\rho}\left(\cdot\right)\left\|P_L^{\mathrm{T}}\right\|\left\|x^{(0\sim n-1)}\right\|^2 + \hat{\sigma}\left(\cdot\right)\left\|P_L^{\mathrm{T}}x^{(0\sim n-1)}\right\|\hat{\kappa}\left(\left\|z^{(0\sim m-1)}\right\|\right) \\ &\leq \left(\hat{\rho}\left(\cdot\right)\left\|P_L^{\mathrm{T}}\right\| + \frac{1}{2}\hat{\sigma}^2\left(\cdot\right)\right)\left\|x^{(0\sim n-1)}\right\|^2 + \frac{1}{2}\left\|P_L^{\mathrm{T}}\right\|^2\hat{\kappa}^2\left(\left\|z^{(0\sim m-1)}\right\|\right) \\ &= \vartheta\left\|x^{(0\sim n-1)}\right\|^2 + \frac{1}{2}\left\|P_L^{\mathrm{T}}\right\|^2\hat{\kappa}^2\left(\left\|z^{(0\sim m-1)}\right\|\right). \end{aligned} \tag{B.253}$$

Combining the above two obtained relations (B.245) and (B.252), gives

$$\begin{aligned} W_1 + W_{21} &= -\frac{1-c}{\varepsilon}\frac{1}{\sigma_{\min}^2\left(P_L\right)}\vartheta\left\|P_L^{\mathrm{T}}x^{(0\sim n-1)}\right\|^2 \\ &\leq -\frac{1-c}{\varepsilon}\vartheta\left\|x^{(0\sim n-1)}\right\|^2. \end{aligned} \tag{B.254}$$

This, together with (B.244) and (B.253), finally produces

$$\begin{aligned} \dot{V} \leq\ & -\frac{(1-c)-\varepsilon}{\varepsilon}\vartheta\left\|x^{(0\sim n-1)}\right\|^2 + \frac{1}{2}\left\|P_L^{\mathrm{T}}\right\|^2\hat{\kappa}^2\left(\left\|z^{(0\sim m-1)}\right\|\right) \\ & + \frac{1}{2}\frac{\left\|P_L\right\|^2}{\mu\sigma_{\min}\left(P\right)}\kappa_h^2\left(\left\|z^{(0\sim m-1)}\right\|\right). \end{aligned}$$

Therefore the conclusion follows from Lemma 12.1.

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